A Novel Node Placement for Long Belt Coverage in Wireless Networks

Bang Wang, Han Xu, *Student Member*, *IEEE*, Wenyu Liu, *Member*, *IEEE*, and Hui Liang

Abstract—Coverage is an important issue in many wireless networks. In this paper, we address the problem of node placement for ensuring complete coverage in a long belt scenario and propose a novel placement approach to minimize the number of nodes needed. In our work, each node is assumed to be able to cover a disk area centered at itself with a fixed radius, then a divide-and-cover node placement method is proposed. In the proposed method, a long belt is divided into some sub-belts (if necessary), and then a string of nodes are placed parallel to the long side of each sub-belt to completely cover the sub-belt. We then determine the optimal distance between two adjacent nodes in a string and the number of such strings to minimize the number of nodes for complete belt coverage. Theoretical proofs and analysis show that compared with other node placement including the well-known regular triangular-lattice placement, the proposed method can achieve lower node density in some cases when the belt height is not very large. A combination of the proposed method and the triangular-lattice placement is then proposed, and the optimal ranges of the belt height for their respective applications to achieve the lowest node density are computed.

Index Terms—Belt coverage, divide-and-cover placement, node placement, wireless networks

1 Introduction

Coverage is an important issue in many wireless networks, including cellular mobile networks, wireless local access networks (WLANs), and wireless sensor networks (WSNs) [1]. In these wireless networks, each node is often assumed to be able to cover a disk area centered at itself with a radius r. For example, a base station in cellular mobile networks can transmit or receive radio signals for those mobile phones within a disk centered at itself with the radius of its transmission range. A wireless sensor node in WSNs can sense and process environmental information from those points within a disk centered at itself with radius of its sensing range.

Network coverage is a collective measure about how an area of interests is covered by nodes within the area but with different geographical locations. An area is said completely covered if any of the space points within the area is covered by at least one node. Deterministic network deployment is to place nodes at planned, predetermined locations. When using deterministic deployment, it is often desirable to find a placement pattern such that the nodes' locations can be easily found for placing nodes. Furthermore, it is also desirable that such a placement pattern can achieve the lowest node density (the number of nodes per unit area) for complete coverage.

In this paper, we consider a network coverage problem in the long bounded belt scenario. In practice, placing transceivers to provide radio coverage for a long-distance tunnel is not uncommon in cellular mobile networks. A tunnel for transceiver placement is often abstracted as a *bounded long belt* from the engineering viewpoint. For example, the New York City subway is of around 337-km long, and about half of its routes are underground tunnels. Another famous example is the Channel Tunnel between Britain and French, which is the longest undersea rail tunnel (50.5 km) in the world. In China, the high-speed railway between Wuhan and Guangzhou contains 226 tunnels with the total length of 177.2 km or 16 percent of the total rail length. Moreover, underground mine is also a typical long bounded belt area. Sensors and transmitters can be placed within such scenarios for disaster (gas, fire or other disasters) monitoring and communication.

In this paper, we study how to place nodes as few as possible for completely covering a bounded long belt. Generally, the distance of a tunnel is much larger than the diameter of the tunnel. For example, the length of Channel Tunnel is 50.5 km, but the diameter of the tunnel is only 7.6 m. Thus, in our problem, we consider such a long bounded belt scenario with width D and height H, where $D \gg H$ and $D \gg r$ and r is the coverage radius of each node. The problem of node placement for complete coverage has been studied for scenarios of very large (or infinite) regions or small bounded rectangles. However, to the best of our knowledge, the problem of placing minimal number of nodes (disks[1]) in a bounded long belt to guarantee complete coverage has not been studied before.

In our work, we first propose a novel divide-and-cover disk placement method. In the proposed method, we divide a belt with height H into some sub-belts each with height H/k, $k = 1, 2, \dots$, and then place a string of disks with

• *The authors are with the Broadband Center, Department of Electronics and Information Engineering, Huazhong University of Science and Technology (HUST), 1037# Luoyu Road, Wuhan, Hubei 430074, China. E-mail: wangbang@hust.edu.cn, xuhan2000@hotmail.com, liuwy@mail.hust.edu.cn, jiaozhi100@163.com.*

Manuscript received 8 Jan. 2012; revised 5 May 2012; accepted 28 May 2012; published online 12 June 2012.
Recommended for acceptance by K. Roemer.
For information on obtaining reprints of this article, please send e-mail to: tc@computer.org, and reference IEEECS Log Number TC-2012-01-0022.
Digital Object Identifier no. 10.1109/TC.2012.145.

1. In this paper, each node is assumed to cover a disk centered at itself with coverage radius r, and the word "node" and "disk" are used interchangeably in this paper.

interdisk distance d parallel to the long side of each sub-belt to completely cover the sub-belt. We first prove what is the optimal interdisk distance, i.e., the optimal d, if k sub-belts are needed for complete belt coverage. For different values of belt height H, we then determine the optimal number of such strings to minimize the node density for complete belt coverage, i.e., the optimal k. Compared with the regular triangular-lattice placement [2], [3], [4] that is the optimal placement in an unbounded region, we prove that the proposed method can achieve a lower node density in some cases when the belt height is not very large. The key idea behind the improvement is that the interdisk distance should be properly determined according to the belt height, other than using a fixed one. In the long bounded belt scenario, the interdisk distance in the triangular-lattice placement might not be the optimal one for some values of the belt height. We then propose to use a combination of the proposed method and the regular triangular-lattice placement to provide complete belt coverage, where the optimal belt height ranges for their respective applications (divide-and-cover placement or triangular-lattice placement) to achieve the lowest node density are also computed. We compare its coverage performance with two other commonly used placement schemes, and the results prove its superiority in terms of lower node density for complete belt coverage. We also discuss the optimality factor, the impacts of left and right-boundary effects, and higher degree coverage issues for the combined placement scheme.

The rest of the paper is organized as follows: We discuss the related work in Section 2, present our method and performance analysis in Section 3, and provide some discussions in Section 4. The paper is concluded in Section 5.

2 Related Work

How to find an optimal node placement pattern in wireless networks has been widely studied in the literature [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. A well-known result is that the regular triangular-lattice pattern achieves the minimum node density to completely cover a very large region [2], [3]. In this pattern, each disk overlaps with six others, and the locations of the disks form a grid in which the nearest pair of disks are $\sqrt{3}r$ apart, where r is the radius of the disks. Brown et al. [5] propose a disk placement for 2-coverage, i.e., each point is covered at least by two disks. In their placement, starting with a single disk being placed within the region, three disks are then added, one centered at the 0 degree point on its circumference, the next at 120 degrees, and another at 240 degrees. And for each newly added disk, add three another disks centered at their circumference (if there are no such disks), at the 180, -60, and 60 degrees point. By this way, the whole region can achieve 2-coverage.

Recently, some researchers have proposed new placement patterns for guaranteeing both coverage and connectivity [7], [9], [10], [13]. For example, Kar and Banerjee [7] propose a strip pattern where nodes are first placed as horizontal strips to provide complete coverage, and then a vertical strip is added to guarantee network connectivity. Bai et al. [9] propose a strip-based pattern. In this pattern, some nodes are first placed as horizontal strips separated by $\sqrt{3}r$ apart, where r is the sensing radius, then two vertical strips are placed at the left and right boundary of horizontal strips, respectively, to connect the horizontal strips. Thus, both complete coverage and network connectivity are achieved. In [10], Bai et al. extend their study of optimal node placement pattern to higher connectivity requirement (up to 6-connectivity).

All of the above mentioned studies have assumed an unbounded region scenario when determining placement patterns. However, in many practical placements, nodes are often needed to be placed in some bounded area. For example, in cellular networks and WLANs, with the requirement for high quality of services, transmitters are required to be placed within buildings to increase indoor radio coverage [14], [15]. In WSNs, sensor nodes are widely deployed within buildings for fire monitoring or other applications [16]. For covering a bounded field, in [4] the field is partitioned into single-row regions and multirow regions: A single row of nodes are placed along the bisector of the single-row region; In multirow regions, nodes are first placed according to a regular triangular-lattice pattern, then some extra nodes are placed along the boundaries of each multirow region to ensure complete coverage.

The problem of barrier coverage [17], [18], [19], [20], [21], [22], [23], [24], [25], [26] has a similar placement scenario to the belt coverage problem, but there are some distinct differences in the two problems. The main purpose of the belt coverage is to cover all the points within the belt. Barrier coverage aims at constructing a chain of sensors connecting two points or enclosing a protected region, with the sensing areas of any two adjacent sensors overlapping with each other's. For a given deployment field, barrier coverage normally does not require that all points of the field to be covered. For example, Chen et al. [21] develop a novel sleep-wake up algorithm to construct barriers that can maximize the network barrier lifetime.

From the viewpoint of geometry [27], [28], [29], our belt coverage problem resembles the problem of placing disks to completely cover a rectangle [30], [31], [32], [33]. The objective is to minimize the radius of disks for completely covering a small rectangle, when a fixed number of disks are used. In [30], based on a graph theoretic approach, a locally optimal circle placement pattern for a square with up to 10 equal circles has been found. In [31], when the width and the length of the rectangle are comparable, several deterministic disk placements have been proposed, and an optimal placement with the minimal radius can be obtained with less than or equal to 5 and 7 disks. Melissen and Schuur [33] show the optimal placement of six and eight disks with the minimal radius. Based on the simulated annealing method, a new optimal placement with eleven disks is also presented. Nurmela and Ostergard [32] use a quasi-Newton method to minimize the uncovered area by moving the disks, and the radii of the disks are further adapted to find locally optimal placement. They present the best placement of a unit-area square with up to 30 disks. However, the above-mentioned exact solutions for small number of disks do not provide guidance for covering a long belt scenario, since the aims of the two problems are different.

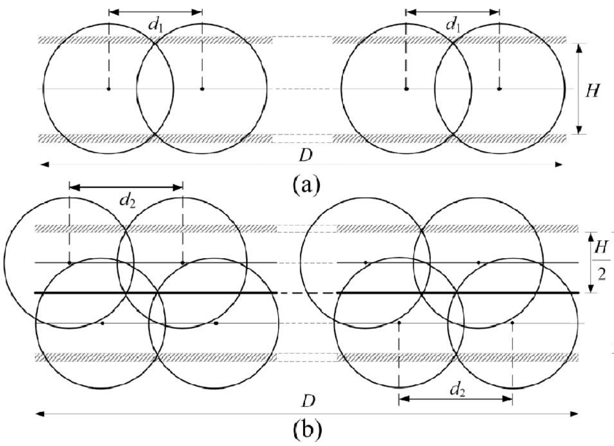

Fig. 1. Equipartition (d, r)-strip placement. (a) $k = 1$. (b) $k = 2$.

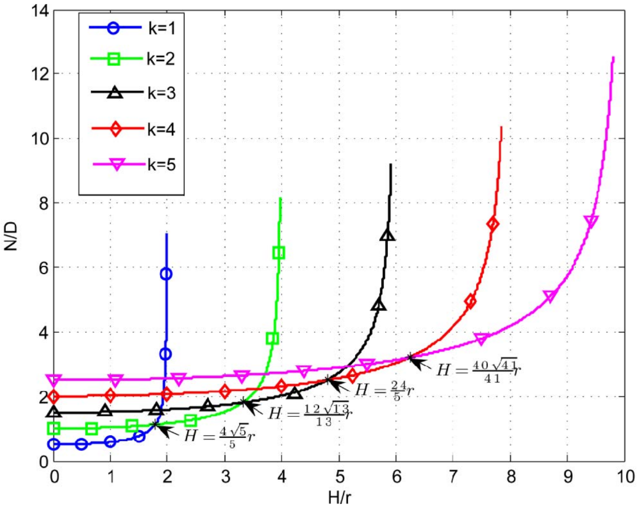

Fig. 2. The normalized number of disks used in equipartition (d, r)-strip placement ($k = 1, 2, 3, 4, 5$) versus the normalized height H of the long belt area.

3 A Novel Disk Placement for Complete Belt Coverage

In this section, we introduce a novel disk placement for complete belt coverage. In our method, we place strings of disks parallel to the long side of the belt region. We then determine the optimal distance between two adjacent disks in a string and the number of such strings to guarantee complete belt coverage. In determining the placement, we assume that the length of the belt is much larger than the height of the belt such that the left and right-boundary effect can be neglected. In this section, we consider to provide 1-coverage, i.e., each point in the belt should be covered by at least one disk. The left and right-boundary effect and the k-coverage problem will be discussed in the next section.

3.1 Preliminary

Definition 1 ((d, r)-strip). *A (d, r)-strip is a string of identical disks each with radius r placed along a line such that the distance between the centers of any two adjacent disks is d.*

We call such a line the strip *center line* and d the strip *disk distance*. The *maximal effective coverage region* of a (d, r)-strip depends on the *critical height* of the strip, which is the distance between the two intersections of two adjacent disks. The critical height can be computed by

$$h = \sqrt{4r^2 - d^2}. \tag{1}$$

Thus, the maximal effective coverage region of a (d, r)-strip is a rectangular region with the same height $\sqrt{4r^2 - d^2}$. Obviously, a (d, r)-strip can only provide complete coverage for a belt with height less than $2r$.

A divide-and-cover method to provide complete coverage for a belt is to divide the belt into many sub-belts parallel to its long side such that the height of each sub-belt is less than $2r$ and each sub-belt can be completely covered by a single (d, r)-strip. Suppose that a belt with height $H < 2kr$ is to be divided into k sub-belts or more. There are many potential partitions satisfying that the height of each sub-belt is less than $2r$. Among many potential partitions, we consider an equipartition such that all sub-belts have the same height. Correspondingly, we call other potential partitions as nonequipartition.

Definition 2 (Equipartition (d, r)-strip placement). *Given a belt with height H, divide this belt into k equal sub-belts by $k - 1$ lines parallel to the longer side of the belt, and place one (d, r)-strip in each sub-belt such that the strip center line is on the bisector of this sub-belt and the strip disk distance is*

$$d_k = \sqrt{4r^2 - (H/k)^2}. \tag{2}$$

Here, H meets the condition $H < 2kr$.

Fig. 1a shows the disk placement when $k = 1$. Fig. 1b shows the disk placement when $k = 2$, where the belt is divided into two equal parts. The maximal effective coverage region of such an equipartition (d_k, r)-strip is a rectangular region with height $\sqrt{4r^2 - d_k^2}$. According to (2), we have

$$k\sqrt{4r^2 - d_k^2} = k\sqrt{4r^2 - \left(\sqrt{4r^2 - (H/k)}\right)^2} = H.$$

That is, k equipartition (d_k, r)-strip placement can provide complete belt coverage. The total number of disks used in equipartition (d, r)-strip placement is given by

$$N_k^{(e)} = \frac{kD}{d_k} = \frac{kD}{\sqrt{4r^2 - (H/k)^2}}. \tag{3}$$

Fig. 2 shows the normalized number of disks used in equipartition (d, r)-strip placement $(k = 1, 2, 3, 4, 5)$ with respect to different values of the normalized belt height H.

Another commonly used placement is the triangle tessellation, where the centers of disks form an equal triangular-lattice with side length $\sqrt{3}r$. It is well known that such a triangle tessellation achieves the minimum number of disks to provide complete coverage for a very large plane [2]. In our context of long belt region, we consider the following adapted triangular-lattice placement.

Definition 3 (Triangular-lattice placement). *Case 1: $0 < H \le r$, place disks on the bisector of the belt area separated by a distance of $\sqrt{3}r$.*

Case 2: $r < H$, place an initial disk in the belt area such that the distance between the center of the disk and one of the

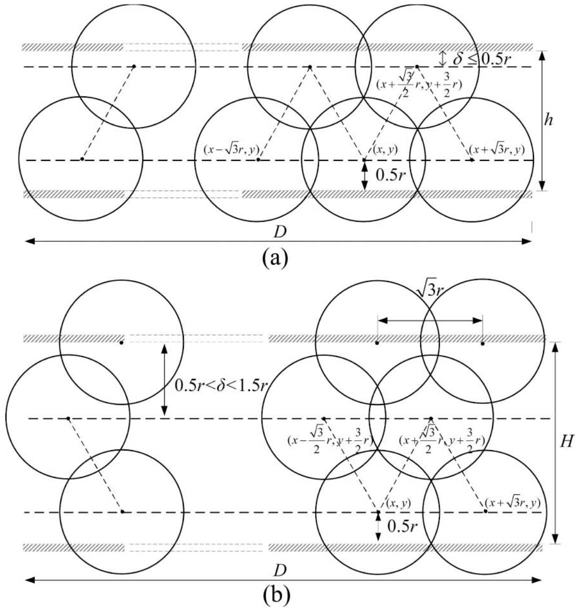

Fig. 3. Triangular-lattice placement.

longer side of the belt area is $0.5r$, as shown in Fig. 3. Suppose that the coordinates of the center of the initial disk are (x, y). Then, the points $(x \pm \sqrt{3}mr, y \pm 3nr)$ and

$$\left(x + \sqrt{3}\left(\frac{1}{2} \pm m\right)r, y + 3\left(\frac{1}{2} \pm n\right)r\right)$$
$$(m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots)$$

are the locations of other disks' centers if these points are inside the belt area. If the distance δ between the last row of disks and the other longer side of the belt area is no larger than $0.5r$, as shown in Fig. 3a, the given placement can provide complete belt coverage. Otherwise, when $0.5r < \delta < 1.5r$, for complete belt coverage, we place extra disks along the longer side at the intersection points of the side and the mid-perpendiculars of any two adjacent disks in the adjacent strip. Thus, extra disks are separated by a distance of $\sqrt{3}r$, as shown in Fig. 3b.

The number of disks used in the triangular-lattice placement can be computed by

$$N_k^{(t)} = \begin{cases} \dfrac{D}{\sqrt{3}r}, 0 < H \leq r \\ \dfrac{kD}{\sqrt{3}r}, (1.5k-2)r < H \leq (1.5k-0.5)r, k = 2, 3, \ldots \end{cases} \tag{4}$$

Fig. 4 plots the normalized number of disks used in the triangular-lattice placement with respect to different values of the normalized belt height.

Definition 4 (Coverage density, Node (disk) density). *Given a belt to be covered by disks, the* coverage density *ρ of a placement of disks is defined as the ratio of the union of the coverage area of all disks to the area of the region to be covered; the* node (disk) density *λ is defined as the number of nodes (disks) per unit area.*

Suppose that the area of a given region is A and there are m disks that can completely cover the region. According to Definition 4, we have

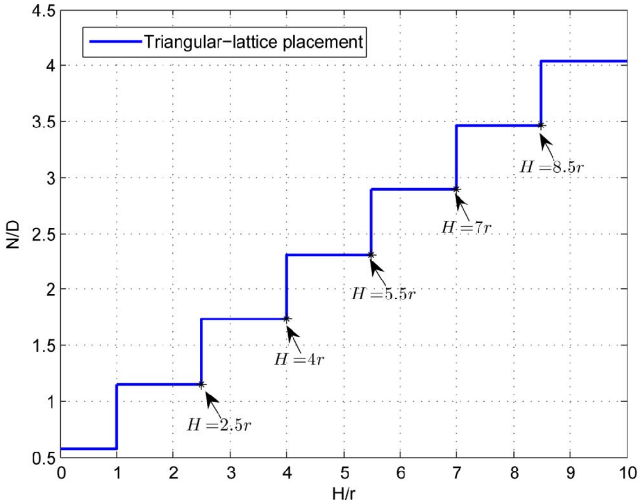

Fig. 4. The number of disks used in triangular-lattice placement change with height H of the long belt area.

$$\rho = m\pi r^2/A, \quad \lambda = m/A, \quad \rho = \lambda\pi r^2.$$

Since A, π, and r are constant, thus minimizing m is equivalent to minimizing ρ and λ.

Below, Lemma 1 and Lemma 2 show the minimum number of disk strips with respect to different belt heights to guarantee complete belt coverage for the equipartition placement and triangular-lattice placement, respectively. Lemma 3 states what is the optimal placement pattern if a belt is only covered by a single (d, r)-strip.

Lemma 1. *If the equipartition (d, r)-strip placement is used, then for a belt with height $H > 0$, at least*

$$K_{\min}^{(e)} = \lfloor H/2r \rfloor + 1 \tag{5}$$

rows of (d, r)-strips are needed to provide complete belt coverage.

Proof of lemma 1. According to Definition 1, the maximal effective coverage region of a (d, r)-strip is a rectangular area with height $\sqrt{4r^2 - d^2}$. Thus, when the height of the belt is H, the number of (d, r)-strip is

$$K^{(e)} = \left\lfloor H/\sqrt{4r^2 - d^2} \right\rfloor + 1.$$

Since $d > 0$, when $d \to 0$, we can get

$$K_{\min}^{(e)} = \lim_{d \to 0} K^{(e)} = \lfloor H/2r \rfloor + 1.$$

□

Lemma 2. *If the triangular-lattice placement is used, then for a belt with height H, if $0 < H \leq r$, at least $K_{\min}^{(t)} = 1$ row of disks is needed. And if*

$$(1.5k-2)r < H \leq (1.5k-0.5)r, k = 2, 3, \ldots, \tag{6}$$

at least $K_{\min}^{(t)} = k$ rows of disks are needed to provide complete belt coverage.

Proof of Lemma 2. In the triangular-lattice placement, the maximal effective coverage region of using only a row of sensors is a rectangular area with height

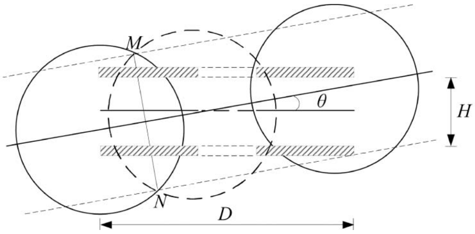

Fig. 5. Using one (d, r)- strip to cover a long belt area.

$$h = \sqrt{4r^2 - d^2} = \sqrt{4r^2 - (\sqrt{3}r)^2} = r. \tag{7}$$

Therefore, when $0 < H \leq r$, only one row of disks is needed.

When $r < H$, it is evident that we need more than one row of (d, r)-strip. If there are k $(k \geq 2)$ rows of disks in the triangular-lattice placement, then the maximal completely covered region is a rectangular area with height

$$h_k = (1.5k - 0.5)r. \tag{8}$$

Substitute $k = k - 1$ into (8), we have

$$h_{k-1} = (1.5k - 2)r. \tag{9}$$

To provide complete belt coverage with at least k rows of (d, r)-strips, the following condition must be satisfied:

$$h_{k-1} < H \leq h_k. \tag{10}$$

That is, if

$$(1.5k - 2)r < H \leq (1.5k - 0.5)r, k = 2, 3, \ldots,$$

at least k rows of disks are needed. □

Lemma 3. *For a belt with width D and height $H < 2r$, if only one (d, r)-strip is used to completely cover the belt, placing the strip on the bisector of the belt with $d = \sqrt{4r^2 - H^2}$ minimizes the number of disks for complete belt coverage.*

Proof of Lemma 3. As shown in Fig. 5, there are n disks in a (d, r)-strip and the angle contained by the bisector of the belt and the center line of the strip is θ. M and N are two intersection points of two adjacent disks.

According to Definition 1, we have $|MN| < 2r$. To cover the belt area completely, the following conditions must be satisfied:

$$|MN| \cos\theta \geq H \tag{11}$$

$$nd\cos\theta \geq D. \tag{12}$$

According to (12), we have $n \geq D/d\cos\theta$. Since

$$d = \sqrt{4r^2 - |MN|^2},$$

we can get

$$n \geq \frac{D}{\cos\theta\sqrt{4r^2 - |MN|^2}} \geq \frac{D}{\sqrt{4r^2\cos^2\theta - H^2}}.$$

Thus, the minimal number of disks is obtained when $\theta = 0$ and $|MN| = H$. In this case,

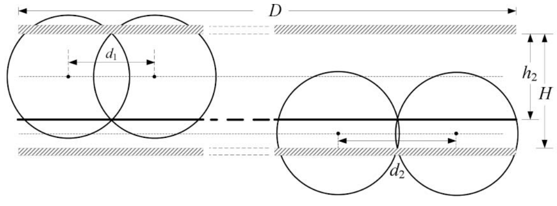

Fig. 6. A subdomain of the belt area when $k = 2$.

$$n_{\min} = \frac{D}{\sqrt{4r^2 - H^2}}.$$

From this equation, we know, to minimize n, the center line of the strip should coincide with the bisector of the given belt area and the strip disk distance should be $\sqrt{4r^2 - H^2}$. Therefore, Lemma 3 is true. □

The following lemma states that in the divide-and-cover method, the equipartition placement needs the minimum number of disks, compared with all other nonequipartition placements.

Lemma 4. *Given a belt with height $H < 2kr$, if the belt is to be divided into k sub-belts such that each sub-belt should be completely covered by one (d, r)-strip, then the equipartition (d, r)-strip placement with*

$$d = \sqrt{4r^2 - (H/k)^2}$$

needs the minimum number of disks for complete belt coverage. In this case, the coverage density is

$$\rho_k^{(e)} = \frac{k^2\pi r^2}{H\sqrt{4k^2r^2 - H^2}}. \tag{13}$$

The minimum coverage density is $\pi/2$ when $H = \sqrt{2}kr$.

Proof. Lemma 4 will be proved by Mathematical Induction.

Initial step. We first verify that Lemma 4 is true when $k = 2$. Note that the case of $k = 1$ is proven in Lemma 3. If $k = 2$, the belt is divided into two sub-belts, as shown in Fig. 6. Suppose that the height of the first sub-belt is h_2. Thus, the height of the second sub-belt is $H - h_2$. To completely cover the belt with a minimal number of disks, the strip disk distance in the first sub-belt is $\sqrt{4r^2 - h_2^2}$. So the strip disk distance in the second sub-belt is $\sqrt{4r^2 - (H - h_2)^2}$.

Suppose D is large enough; thus, the numbers of disks in the first and second part can be computed by

$$N_1^{(2)} = \frac{D}{d_1} = \frac{D}{\sqrt{4r^2 - h_2^2}},$$

and

$$N_2^{(2)} = \frac{D}{d_2} = \frac{D}{\sqrt{4r^2 - (H - h_2)^2}}.$$

Thus, the total number of nodes is

$$\begin{aligned} N^{(2)} &= N_1^{(2)} + N_2^{(2)} \\ &= \frac{D}{\sqrt{4r^2 - h_2^2}} + \frac{D}{\sqrt{4r^2 - (H - h_2)^2}}. \end{aligned} \tag{14}$$

To minimize $N^{(2)}$, we let $\frac{\mathrm{d}N^{(2)}}{\mathrm{d}h_2} = 0$. Thus, we have

$$\left(4r^2 - h_2^2\right)^{-\frac{3}{2}} h_2 = \left(4r^2 - (H - h_2)^2\right)^{-\frac{3}{2}} (H - h_2).$$

With some algebra calculation, we have

$$h_2 = \frac{H}{2}.$$

Therefore, Lemma 4 is true, when $k = 2$. We are done with the initial step.

Inductive Step. Here, we must prove the following assertion: If Lemma 4 is true when $k = n, n \geq 2$, then (for this same k) when $k = n + 1$, Lemma 4 is also true.

When $k = n + 1$, the belt area is divided into $n + 1$ parts. Suppose that the height of the first part is h_{n+1}. Thus, h_{n+1} meets the condition $h_{n+1} < 2r$. To completely cover the area with minimal number of disks, we should divide the rest part into n equal parts. Thus, the height of each rest part is $(H - h_{n+1})/n$. In this case, the minimal numbers of disks used in the first part and in the rest parts are

$$N_1^{(n+1)} = \frac{D}{\sqrt{4r^2 - h_{n+1}^2}},$$

and

$$N_2^{(n+1)} = N_3^{(n+1)} = \cdots = N_{n+1}^{(n+1)} = \frac{D}{\sqrt{4r^2 - \left(\frac{H - h_{n+1}}{n}\right)^2}}.$$

The total number of nodes $N^{(n+1)}$ is

$$N^{(n+1)} = N_1^{(n+1)} + \sum_{m=2}^{n+1} N_m^{(n+1)}.$$

To minimize $N^{(n+1)}$, we let $\frac{\mathrm{d}N^{(n+1)}}{\mathrm{d}h_n+1} = 0$. Thus, we get

$$\begin{aligned}&\left(4r^2 - h_{n+1}^2\right)^{-\frac{3}{2}} h_{n+1} \\ &\quad = \left(4r^2 - \left(\left(\frac{H - h_{n+1}}{n}\right)^2\right)\right)^{-\frac{3}{2}} \frac{(H - h_{n+1})}{n}.\end{aligned}$$

With some algebra calculation, we have

$$h_{n+1} = \frac{H}{n+1}.$$

Therefore, Lemma 4 is true, when $k = n + 1$. We are done with the inductive step.

Next, we compute the coverage density. Suppose there are m disks in a strip. According to Definition 4, the coverage density is

$$\rho_k^{(e)} = \frac{mk\pi r^2}{mk\frac{H}{k}\sqrt{4r^2 - \left(\frac{H}{k}\right)^2}} = \frac{k^2 \pi r^2}{H\sqrt{4k^2r^2 - H^2}}.$$

To minimize $\rho_k^{(e)}$, we need to maximize $(4r^2 - (\frac{H}{k})^2)(\frac{H}{k})^2$. We let

$$\frac{\mathrm{d}\left(4r^2 - \left(\frac{H}{k}\right)^2\right)\left(\frac{H}{k}\right)^2}{\mathrm{d}H} = 0.$$

Thus, we have

$$H = \sqrt{2}kr,$$

and

$$\rho_{\min}^{(e)} = \frac{\pi r^2}{\sqrt{(4r^2 - 2r^2)2r^2}} = \frac{\pi}{2}. \tag{15}$$

Therefore, Lemma 4 is true. □

Given a belt with height $H < 2kr$, we can equally partition the belt into $k, k+1, \ldots$ sub-belts. The following lemma states the relation between the belt height and the optimal equipartition placement.

Lemma 5. *Given a belt with height*

$$\frac{-2k + 2k^2}{\sqrt{1 - 2k + 2k^2}} r \leq H \leq \frac{2k + 2k^2}{\sqrt{1 + 2k + 2k^2}} r, k = 1, 2, \ldots, \tag{16}$$

the number of disks used to completely cover the area by k-equipartition placement is always no more than by m-equipartition placement ($m \neq k$).

Proof. We first consider the case that only one strip is enough to completely cover a belt. In this case, we have

$$\begin{aligned}N_1^{(e)} &= D/d_1^{(e)} = D/\sqrt{4r^2 - H^2} \\ N_2^{(e)} &= 2D/d_2^{(e)} = 2D/\sqrt{4r^2 - (H/2)^2} \\ N_1^{(e)} &\leq N_2^{(e)} \Rightarrow H \leq \frac{4\sqrt{5}}{5} r.\end{aligned}$$

Thus, Lemma 5 is true when $k = 1$.

For $k = 2, 3, \ldots$, we have

$$\begin{aligned}N_{k-1}^{(e)} &= (k-1)D/\sqrt{4r^2 - (H/(k-1))^2} \\ N_k^{(e)} &= kD/\sqrt{4r^2 - (H/k)^2} \\ N_{k+1}^{(e)} &= (k+1)D/\sqrt{4r^2 - (H/(k+1))^2}.\end{aligned}$$

$$N_k^{(e)} \leq N_{k-1}^{(e)} \Rightarrow \frac{-2k + 2k^2}{\sqrt{1 - 2k + 2k^2}} r \leq H,$$

and

$$N_k^{(e)} \leq N_{k+1}^{(e)} \Rightarrow H \leq \frac{2k + 2k^2}{\sqrt{1 - 2k + 2k^2}} r.$$

Let

$$f(k) = \frac{2k + 2k^2}{\sqrt{1 + 2k + 2k^2}} r, k = 1, 2, \ldots \tag{17}$$

When H meets the condition in (16), i.e.,

$$f(k-1) \leq H \leq f(k), \tag{18}$$

we can get

$$N_k^{(e)} \leq N_{k+1}^{(e)}, \tag{19}$$

and

$$N_k^{(e)} \leq N_{k-1}^{(e)}. \tag{20}$$

Since

$$f'(k) = 2(2k+1)(1+2k+2k^2)^{-\frac{1}{2}}\left(\frac{1}{2}+\frac{1}{1+2k+2k^2}\right) > 0,$$

thus $f(k)$ is monotonic increasing. We have

$$f(k+m) < f(k+m+1), m = 0,1,2,\ldots,$$

and

$$f(k-n) > f(k-(n+1)), n = 0,1,2,\ldots$$

According to (18), (19), and (20) we can get

$$N^{(e)}_{k+m} \leq N^{(e)}_{k+m+1}, m = 0,1,2,\ldots, \tag{21}$$

and

$$N^{(e)}_{k-n} \leq N^{(e)}_{k-(n+1)}, n = 0,1,2,\ldots \tag{22}$$

According to (21) and (22), we show that for any $m \neq k$, if H meets the condition in (16), then $N^{(e)}_k \leq N^{(e)}_m$. This completes the proof. □

Lemma 6. *Given a belt with width D and height H with very large D and $D \gg H$, if k rows of disks are used in the triangular-lattice placement to completely cover the belt, the coverage density is*

$$\rho^{(t)}_k = \frac{k\pi r}{\sqrt{3}H}, k = 1,2,\ldots \tag{23}$$

Proof. Suppose there are m_k disks in kth strips. For a very large D, we can compute the number of nodes in a row by

$$m_k = D/d^{(t)} = D/\sqrt{3}r. \tag{24}$$

Thus, the coverage density is

$$\rho^{(t)}_k = \frac{m_k\pi r^2 k}{HD} = \frac{m_k\pi r^2 k}{Hm_k\sqrt{3}r} = \frac{k\pi r}{\sqrt{3}H}, k = 1,2,\ldots$$

□

Lemma 7. *Given a belt with width D and height H, if $D \to \infty$, $H \to \infty$, the triangular-lattice placement can completely cover the area with the minimal number of disks. In this case, the coverage density $\rho(T)$ meets $\rho(T) = \frac{2\pi}{3\sqrt{3}}$.*

Proof. The first part has been proved in [2], [3]. Here, we provide another approach to prove the second part of Lemma 7. According to (6) in Lemma 2, we can get

$$\left(1.5 - \frac{2}{k}\right)r < \frac{H}{k} \leq \left(1.5 - \frac{0.5}{k}\right)r. \tag{25}$$

Since $H \to \infty, k \to \infty$ and

$$\lim_{k\to\infty}\left(1.5r - \frac{2r}{k}\right) = 1.5r, \lim_{k\to\infty}\left(1.5r - \frac{0.5r}{k}\right) = 1.5r.$$

According to the Squeeze Theorem [34], from (25) we have

TABLE 1
Range of H and Corresponding k in the Equipartition Placement

Range of H	Value of k	Range of H	Value of k
$(0, \frac{4\sqrt{5}}{5}r]$	1	$(\frac{40}{\sqrt{41}}r, \frac{60}{\sqrt{61}}r]$	5
$(\frac{4\sqrt{5}}{5}r, \frac{12}{\sqrt{13}}r]$	2	$(\frac{60}{\sqrt{61}}r, \frac{84}{\sqrt{85}}r]$	6
$(\frac{12}{\sqrt{13}}r, \frac{24}{5}r]$	3	$(\frac{84}{\sqrt{85}}r, \frac{112}{\sqrt{113}}r]$	7
$(\frac{24}{5}r, \frac{40}{\sqrt{41}}r]$	4	$(\frac{112}{\sqrt{113}}r, \frac{144}{\sqrt{145}}r]$	8

$$\lim_{k\to\infty}\frac{H}{k} = 1.5r. \tag{26}$$

Substitute (26) into (23), we can get

$$\rho(T) = \lim_{H\to\infty, D\to\infty}\rho = \frac{2\pi}{3\sqrt{3}}. \tag{27}$$

□

3.2 The Proposed Placement

According to Lemma 7, we know that if H is large enough, the triangular-lattice placement is an optimal disk placement, because it achieves the lowest disk density for complete coverage. However, if H is not large enough, the triangular-lattice placement may be suboptimal due to the boundary effect of the belt longer sides. Recall that the strip disk distance d in the triangular-lattice placement is fixed as $\sqrt{3}r$. When H is not very large, we can stretch d a little bit to use fewer disks for complete belt coverage. In other words, when H is not large enough, the equipartition placement may be better than the triangular-lattice placement. In what follows, we study which is the optimal placement for different values of H.

We first determine the optimal equipartition (d, r)-strip placement. Table 1 summarizes different ranges of H and the corresponding k according to (16) in lemma 5 for $k \leq 8$.

We next propose a placement scheme combining both the optimal equipartition (d, r)-strip placement and the triangular-lattice placement. According to the belt height range, which one of them (equipartition or triangular-lattice) should be used is determined, and the results are summarized in the following theorem.

Theorem 1. *Consider a long belt with width D much larger than its height H, $D \gg H$, such that the left and right-boundary effects can be neglected. The node deployment shown in Table 2 combines both the equipartition (d, r)-strip placement in Definition 2 and the triangular-lattice placement in Definition 3 for complete belt coverage. The optimal applications of which of them have also been computed as in Table 2 for different belt height H.*

Proof. According to (4), we know that the number of disks in the triangular-lattice placement is a step function. Fig. 7 shows the number of disks used in the optimal equipartition (d, r)-strip placement and in the triangular-lattice placement. From the figure, we can see there are eight intersections between the two curves.

Now, we compare the optimal equipartition (d, r)-strip placement and the triangular-lattice placement for different ranges of H:

TABLE 2
Belt Height Ranges, Methods, and Coverage Densities

Range of H	Method	Coverage Density
$(0, \frac{4\sqrt{5}}{5}r]$	Equipartition (d, r)-strip placement, $k=1$	$\frac{\pi r^2}{H\sqrt{4r^2-H^2}}$
$(\frac{4\sqrt{5}}{5}r, 2r]$	Equipartition (d, r)-strip placement, $k=2$	$\frac{4\pi r^2}{H\sqrt{16r^2-H^2}}$
$(2r, 2.5r]$	Triangular-lattice placement, 2 strips	$\frac{2\pi r}{\sqrt{3}H}$
$(2.5r, \frac{4\sqrt{6}}{3}r]$	Equipartition (d, r)-strip placement, $k=2$	$\frac{4\pi r^2}{H\sqrt{16r^2-H^2}}$
$(\frac{4\sqrt{6}}{3}r, 4r]$	Triangular-lattice placement, 3 strips	$\frac{\sqrt{3}\pi r}{H}$
$(4r, \frac{3}{4}\sqrt{37}r]$	Equipartition (d, r)-strip placement, $k=3$	$\frac{9\pi r^2}{H\sqrt{36r^2-H^2}}$
$(\frac{3}{4}\sqrt{37}r, 5.5r]$	Triangular-lattice placement, 4 strips	$\frac{4\pi r}{\sqrt{3}H}$
$(5.5r, \frac{8}{5}\sqrt{13}r]$	Equipartition (d, r)-strip placement, $k=4$	$\frac{16\pi r^2}{H\sqrt{64r^2-H^2}}$
$(\frac{8}{5}\sqrt{13}r, \infty)$	Triangular-lattice placement, k strips	$\frac{k\pi r}{\sqrt{3}H}$

1. When $0 < H \leq r$, the equipartition placement with $k=1$ and the triangular-lattice placement with one row of disks can be used, and we have
$$N_1^{(e)} = \frac{D}{\sqrt{4r^2 - H^2}}, \quad N_1^{(t)} = \frac{D}{\sqrt{3}r}.$$
Since $0 < H \leq r$, we can get $N_1^{(e)} \leq N_1^{(t)}$. Thus in this range, we always choose the equipartition placement with $k=1$.
2. When $r < H \leq \frac{4\sqrt{5}}{5}r$, the equipartition placement with $k=1$ and triangular-lattice placement with two rows of disks can be used as
$$N_2^{(t)} = \frac{2D}{\sqrt{3}r}.$$
Since $r < H \leq \frac{4\sqrt{5}}{5}r$, we can get $N_1^{(e)} \leq N_2^{(t)}$. Thus in this range, we always choose the equipartition placement with $k=1$.
3. When $\frac{4\sqrt{5}}{5}r < H \leq 2.5r$, the equipartition placement with $k=2$ and the triangular-lattice placement with two rows of disks can be used as
$$N_2^{(e)} = \frac{2D}{\sqrt{4r^2 - (H/2)^2}}.$$
If $N_2^{(e)} \leq N_2^{(t)}$, we can get
$$\frac{2D}{\sqrt{4r^2 - (H/2)^2}} \leq \frac{2D}{\sqrt{3}r} \Rightarrow H \leq 2r.$$
Thus, if $\frac{4\sqrt{5}}{5}r < H \leq 2r$, we choose the equipartition placement with $k=2$. If $2r < H \leq 2.5r$, we choose the triangular-lattice placement with two rows of disks.

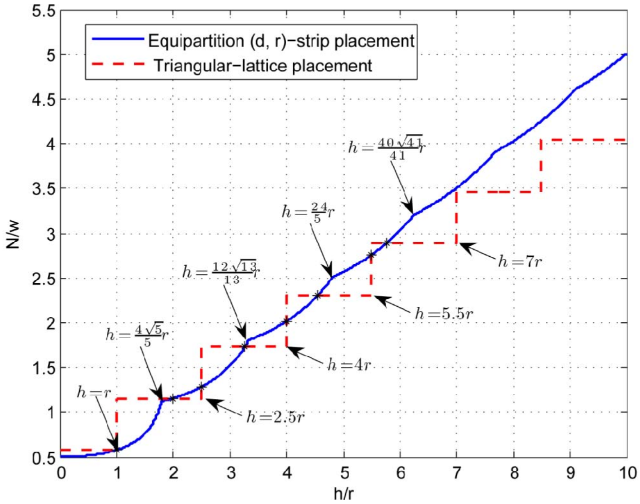

Fig. 7. The number of disks used in optimal equipartition (d, r)-strip placement and in triangular-lattice placement. The filled dots are intersection points of the two curves.

4. When $2.5r < H \leq \frac{12\sqrt{13}}{13}r$, the equipartition placement with $k=2$ and the triangular-lattice placement with three rows of disks can be used as
$$N_3^{(t)} = \frac{3D}{\sqrt{3}r}.$$
If $N_2^{(e)} \leq N_3^{(t)}$, we can get
$$\frac{2D}{\sqrt{4r^2 - (H/2)^2}} \leq \frac{3D}{\sqrt{3}r} \Rightarrow H \leq \frac{4\sqrt{6}}{3}r.$$
Therefore, if $2.5r < H \leq \frac{4\sqrt{6}}{3}r$, we choose the equipartition placement with $k=2$. If $\frac{4\sqrt{6}}{3}r < H \leq \frac{12\sqrt{13}}{13}r$, we choose the triangular-lattice placement with three rows of disks.
5. When $\frac{12\sqrt{13}}{13}r < H \leq 4r$, the equipartition placement with $k=3$ and the triangular-lattice placement with three rows of disks can be used as
$$N_3^{(e)} = \frac{3D}{\sqrt{4r^2 - (H/3)^2}}.$$
Since $\frac{12\sqrt{13}}{13}r < H \leq 4r$, we can get $N_3^{(t)} \leq N_3^{(e)}$. Thus in this range, we always choose the triangular-lattice placement with $k=3$.
6. When $4r < H \leq 4.8r$, the equipartition placement with $k=3$ and the triangular-lattice placement with four rows of disks can be used as
$$N_4^{(t)} = \frac{4D}{\sqrt{3}r}.$$
If $N_3^{(e)} \leq N_4^{(t)}$, we can get
$$\frac{3D}{\sqrt{4r^2 - (H/3)^2}} \leq \frac{4D}{\sqrt{3}r} \Rightarrow H \leq \frac{3\sqrt{37}}{4}r.$$
Therefore, if $4r < H \leq \frac{3\sqrt{37}}{4}r$, we choose the equipartition placement with $k=3$. If $\frac{3\sqrt{37}}{4}r < H \leq 4.8r$, we choose the triangular-lattice placement with four rows of disks.

7. When $4.8r < H \le 5.5r$, the equipartition placement with $k = 4$ and the triangular-lattice placement with four rows of disks can be used as

$$N_4^{(e)} = \frac{4D}{\sqrt{4r^2 - (H/4)^2}}.$$

Since $4.8r < H \le 5.5r$, we can get $N_3^{(t)} \le N_4^{(e)}$. Thus in this range, we always choose the triangular-lattice placement with $k = 4$.

8. When $5.5r < H \le \frac{40\sqrt{41}}{41} r$, the equipartition placement with $k = 4$ and the triangular-lattice placement with five rows of disks can be used as

$$N_5^{(t)} = \frac{5D}{\sqrt{3}r}.$$

If $N_4^{(e)} \le N_5^{(t)}$, we can get

$$\frac{4D}{\sqrt{4r^2 - (H/4)^2}} \le \frac{5D}{\sqrt{3}r} \Rightarrow H \le \frac{8\sqrt{13}}{5} r.$$

Therefore, if $5.5r < H \le \frac{8\sqrt{13}}{5} r$, we choose the equipartition placement with $k = 4$. If $\frac{8\sqrt{13}}{5} r < H \le \frac{40\sqrt{41}}{41} r$, we choose the triangular-lattice placement.

9. When $\frac{40\sqrt{41}}{41} r < H \le 7r$, the equipartition placement with $k = 5$ and the triangular-lattice placement with five rows of disks can be used as

$$N_5^{(e)} = \frac{5D}{\sqrt{4r^2 - (H/5)^2}}.$$

In this range, $N_5^{(e)} > N_5^{(t)}$, and we choose the triangular-lattice placement.

10. When $7r < H$, according to Lemma 2, we know that if $(1.5k - 2)r < H \le (1.5k - 0.5)r, k = 2, 3, \ldots,$ at least k rows of disks are needed in the triangular-lattice placement. Since $7r < H$, we can get $k \ge 6$. According to Lemma 1, we know that if H is with the same range, at least

$$m = \lfloor (1.5k - 2)/2 \rfloor + 1$$

rows of strips are needed in the equipartition placement.

Thus, we can get

$$0.75k - 1 \le m \le 0.75k.$$

Note that m can take the only one integer value in between $0.75k - 1$ and $0.75k$, $k = 6, 7, \ldots$ For simplicity, we use the subscript $0.75k - 1$ and $0.75k$ to denote the possible integer within such a range. We have

$$N_k^{(t)} = \frac{kD}{\sqrt{3}r}, \quad N_{m \min}^{(e)} = \min\left(N_{0.75k}^{(e)}, N_{0.75k-1}^{(e)}\right).$$

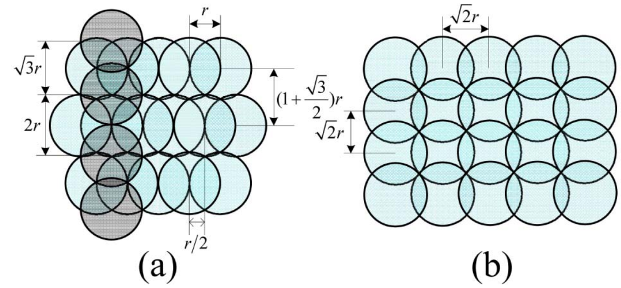

Fig. 8. Two common placements: (a) Kar's deployment pattern, (b) Square deployment pattern.

Here,

$$N_{0.75k}^{(e)} = \frac{0.75kD}{\sqrt{4r^2 - \left(\frac{H}{0.75k}\right)^2}}, N_{0.75k-1}^{(e)} = \frac{(0.75k - 1)D}{\sqrt{4r^2 - \left(\frac{H}{0.75k-1}\right)^2}}.$$

Since $k \ge 6$, we can get

$$\frac{3\sqrt{37}}{16} kr < (1.5k - 2)r.$$

Thus, if $(1.5k - 2)r < H$, we have $\frac{3\sqrt{37}}{16} kr < H$. Since

$$\frac{3\sqrt{37}}{16} kr < H \Rightarrow \frac{kD}{\sqrt{3}r} < \frac{0.75kD}{\sqrt{4r^2 - (H/0.75k)^2}},$$

we can get $N_k^{(t)} < N_{0.75k}^{(e)}$.

Similarly, if $k \ge 6$, we can get

$$\frac{\sqrt{4k^2 - 3(0.75k - 1)^2}}{k} (0.75k - 1) < 1.5k - 2.$$

If $(1.5k - 2)r < H$, we have

$$r\sqrt{4k^2 - 3(0.75k - 1)^2} < kH/(0.75k - 1)$$
$$\Longrightarrow \frac{k}{\sqrt{3}r} < \frac{(0.75k - 1)}{\sqrt{\left(4r^2 - \frac{H^2}{(0.75k-1)^2}\right)}}.$$

Thus, $N_k^{(t)} < N_{0.75k-1}^{(e)}$.

So when $7r < H$, $N_k^{(t)} < N_m^{(e)}$, the number of disks used in the triangular-lattice placement is always less than in the equipartition placement.

In conclusion, we always choose the placement which uses the fewer disks with different ranges of H, and the results are summarized in Table 2. □

3.3 Performance Evaluation and Analysis

We compare our node deployment scheme with some common placement schemes, including Kar's placement [7], the very popular regular square deployment pattern [9] and the triangular-lattice placement [2], [4].

In the Kar's placement pattern, a string of disks are placed along a line such that the distance between the centers of any two adjacent disks is r, then the whole plane are tiled with these strips. Here, the strip distance is $(1 + \frac{\sqrt{3}}{2})r$. Note that for every even integer k, the kth strip should be translated by distance $r/2$ along the strip line, as shown in Fig. 8a. In addition, some extra disks should be placed along the direction perpendicular to the strip line

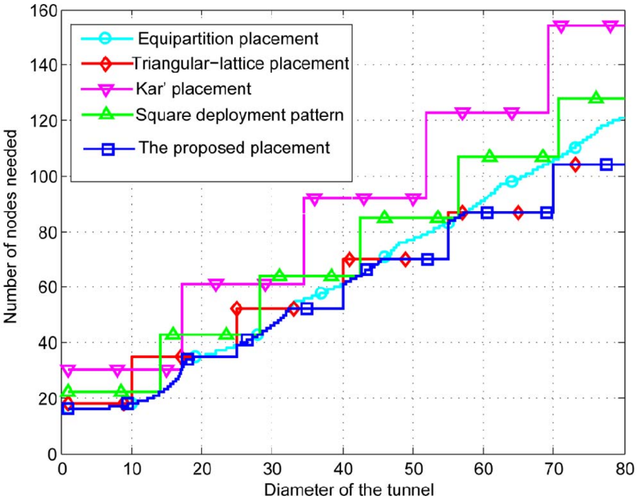

Fig. 9. The number of nodes needed in the proposed scheme and in other placements. Here the length of the belt is 300 m, and the belt width H varies from 1 to 80 m. The coverage radius of each node is 10 m.

with specific distance, as the shaded disks shown in the figure. Here, the extra strip of nodes are not for complete coverage, but for connectivity. When the belt is long enough, the number of disks added by this extra vertical strip can be neglected.

In the regular square deployment pattern, disks are placed such that the centers of any four neighbor disks can form a regular square with side length $\sqrt{2}r$, as shown in Fig. 8b. The triangular-lattice placement has been introduced in Definition 3. Moreover, in some schemes the relationship between the communication range r_c and sensing range r_s will impact the nodes' pattern. For fairness, here we assume that $r_c \geq \sqrt{3}r_s$. In this case, node density is determined by ensuring complete coverage and the coverage density of such placements will be smaller than in the other cases where $r_c < \sqrt{3}r_s$. Under this assumption of $r_c \geq \sqrt{3}r_s$, the node placement pattern in [9] and the Diamond pattern [10] both coincide with the triangular-lattice placement pattern.

The comparison results are shown in Fig. 9. From the figure, we have the following observations: First, the number of nodes needed by the proposed combination deployment requires the smallest number of nodes. Note that when H/r belongs to the following four open intervals $(2, 2.5)$, $(\frac{4\sqrt{6}}{3}, 4)$, $(\frac{3\sqrt{37}}{4}, 5.5)$, and $(\frac{8\sqrt{13}}{5}, +\infty)$, the proposed deployment coincides with the triangular-lattice placement. Second, the number of nodes needed by regular square deployment is always more than by the triangular-lattice placement, except when H/r belongs to the intervals $(1, \sqrt{2})$, $(2.5, 2\sqrt{2})$, $(4, 3\sqrt{2})$, and $(5.5, 4\sqrt{2})$. Third, the number of nodes needed by Kar's placement is more than other placements except when H/r belongs to the intervals $(\sqrt{2}, \sqrt{3})$ and $(2\sqrt{2}, 2\sqrt{3})$. In these two intervals, the Kar's placement is better than the square deployment. Moreover, in the first interval, the Kar's placement is even better than the triangular-lattice placement.

We next provide the analytical results that compare the number of nodes for complete belt coverage by the four deployments, when the belt height is in different ranges. For the Kar's placement, as shown in Fig. 8a, the disk distance is $d_{Kar} = r$, and the distance between two adjacent strips is $(1 + \frac{\sqrt{3}}{2})r$. Thus, the number of nodes used by this scheme can be computed by

$$N_k^{(Kar)} = \frac{kD}{r} + k - 1, (k-1)\sqrt{3}r < H \leq k\sqrt{3}r, k = 1, 2, \ldots \tag{28}$$

For the regular square deployment, as shown in Fig. 8b, the disk distance is $d_{square} = \sqrt{2}r$, and the distance between two adjacent strips is also $\sqrt{2}r$. In this case, the nodes used by this scheme is given by

$$N_k^{(s)} = \frac{kD}{\sqrt{2}r}, (k-1)\sqrt{2}r < H \leq k\sqrt{2}r, k = 1, 2, \ldots \tag{29}$$

In summary, according to (3), (4), (28), and (29), we can get the following comparison results:

1. $0 < H \leq r, N^{(e)} \leq N^{(t)} < N^{(s)} < N^{(Kar)}$
2. $r < H \leq \sqrt{2}r, N^{(e)} \leq N^{(s)} < N^{(Kar)} < N^{(t)}$
3. $\sqrt{2}r < H \leq \sqrt{3}r, N^{(e)} \leq N^{(Kar)} < N^{(t)} < N^{(s)}$
4. $\sqrt{3}r < H \leq 2r, N^{(e)} \leq N^{(t)} < N^{(s)} < N^{(Kar)}$
5. $2r < H \leq 2.5r, N^{(t)} \leq N^{(e)} < N^{(s)} < N^{(Kar)}$
6. $2.5r < H \leq 2\sqrt{2}r, N^{(e)} \leq N^{(s)} < N^{(t)} < N^{(Kar)}$
7. $2\sqrt{2}r < H \leq \frac{4\sqrt{6}}{3}r, N^{(e)} \leq N^{(t)} < N^{(Kar)} < N^{(s)}$
8. $\frac{4\sqrt{6}}{3}r < H \leq 2\sqrt{3}r, N^{(t)} \leq N^{(e)} < N^{(Kar)} < N^{(s)}$
9. $2\sqrt{3}r < H \leq 4r, N^{(t)} \leq N^{(e)} < N^{(s)} < N^{(Kar)}$
10. $4r < H \leq 3\sqrt{2}r, N^{(e)} \leq N^{(s)} < N^{(t)} < N^{(Kar)}$
11. $3\sqrt{2}r < H \leq \frac{3\sqrt{37}}{4}r, N^{(e)} \leq N^{(t)} < N^{(s)} < N^{(Kar)}$
12. $\frac{3\sqrt{37}}{4}r < H \leq 5.5r, N^{(t)} \leq N^{(e)} < N^{(s)} < N^{(Kar)}$
13. $5.5r < H \leq 4\sqrt{2}r, N^{(e)} \leq N^{(s)} < N^{(t)} < N^{(Kar)}$
14. $4\sqrt{2}r < H \leq \frac{8\sqrt{13}}{5}r, N^{(e)} \leq N^{(t)} < N^{(s)} < N^{(Kar)}$
15. $\frac{8\sqrt{13}}{5}r < H, N^{(t)} < N^{(e)} \leq N^{(s)} < N^{(Kar)}$.

4 Discussion

4.1 Optimality Factor of the Solution

In this section, we discuss the approximation factor of our scheme to the optimal one. It is well known that given an unbounded area, the triangular-lattice placement achieves the minimum node density to completely cover the area. In this case, according to Definition 4, the critical node density is $\frac{2\sqrt{3}}{9} \times \frac{1}{r^2}$, and the coverage density (simply written as $\rho \equiv \frac{A_{disks}}{A_{covered}}$) is given by $\frac{2\sqrt{3}\pi}{9}$. Note that this is the lowest bound (the optimal one) for all the coverage problems of 1-coverage and with disk model, in an unbounded scenario with the number of sensors tending to infinity. However, this may not be the optimal one in the bounded belt scenario, because $A_{\text{covered}} \geq A_{belt}$ in all possible deployments, even with the triangular-lattice placement; while our coverage density defined in belt scenario is as $\rho \equiv \frac{A_{disks}}{A_{belt}}$. So in this case, we consider using an optimality scaling factor $\alpha \equiv \frac{A_{belt}}{A_{\text{covered}}}$ to infer the optimal coverage density in a bounded belt scenario. That is, suppose that k rows of disks are used in the triangular-lattice placement, we can compute an optimal density as $\rho_o = \alpha \times \frac{2\sqrt{3}\pi}{9}$.

For k rows of disks in the triangular-lattice deployment, the area covered by these disks are given by

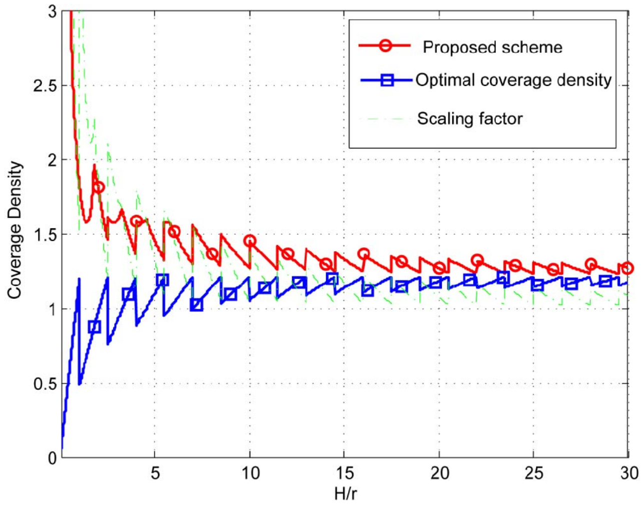

Fig. 10. The value of ρ_p, ρ_o, and α.

$$A_{disks} = kDr + (k-1)\frac{Dr}{2} = \frac{3k-1}{2}Dr, k = 1, 2, \ldots,$$

where D is the belt length (very large).

Note that this computation is consistent with the optimal one. That is, the coverage density of using k-strips of triangulation-lattice is given by

$$\rho_k = \frac{\frac{D}{\sqrt{3}r}\pi r^2 k}{\frac{3k-1}{2}Dr} = \frac{2\sqrt{3}\pi}{9}\frac{3k}{3k-1}. \tag{30}$$

Therefore, as $k \to \infty$, we have $\rho_k \to \rho_{opt} = \frac{2\sqrt{3}}{9}\pi$. In this case, we get

$$\alpha = \frac{A_{belt}}{A_{\text{covered}}} = \frac{DH}{\frac{3k-1}{2}Dr} = \frac{2H}{(3k-1)r} \tag{31}$$

$$\rho_o = \alpha\frac{2\sqrt{3}\pi}{9} = \frac{4\sqrt{3}H}{9(3k-1)r}. \tag{32}$$

According to Table 2, we can get the coverage density ρ_p for our proposed scheme. When $k \geq 5$, $\rho_p = \frac{k\pi r}{\sqrt{3}H}$, and we have

$$\frac{\rho_p}{\rho_o} = \frac{3k(3k-1)r^2}{4H^2}. \tag{33}$$

From Lemma 2, we have $1.5k - 2 < \frac{H}{r} < 1.5k - 0.5$. Thus, we get when $k \geq 5$,

$$\frac{1}{2.25k^2 - 6k + 4} > \frac{r^2}{H^2} > \frac{1}{2.25k^2 - 1.5k + 0.25}. \tag{34}$$

Substitute (34) into (33), we get

$$\frac{9k^2 - 3k}{9k^2 - 24k + 16} > \frac{\rho_p}{\rho_o} > \frac{9k^2 - 3k}{9k^2 - 6k + 1} = 1 + \frac{1}{3k-1}. \tag{35}$$

Apparently, we have $\frac{\rho_p}{\rho_o} > 1, \rho_p > \rho_o$.

We next consider the limit of ρ_p. Substitute (26) into (32), we can get

$$\lim_{k\to\infty} \rho_o = \frac{2\sqrt{3}\pi}{9}, \quad \lim_{k\to\infty} \alpha = 1,$$

When $k \to \infty$, $\rho_p \to \rho_o \to \frac{2\sqrt{3}\pi}{9}$. That is to say, coverage density of the proposed combined scheme tends to the optimal one in unbounded scenario (convergence and consistence), as shown in Fig. 10.

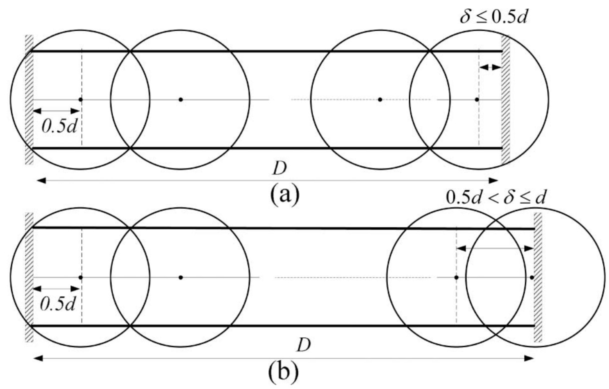

Fig. 11. A (d, r)-strip is placed when left and right-boundaries are considered.

4.2 Left and Right-Boundary Effect

In this section, we discuss the left and right-boundary effect. As we mentioned in Section 3, if the left and right-boundary effects are not considered, the number of disk in the kth strip, m_k, can be computed by

$$m_k = D/d.$$

Here, $d = d^{(e)} = \sqrt{4r^2 - (H/k)^2}$ for the equipartition placement and $d = d^{(t)} = \sqrt{3}r$ for the triangular-lattice placement.

Now, we consider the left and right-boundaries. Fig. 11 shows that a (d, r)-strip is placed in a bounded belt. In this case, to minimize the number of disks in the strip, no matter which placement we choose, we need to find an initial point where the first disk in the strip should be placed. To maximize the coverage region of a strip, an initial disk should be placed that the distance between the center of the disk and one boundary is $0.5d$. As shown in Fig. 11, after placing the initial disk, the locations of other disks in a strip are then determined. If the distance ϕ between the last disk and the other boundary is not larger than $0.5d$, as shown in Fig. 11a, the given placement can provide complete belt coverage. Otherwise, when $0.5d < \phi < d$, for complete belt coverage, we place an extra disk at the intersection point of the other side boundary and the bisector, as shown in Fig. 11b. Thus, the number of disks in a strip meets the following condition:

$$m'_k = \begin{cases} D/d, D/d \in \mathcal{Z}^+ \\ \lfloor D/d \rfloor + 1, others. \end{cases} \tag{36}$$

Note that in (36) if D/d is not an integer, the number of disks is $\lfloor D/d \rfloor + 1$.

Let

$$\sigma = \frac{m'_k - m_k}{m_k} \tag{37}$$

denote the normalized number of additional disks for belt coverage with left and right-boundary. The value of σ is plotted in Fig. 12. From this figure, we can see that when D/d is not very large, σ is close to 1. Thus, there is a sensible difference between m_k and m'_k. But when $D/d > 15, \sigma < 0.1$. In this case, the difference between m_k and m'_k is very slight.

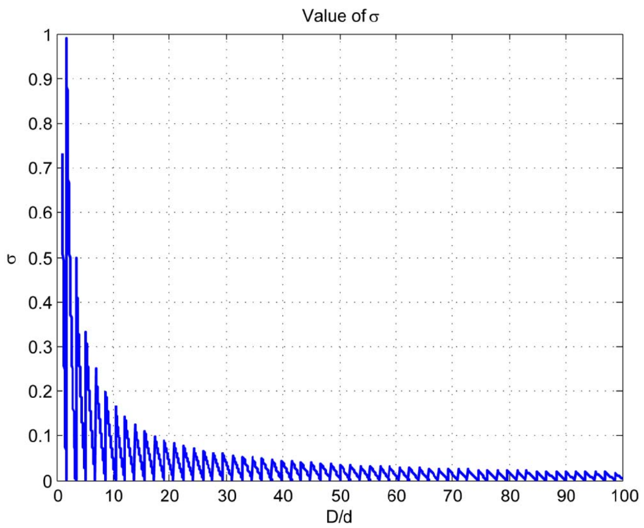

Fig. 12. The value of σ.

All the discussions in the paper assume that $D \gg d$. We can get $\sigma \rightarrow 0$ and $m'_k \approx m_k$. That is, when D is large enough, even there exists left and right boundaries in the belt, the conclusion in this paper is still valid. If D is not very large and comparable to H, an exhaustive search or a heuristic method can be used to obtain an optimal disk placement. For example, as mentioned in Section 2, a simulated annealing method [33] or a quasi-Newton method [32] can be used to find an optimal placement with a given number of disks for a small rectangle.

4.3 K-Coverage

Providing K-coverage for a region means that every point of this region is covered by at least K disks. The proposed scheme in our work can be easily extended to provide K-coverage. For example, we can simply superimpose K copies of the 1-coverage placement, one on top of another, and K-coverage can be achieved. Apparently, this is the simplest way to provide K-coverage. However, it may not be a good scheme, because many nodes would have to be piled at exactly the same location. An alternative way is to put extra $K-1$ strips on each existing strip line and set the offset distance between any two strips by d/K, where d is given by (2). Moreover, in the proposed equipartition scheme, each (d, r)-strip can be placed independently. Therefore, we can easily change the coverage degree of different parts of the belt. For example, given a belt with height H, if 2-equipartition placement can be used, then the strip distance is $\sqrt{16r^2 - H^2}/2$. If we want to achieve 2-coverage on the left half (top half) and 1-coverage on the right half (bottom half) of the belt, we can place two extra strips on the strip line of left half of the belt (one extra strip on the strip line of bottom half of the belt), and make the new left half (top half) strip distance $\sqrt{16r^2 - H^2}/4$, as shown in Fig. 13. By this way, each point in the left half (top half) of the belt can be covered by at least two disks.

5 Conclusion

In this paper, we have proposed a novel solution to the long belt node coverage problem in wireless networks. In our work, the belt is divided into some sub-belts, and then a

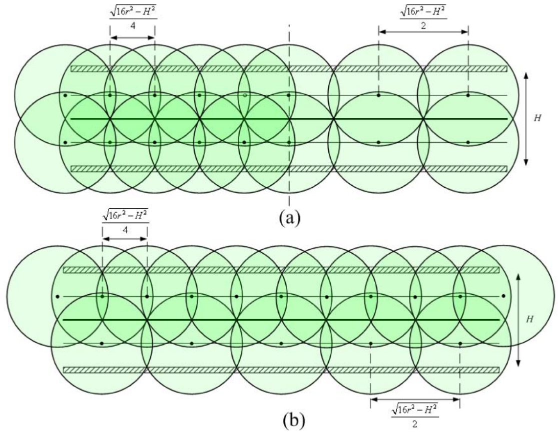

Fig. 13. 2-coverage is achieved in some parts of the belt. (a) Left-half of the belt is 2-covered. (b) Top-half of the belt is 2-covered.

string of nodes (disks) are placed parallel to the long side of each sub-belt to completely cover the sub-belt. The optimal distance between two adjacent disks in a string and the number of such strings to minimize the number of disks for complete belt coverage are then determined. Theoretical analysis and numerical results show that the proposed placement scheme requires fewer nodes to ensure complete belt coverage as compared with the well-known regular triangular-lattice pattern placement scheme, when the height of the belt H is less than $\frac{8}{5}\sqrt{13}r$, where r is the disk radius. A combination of the proposed method and the triangular-lattice placement has been proposed, and the optimal ranges of the belt height for their respective applications have computed.

Acknowledgments

The authors would like to thank the editors and anonymous reviewers for their valuable comments that have helped to improve the paper. This work was partly supported by National Natural Science Foundation of China (Grant No. 61173120) and Doctoral Fund of Ministry of Education of China (Grant No. 20110142120078). The corresponding author is Han Xu.

References

[1] B. Wang, *Coverage Control in Sensor Networks.* Springer-Verlag, 2010.
[2] H. Zhang and J. Hou, "Maintaining Sensing Coverage and Connectivity in Large Sensor Networks," *J. Ad Hoc and Sensor,* vol. 51, p. 61801, 2005.
[3] R. Kershner, "The Number of Circles Covering a Set," *Am. J. Math.,* vol. 61, no. 3, pp. 665-671, 1939.
[4] Y. Wang, C. Hu, and Y. Tseng, "Efficient Placement and Dispatch of Sensors in a Wireless Sensor Network," *IEEE Trans. Mobile Computing,* vol. 7, no. 2, pp. 262-274, Feb. 2008.
[5] T. Brown, D. Sarioz, A. Bar-Noy, T. LaPorta, D. Verma, M. Johnson, and H. Rowaihy, "Geometric Considerations for Distribution of Sensors in Ad-Hoc Sensor Networks," *Proc. SPIE DSS,* vol. 6562, 2007.
[6] G. Das, S. Das, S. Nandy, and B. Sinha, "Efficient Algorithm for Placing a Given Number of Base Stations to Cover a Convex Region," *J. Parallel and Distributed Computing,* vol. 66, no. 11, pp. 1353-1358, 2006.
[7] K. Kar and S. Banerjee, "Node Placement for Connected Coverage in Sensor Networks," *Proc. Workshop Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt),* vol. 3, 2003.

[8] K. Chakrabarty, S. Iyengar, H. Qi, and E. Cho, "Grid Coverage for Surveillance and Target Location in Distributed Sensor Networks," *IEEE Trans. Computers,* vol. 51, no. 12, pp. 1448-1453, Dec. 2002.

[9] X. Bai, S. Kumar, D. Xuan, Z. Yun, and T. Lai, "Deploying Wireless Sensors to Achieve Both Coverage and Connectivity," *Proc. Seventh ACM Int'l Symp. Mobile Ad Hoc Networking and Computing,* pp. 131-142, 2006.

[10] X. Bai, Z. Yun, D. Xuan, T. Lai, and W. Jia, "Optimal Patterns for Four-Connectivity and Full Coverage in Wireless Sensor Networks," *IEEE Trans. Mobile Computing,* vol. 9, no. 3, pp. 435-448, Mar. 2010.

[11] X. Xu and S. Sahni, "Approximation Algorithms for Sensor Deployment," *IEEE Trans. Computers,* vol. 56, no. 12, pp. 1681-1695, Dec. 2007.

[12] M. Ma and Y. Yang, "Adaptive Triangular Deployment Algorithm for Unattended Mobile Sensor Networks," *IEEE Trans. Computers,* vol. 56, no. 7, pp. 946-847, July 2007.

[13] H. Ammari and S. Das, "Integrated Coverage and Connectivity in Wireless Sensor Networks: A Two-Dimensional Percolation Problem," *IEEE Trans. Computers,* vol. 57, no. 10, pp. 1423-1434, Oct. 2008.

[14] M. Tolstrup, *Indoor Radio Planning: A Practical Guide for Gsm, DCS, UMTS and HSPA.* Wiley Publishing, 2008.

[15] D. Cavalcanti, D. Agrawal, C. Cordeiro, B. Xie, and A. Kumar, "Issues in Integrating Cellular Networks WLANs, AND MANETs: A Futuristic Heterogeneous Wireless Network," *IEEE Wireless Comm.,* vol. 12, no. 3, pp. 30-41, June 2005.

[16] I. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, "Wireless Sensor Networks: A Survey," *Computer Networks,* vol. 38, no. 4, pp. 393-422, 2002.

[17] S. Kumar, T. Lai, and A. Arora, "Barrier Coverage with Wireless Sensors," *Proc. 11th Ann. Int'l Conf. Mobile Computing and Networking,* pp. 284-298, 2005.

[18] G. Yang and D. Qiao, "Multi-Round Sensor Deployment for Guaranteed Barrier Coverage," *Proc. IEEE INFOCOM '10,* pp. 1-9, 2010.

[19] B. Liu, O. Dousse, J. Wang, and A. Saipulla, "Strong Barrier Coverage of Wireless Sensor Networks," *Proc. Ninth ACM Int'l Symp. Mobile Ad Hoc Networking and Computing,* pp. 411-420, 2008.

[20] A. Chen, T. Lai, and D. Xuan, "Measuring and Guaranteeing Quality of Barrier-Coverage in Wireless Sensor Networks," *Proc. Ninth ACM Int'l Symp. Mobile Ad Hoc Networking and Computing,* pp. 421-430, 2008.

[21] A. Chen, S. Kumar, and T. Lai, "Designing Localized Algorithms for Barrier Coverage," *Proc. 13th Ann. ACM Int'l Conf. Mobile Computing and Networking,* pp. 63-74, 2007.

[22] A. Saipulla, C. Westphal, B. Liu, and J. Wang, "Barrier Coverage of Line-Based Deployed Wireless Sensor Networks," *Proc. IEEE INFOCOM '09,* pp. 127-135, 2009.

[23] A. Saipulla, B. Liu, G. Xing, X. Fu, and J. Wang, "Barrier Coverage with Sensors of Limited Mobility," *Proc. 11th ACM Int'l Symp. Mobile Ad Hoc Networking and Computing,* pp. 201-210, 2010.

[24] L. Li, B. Zhang, X. Shen, J. Zheng, and Z. Yao, "A Study on the Weak Barrier Coverage Problem in Wireless Sensor Networks," *Computer Networks,* vol. 55, pp. 711-721, 2011.

[25] M. Noori, S. Movaghati, and M. Ardakani, "Characterizing the Path Coverage of Random Wireless Sensor Networks," *EURASIP J. Wireless Comm. and Networking,* vol. 2010, p. 14, 2010.

[26] D. Ban, J. Jiang, W. Yang, W. Dou, and H. Yi, "Strong K-Barrier Coverage with Mobile Sensors," *Proc. Sixth Int'l Wireless Comm. and Mobile Computing Conf.,* pp. 68-72, 2010.

[27] R. Williams, *The Geometrical Foundation of Natural Structure: A Source Book of Design.* Dover Publications, 1979.

[28] J. Pach and P. Agarwal, *Combinatorial Geometry.* Wiley, 1995.

[29] P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry.* Springer Verlag, 2005.

[30] T. Tarnai and Z. Gáspár, "Covering a Square by Equal Circles," *Elemente Der Mathematik,* vol. 50, no. 4, pp. 167-170, 1995.

[31] A. Heppes and H. Melissen, "Covering a Rectangle with Equal Circles," *Periodica Mathematica Hungarica,* vol. 34, no. 1, pp. 65-81, 1997.

[32] K. Nurmela and P. Ostergard, "Covering a Square with up to 30 Equal Circles," Research Report, Helsinki Univ. of Technology, 2000.

[33] J. Melissen and P. Schuur, "Improved Coverings of a Square with Six and Eight Equal Circles," *Electronic J. Combinatorics,* vol. 3, no. 1, 1996.

[34] W. Rudin, *Principles of Math. Analysis,* vol. 1. McGraw-Hill, 1976.

Bang Wang received the BS and MS degrees from the Department of Electronics and Information Engineering (E&I) of Huazhong University of Science and Technology (HUST) Wuhan, China, in 1996 and 2000, respectively, and the PhD degree from Electrical and Computer Engineering (ECE) Department of National University of Singapore (NUS), Singapore in 2004. He is currently working as an associate professor in the E&I Department, HUST. His research interests include coverage issues, network localization, and resource allocation in wireless networks. He had published one European patent, two books, and more than 60 technical papers in international conferences and journals.

Han Xu received the BE degree in electrical engineering from Wuhan University, China, in 2005, the ME degree in electrical engineering from Research Institute of Post & Telecommunication (WRI), Wuhan, China, in 2008, and is currently working toward the PhD degree in the Department of Electronics and Information Engineering, Huazhong University of Science and Technology (HUST). His research interests include coverage issues, wireless networks, and sensor networks. He is a student member of the IEEE.

Wenyu Liu received the BS degree in computer science from Tsinghua University, Beijing, China, in 1986, and the MS and PhD degrees, both in electronics and information engineering, from Huazhong University of Science & Technology (HUST), Wuhan, China, in 1991 and 2001, respectively. He is currently a professor and an associate dean of the Department of Electronics & Information Engineering, HUST. His current research areas include computer networking, multimedia information processing. He is a member of the IEEE.

Hui Liang received the BE and ME degrees in electronics and information engineering from the Huazhong University of Science and Technology, China in 2008 and 2011, respectively. His research interests include wireless coverage optimization and computer vision.