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Almost-triangular Hopf Algebras

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Abstract In this paper, we consider a finite dimensional semisimple cosemisimple quasitriangular Hopf algebra (H, R) with $R^{21}R \in C(H \otimes H)$ (we call this type of Hopf algebras almost-quasitriangular) over an algebraically closed field k. We denote by B the vector space generated by the left tensorand of $R^{21}R$. Then B is a sub-Hopf algebra of H. We proved that when $\dim B$ is odd, H has a triangular structure and can be obtained from a group algebra by twisting its usual comultiplication [14]; when $\dim B$ is even, H is an extension of an abelian group algebra and a triangular Hopf algebra, and may not be triangular. In general, an almost-triangular Hopf algebra can be viewed as a cocycle bicrossproduct.

AMS Subject Classifications 16W30

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1 Introduction

The classification of finite-dimensional Hopf algebras is one of the most fundamental problems in the theory of Hopf algebras, and this problem is so difficult that only recently some remarkable results, e.g. [3–6, 13–14, 16] are well developed. Many

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of these results depend heavily on the notion of quasitriangular Hopf algebras, a concept introduced by Drinfel'd in the middle 1980s [12] for constructing solutions to the quantum Yang–Baxter equation which arises in mathematical physics. In categorical language, a quasitriangular Hopf algebra is a Hopf algebra whose category of finite-dimensional representations is braided rigid. A quasitriangular Hopf algebra is triangular if the corresponding braided rigid category is symmetric. Quasitriangular Hopf algebras are not obscure, as Drinfel'd shows that any finite-dimensional Hopf algebras can be embedded into a finite-dimensional quasitriangular Hopf algebra known as its quantum double. Thus, prior to general classification, it would be better to focus on quasitriangular Hopf algebras.

In [14], Etingof and Gelaki completely classified semisimple cosemisimple triangular Hopf algebras by means of group-theoretical data. They showed that over an algebraically closed field k there is a bijection between the set of isomorphism classes of semisimple and cosemisimple triangular Hopf algebra of dimension N and the set of isomorphism classes (G, H, V, u), where G is a group of order N, H is a subgroup of G, V is an irreducible projective representation of dimension $(|H|)^{\frac{1}{2}}$, and $u \in G$ is a central element. This classification implies that any semisimple and cosemisimple triangular Hopf algebra over a field can be obtained from a group algebra by twisting its coproduct.

By definition, a quasitriangular Hopf algebra (H, R) is triangular if $R^{21}R = 1 \otimes 1$. To generalize this concept, we call a quasitriangular Hopf algebra (H, R) *almost-triangular* if $R^{21}R \in C(H \otimes H) = C(H) \otimes C(H)$, the center of $H \otimes H$.

The main topic of this paper is to study almost-triangular Hopf algebras. We obtain the following main theorem.

MAIN THEOREM. Let (H, R) be a finite dimensional semisimple and cosemisimple almost-triangular Hopf algebra over an algebraically closed field. Let B be the vector space generated by the left tensorand of $R^{21}R$. Then

1. *$R^{21}R$ is symmetric, that is, invariant under the switch map $H \otimes H \mapsto H \otimes H$.*
2. *$B \subset C(H)$ is a commutative and cocommutative sub-Hopf algebra of H.*
3. *If* dim *B is odd, then H has a triangular structure.*
4. *If* dim *B is even, then there is a quasitriangular structure $\widetilde{R}$ on H, such that $\widetilde{R}R^{-1} \in B \otimes B$, $\widetilde{R}^{21}\widetilde{R} \in B \otimes B$. Let $R_{\widetilde{B}} = \widetilde{R}^{21}\widetilde{R}$ and $\widetilde{B}$ be the vector space generated by the left tensorand of $R_{\widetilde{B}}$. Then,* dim $\widetilde{B} = 2^l$ *for some nonnegative integer l.*

To give the structure of almost-triangular Hopf algebras, we use a well-known construction, the cocycle bicrossproduct, developed by Doi and Takeuchi [10], Blattner, Cohen and Montgomery [7], Majid [17], Andruskiewitsch [1, 2], and other authors [8, 20]. We show that as a Hopf algebra, an almost-triangular Hopf algebra H is isomorphic to a cocycle bicrossproduct $B \#_{\sigma}^{\tau} \overline{H}$.

The paper is organized as follows.

In Section 2, we give some properties of almost-triangular Hopf algebras (H, R) and prove the main result Theorem 2.5.

In Section 3, we briefly recall the product construction introduced in [2] and apply it to almost-triangular Hopf algebras. We prove that an almost-triangular Hopf algebra H is isomorphic to the cocycle bicrossproduct $B \#_{\sigma}^{\tau} \overline{H}$ as a Hopf algebra.

Preliminaries. Throughout this paper k is a field, and we refer the reader to the books [18, 22] as references for the general theory of Hopf algebras. For a coalgebra C and $c \in C$ we write $\triangle(c) = \sum c_1 \otimes c_2$. We write $\rho(a) = \sum a_{(-1)} \otimes a_{(0)}$ for $a \in A$ – a left C-comodule; and $\rho(a) = \sum a_{(0)} \otimes a_{(1)}$ for $a \in A$ – a right C-comodule.

Let H be a Hopf algebra over k with comultiplication $\triangle$, counit ε, and antipode S. The following notion of a twisting deformation of H is due to V. Drinfel'd [11].

A twist for H is an invertible element $J \in H \otimes H$ that satisfies

$$(\triangle \otimes id)(J)(J \otimes 1) = (id \otimes \triangle)(J)(1 \otimes J)$$

We write $J = \sum J^1 \otimes J^2$ and $J^{(-1)} = \sum K^1 \otimes K^1$, we also denote $J^2 \otimes J^1$ by $J_{(21)}$.

Now, we recall the definition of a quasitriangular Hopf algebra and some of its properties. Let H be a Hopf algebra over k and let $R = \sum R^1 \otimes R^2 \in H \otimes H$. Define a linear map $f_R : H^* \longrightarrow H$ by $f_R(p) = \sum \langle p, R^1 \rangle R^2$, where $p \in H^*$. The pair (H, R) is said to be a quasitriangular Hopf algebra if the following axioms hold, where $r = R$.

(QT1) $\sum \triangle(R^1) \otimes R^2 = \sum R^1 \otimes r^1 \otimes R^2 r^2$
(QT2) $\varepsilon(R^1)R^2 = 1$
(QT3) $\sum R^1 \otimes \triangle(R^2) = \sum R^1 r^1 \otimes r^2 \otimes R^2$
(QT4) $\varepsilon(R^2)R^1 = 1$
(QT5) $(\triangle^{cop}(h))R = R(\triangle(h))$ for all $h \in H$.
or equivalently, if $f_R : H^* \longrightarrow H^{cop}$ is a Hopf algebra map and $(QT5)$ is satisfied.
Observe that $(QT5)$ is equivalent to
(QT5′) $\sum \langle p_1, h_2 \rangle h_1 f_R(p_2) = \sum \langle p_2, h_1 \rangle f_R(p_1) h_2$ for all $p \in H^*$ and $h \in H$

Note that for a finite-dimensional quasitriangular Hopf algebra (H, R) the map $f_R^* : H^{*op} \longrightarrow H$ defined by $f_R^*(p) = \sum \langle p, R^2 \rangle R^1$ is a Hopf algebra map satisfying

$$\sum \langle p_1, h_2 \rangle h_1 f_R^*(p_2) = \sum \langle p_2, h_1 \rangle f_R^*(p_1) h_2$$

for all $p \in H^*$ and $h \in H$.

Conversely, for any finite dimensional Hopf algebra H and $f \in \mathrm{Hom}_k(H^*, H)$, the inverse image R_f of f under the canonical isomorphism $H \otimes H \longrightarrow \mathrm{Hom}_k(H^*, H)$ is an element in $H \otimes H$. We say that f determines a quasitriangular structure on H if (H, R_f) is quasitriangular, or equivalently, $f : H^* \longrightarrow H^{cop}$ is a Hopf algebra homomorphism with (QT5′).

A quasitriangular Hopf algebra (H, R) is called triangular if $R^{21}R = 1$ where $R^{21} = \sum R^2 \otimes R^1$. Note that this is equivalent to $R^{21} = R^{-1} = \sum SR^1 \otimes SR^2$ and $f_R * f_{R^{21}} = \varepsilon$ in the convolution algebra $\mathrm{Hom}_k(H^*, H)$.

2 Almost-triangular Hopf Algebras

Let (H, R) be a finite-dimensional quasitriangular Hopf algebra over k. Then (H, R) is said to be triangular if $R^{21}R = 1 \otimes 1$. For a more general consideration, we call (H, R) *almost-triangular* if $R^{21}R \in C(H) \otimes C(H)$.

LEMMA 2.1. Let (H, R) be an almost-triangular Hopf algebra over a field k. Then, $R_B = R^{21}R$ is symmetric and satisfies QT1–QT4.

Proof. First, we have the following equation by $R^{21}R \in C(H) \otimes C(H)$

$$R^{21}RR^{21} = R^{21}R^{21}R$$

Then, $RR^{21} = R^{21}R$ since R^{21} is invertible, i.e. R_B is symmetric.

Let $R = \widetilde{R} = r = \widetilde{r}$. Then we have

$$\begin{aligned}(\Delta \otimes 1)R_B &= (\Delta \otimes 1)R^{21}R = (\Delta \otimes 1)R^{21}(\Delta \otimes 1)R\\ &= \sum (r^2 \otimes R^2 \otimes R^1 r^1)(\widetilde{R}^1 \otimes \widetilde{r}^1 \otimes \widetilde{R}^2\widetilde{r}^2)\\ &= \sum r^2\widetilde{R}^1 \otimes R^2\widetilde{r}^1 \otimes R^1 r^1 \widetilde{R}^2\widetilde{r}^2\\ &= \sum r^2\widetilde{R}^1 \otimes R^2\widetilde{r}^1 \otimes r^1\widetilde{R}^2 R^1\widetilde{r}^2\\ &= \sum R_B^{13}R_B^{23}\end{aligned}$$

The fourth equality follows from $R^{21}R \in C(H) \otimes C(H)$ and we obtain that R_B satisfies QT1. The proof of QT3 is similar and QT2, QT4 are obvious. □

REMARK 2.2. As we can see in the proof, Lemma 2.1 also holds if we assume only that the second tensorand of R_B commutes with the first tensorand of R.

Now assume that (H, R) is an almost-triangular Hopf algebra. Let $R_B = R^{21}R$, and $B = R_B^1\langle H^*, R_B^2\rangle$ be the subspace generated by the first tensorand of R_B. By Lemma 2.1, the symmetry of $R^{21}R$ implies that B is also the subspace spanned by the second tensorand of $R^{21}R$.

PROPOSITION 2.3. B is a commutative and cocommutative sub-Hopf algebra of H. Furthermore, (B, R_B) is minimal quasitriangular [19].

Proof. First B is a sub-Hopf algebra as a consequence of Lemma 2.1.

Since $R_B \in C(H) \otimes C(H)$, we have the natural commutativity, and the cocommutativity via

$$\begin{aligned}\Delta R_B^1 \otimes R_B^2 &= \sum R_B^1 \otimes R_B^1 \otimes R_B^2 R_B^2\\ &= \sum R^2 r^1 \otimes \widetilde{R}^2\widetilde{r}^1 \otimes R^1 r^2 \widetilde{R}^1\widetilde{r}^2\\ &= \sum R^2 r^1 \otimes \widetilde{R}^2\widetilde{r}^1 \otimes \widetilde{R}^1\widetilde{r}^2 R^1 r^2\\ &= \Delta^{cop} R_B^1 \otimes R_B^2\end{aligned}$$

Thus R_B is an R-matrix of B and the minimality of R_B comes from the definition of B. □

Now, we assume further that H is semisimple and cosemisimple over an algebraically closed field k. Then B is a commutative and cocommutative cosemisimple Hopf algebra. We can assume that $B = k[G]$, where G is a finite abelian group. Since

B (spanned by the first tensorand of $R^{21}R$) is a central Hopf subalgebra of H which is clearly normal, we define $\overline{H} = H/HB^+$ to be the corresponding quotient Hopf algebra.

THEOREM 2.4. Let (H, R) be a semisimple and cosemisimple almost-triangular Hopf algebra over an algebraically closed field k. Then we have $H \cong k[G]\#_\sigma k[G']^J$ as an algebra, where both G and G' are finite groups, and $k[G']^J$ is a twist Hopf algebra.

Proof. By Proposition 2.3, B is a commutative sub-Hopf algebra. Write $\overline{H} = H/HB^+$. Then $\overline{H}$ is a triangular semisimple Hopf algebra with its triangular structure $\overline{R}$ being the quotient of R. That $\overline{R}^{21}\overline{R} = 1$ can easily be deduced from $R^{21}R \in B$. By [14], there exist a group G' and a twist J, such that $(\overline{H}, \overline{R}) \cong (k[G']^J, J_{21}^{-1}J)$. Now, using Schneider's normal basis theorem [21], we have $H \cong B\#_\sigma \overline{H}$ as an algebra. □

It is natural to ask when an almost-triangular Hopf algebra is actually triangular? We give the answer in the following theorem.

THEOREM 2.5. Let (H, R) be a finite dimensional semisimple and cosemisimple almost-triangular Hopf algebra over an algebraically closed field, $R_B = R^{21}R$ and $B = R_B^1\langle H^*, R_B^2\rangle$. Then there exists an R-matrix R_1 *over B such that*

1. *if* dim B *is odd then (H, R_1R) is triangular, and thus $(H, R) \cong (k[G_1]^{J_1}, (J_1)_{21}^{-1}J_1)$ for a finite group G_1 and a twist $J_1 \in k[G_1]\otimes k[G_1]$;*
2. *if* dim B *is even then (H, R_1R) is almost-triangular, and $\widetilde{R} = R_1R$ is an R-matrix on H such that the corresponding commutative and cocommutative sub-Hopf algebra $\widetilde{B}$ has dimension* $\dim \widetilde{B} = 2^l$ *for some nonnegative integer l.*

Proof. We know from Proposition 2.3 that $R_B = R^{21}R$ is an R-matrix of the sub-Hopf algebra B of H and B is commutative, cocommutative, cosemisimple and minimal quasitriangular. Thus $B = k\left[G\right]$ for a finite abelian group G. Let $f : B^* \longrightarrow B$ be the Hopf algebra isomorphism $f(b^*)=R_B^1\left\langle b^*, R_B^2\right\rangle$, for $b^* \in B^*$.

1. Assume that $|G| = 2k+1$ for a nonnegative integer k. Then, by Langrange's Theorem $f^{|G|}(g^*) = (f(g^*))^{|G|} = 1_G = 1_G\langle g^*, 1_G\rangle$ for any $g^* \in G(B^*)$, i.e. $f^{|G|} = 1_G\varepsilon_{B^*}$. Therefore, we have

$$f_{(R_B)^{|G|}} = \left(f_{R_B}\right)^{|G|} = f_{1\otimes 1},$$

and thus

$$(R_B)^{2k+1} = (R_B)^{|G|} = 1\otimes 1.$$

Let $R_1 = (R_B)^k$ and $\widetilde{R} = R_1R$. By Lemma 2.1, R_1 is symmetric.
The property that $\widetilde{R}$ is an R-matrix can be deduced from the fact that R and R_1 are R-matrices on H and B, respectively, $R_1 \in C(H)\otimes C(H)$. In addition,

$$\widetilde{R}^{21}\widetilde{R} = (R_1)^{21}R^{21}R_1R = (R_1)^{21}R_1R^{21}R = R_1R_1R^{21}R = R_B^{-1}R_B = 1\otimes 1.$$

This proves that (H, R_1R) is triangular.

2. Assume that $|G| = 2^L m$, where m is odd. Since G is abelian $G = G_1 \times G_2$, for two subgroups G_1 and G_2 of G with $|G_1| = 2^L$ and $|G_2| = m$. For any $g^* \in G(B^*)$, the image $f_{R_B}(g^*)$ is grouplike whence $f_{R_B}(g^*) = g_1 g_2$ for some $g_1 \in G_1$ and $g_2 \in G_2$. By Langrange's Theorem again,

$$\begin{aligned} f_{R_B^m}(g^*) &= \langle g^*, (R_B^m)^1 \rangle (R_B^m)^2 \\ &= \langle g^*, R_B^{1m} \rangle (R_B^m)^2 \\ &= \langle g^*, R_B^1 \rangle^m (R_B^m)^2 \\ &= (f_{R_B}(g^*))^m \\ &= g_1^m g_2^m \\ &= g_1^m \in G_1 \end{aligned}$$

Thus by the symmetry of R_B, we obtain $R_B^m \in k[G_1] \otimes k[G_1]$.
Let $R_1 = (R_B)^{(m-1)/2}$. Then R_1 is an R-matrix of B and $\widetilde{R} = R_1 R$ is an R-matrix of H such that

$$\begin{aligned} \widetilde{R}^{21} \widetilde{R} = (R_1)^{21}\, R^{21} R_1 R &= (R_1)^{21}\, R_1 R^{21} R \\ &= (R_1)^2\, R_B = (R_B)^{m-1}\, R_B \\ &= R_B^m \in k[G_1] \otimes k[G_1]. \end{aligned}$$

This proves that the subspace $\widetilde{B}$ spanned by the left tensorand of $\widetilde{R}^{21} \widetilde{R}$ is contained in $k[G_1]$ and has dimension 2^l for some $l \leq L$. □

REMARK 2.6. For an even-dimensional semisimple quasitriangular Hopf algebra (H, R), the property $R^{21} R \in C(H) \otimes C(H)$ is not sufficient for the existence of a triangular structure on H. A nice construction of Gelaki [15] provides an example of this.

For completeness, we recall here Gelaki's A_{qp}-constructions [15].

Let p and q be prime numbers satisfying $p = 1 \mod q$ and pick $m \in \mathbb{Z}_p$ such that $|m| = q$. Let $\langle b \rangle$ and $\langle h \rangle$ be cyclic groups of orders p and q^2. Assume that the base field k contains primitive pth and q^2th roots of unity. Then $\theta(b) = b^m$ induces a group automorphism of order q of $\langle b \rangle$. Any non-trivial orbit of the action of θ on G is of the form $\left\{ b^k, b^{km}, b^{km^2}, \ldots, b^{km^{q-1}} \right\}$, which is a set of q elements. Choose a set of representatives $\{b_j\}_{i=0}^r$ of all the disjoint orbits, with $b_0 = 1$, $b_1 = b$, and then $r = \dfrac{p-1}{q}$. Take a generator λ in the linear characters of $k\langle \theta \rangle$, and write $b_i^j = \dfrac{1}{q} \sum_{k=0}^{q-1} \langle \lambda^{-i}, \theta^k \rangle \theta^k (b_j)$ $(0 \leq i \leq q-1,\ 0 \leq j \leq r)$. Then there is a Hopf algebra structure on $A_{qp} = k[\langle b \rangle] \times k[\langle h \rangle]$ with multiplication, comultiplication and antipode defined as

$$\begin{aligned} (b_i^j \otimes h^s)(b_k^l \otimes h^t) &= \eta^{sk} (b_i^j b_k^l \otimes h^{s+t}), \\ \Delta(b_i^j \otimes h^s) &= \sum_{t=0}^{q-1} (b_{i-t}^j \otimes h^{qt+s}) \otimes (b_t^j \otimes h^s), \\ S(b_i^j \otimes h^t) &= \eta^{-ti} (S(b_i^j) \otimes h^{-qi-t}), \end{aligned}$$

and

$$\varepsilon\left(b_i^{\,j}\otimes h^t\right)=\delta_{i0},$$

where $\eta=\langle\lambda,\theta\rangle$.

THEOREM 2.7 ([15, Theorem 3.11]) *Let p and q be prime numbers satisfying $p=1$ mod q, and let k be a field containing primitive pth and q^2th roots of unity. Then A_{qp} is a self-dual semisimple Hopf algebra of dimension pq^2, and A_{qp} is quasitriangular if and only if $q=2$. Furthermore, A_{2p} admits exactly $2p-2$ minimal quasitriangular structures and exactly two non-minimal quasitriangular structures with $k\left[G\left(A_{2p}\right)\right]$ as the corresponding minimal quasitriangular sub-Hopf algebra. Moreover, none of the above-mentioned quasitriangular structures is triangular.*

PROPOSITION 2.8. Let p be an odd prime number, and let k be a field containing primitive pth and 4th roots of unity. Denote by i a primitive 4th root of unity. Then A_{2p} is almost-triangular with $R=\sum_{l,k=0}^{3}\frac{i^{-lk}}{4}(1\otimes h^l)\otimes(1\otimes h^k)$, and there is no triangular structure on A_{2p}.

Proof. By [15, Page 249], $k\left[\langle h^2\rangle\right]\in C(A_{2p})$. We have

$$\begin{aligned}
R^{21}R &= \sum_{k,l,r,s=0}^{3}\frac{i^{-kl-rs}}{16}(1\otimes h^{k+r})\otimes(1\otimes h^{l+s})\\
&= \sum_{k,l,u,v=0}^{3}\frac{i^{kv-2kl+lu-uv}}{16}(1\otimes h^{u})\otimes(1\otimes h^{v})\qquad(\text{let}\quad k+r=u,\ l+s=v)\\
&= \sum_{u,v=0}^{3}\left(\sum_{k,l=0}^{3}\frac{i^{kv-2kl+lu-uv}}{16}\right)(1\otimes h^{u})\otimes(1\otimes h^{v})\\
&= \sum_{u,v=0}^{3}\left(\frac{\left(1+i^{2u}\right)\left(1+i^{2v}\right)\left(1+i^{u}+i^{v}-i^{u+v}\right)}{16\,i^{uv}}\right)(1\otimes h^{u})\otimes(1\otimes h^{v})\\
&= \frac{1}{2}\left(1\otimes 1\otimes 1\otimes 1+1\otimes 1\otimes 1\otimes h^2+1\otimes h^2\otimes 1\otimes 1-1\otimes h^2\otimes 1\otimes h^2\right)\\
&\quad\in C(A_{2p})\otimes C(A_{2p}),
\end{aligned}$$

hence, A_{2p} is almost-triangular.

The non-existence of triangular structure on A_{2p} is just a consequence of theorem [15, Theorem 3.11] mentioned above. □

3 Structure of *H* as Cocycle Bicrossproduct

In this section, we will see that as a Hopf algebra, an almost-triangular Hopf algebra H is isomorphic to a cocycle bicrossproduct $B\#_\sigma^\tau\overline{H}$ – a construction developed by Doi and Takeuchi [10], Blattner, Cohen and Montgomery [7], Majid [17], Andruskiewitsch [1, 2], and other authors [9].

First, we recall the bicrossproduct construction which appeared in [2].

Let A, B be two Hopf algebras, $\rightharpoonup: B \otimes A \to A$ a weak action, $\rho : B \to B \otimes A$ a weak coaction and $\sigma : B \times B \to A$ a cocycle in the sense that σ is bilinear map satisfying

(unitary condition)

$$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1;$$

(cocycle condition)

$$(h_{(1)} \rightharpoonup \sigma(l_{(1)}, m_{(1)}))\sigma(h_{(2)}, l_{(2)}m_{(2)}) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m);$$

(twisted module condition)

$$(h_{(1)} \rightharpoonup (l_{(1)} \rightharpoonup a))\sigma(h_{(2)}, l_{(2)}) = \sigma(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \rightharpoonup a),$$

for any $h, l, m \in B$ and $a \in A$.

Let $\tau : B \to A \otimes A$ be a cocycle in the sense that τ is a bilinear map satisfying

(counitary condition)

$$\varepsilon_B(b)1_A = (\varepsilon_A \otimes id)\tau(b) = (id \otimes \varepsilon_A)\tau(b);$$

(cycle condition)

$$m_{A^{\otimes 3}}(\Delta \otimes id \otimes \tau \otimes id)(\tau \otimes \rho)\Delta = (id \otimes m_{A^{\otimes 2}})(id \otimes \Delta \otimes id \otimes id)(\tau \otimes \tau)\Delta;$$

(twisted comodule condition)

$$(id \otimes m_{A^{\otimes 2}})(id \otimes \Delta \otimes id \otimes id)(\rho \otimes \tau)\Delta = m^{13}_{A^{\otimes 2}}(id \otimes id \otimes \rho \otimes id)(\tau \otimes \rho)\Delta,$$

where $m^{13}_{A^{\otimes 2}} : A \otimes A \otimes B \otimes A \otimes A \to B \otimes A \otimes A$ sends $g \otimes h \otimes c \otimes \overline{g} \otimes \overline{h}$ to $c \otimes g\overline{g} \otimes h\overline{h}$.

Let $C = A^\tau \#_\sigma B$ denote the vector space $A \otimes B$ provided with the multiplication

$$(a \otimes b)(\overline{a} \otimes \overline{b}) = a(b_{(1)} \rightharpoonup \overline{a})\sigma(b_{(2)}, \overline{b}_{(1)}) \otimes b_{(3)}\overline{b}_{(2)}$$

and the comultiplication

$$\Delta(a \otimes b) = a_{(1)}\tau(b_{(1)})^1 \otimes b_{(2)(0)} \otimes a_{(2)}\tau(b_{(1)})^2 b_{(2)(1)} \otimes b_{(3)},$$

where $\tau(b) = \tau(b)^1 \otimes \tau(b)^2$.

Let $\iota : A \to C$ and $\pi : C \to B$ be given by $\iota(a) = a \otimes 1$, $\pi(a \otimes b) = \varepsilon(a)b$. $A \#_\sigma B$ (resp., $A^\tau \# B$) denotes the same space considered merely as an algebra (resp., as a coalgebra).

PROPOSITION 3.1. [1, 2, 17] *If σ and τ satisfy the compatibility conditions listed in* [2, *Page* 17], *then $C = A^\tau \#_\sigma B$ is a bialgebra. Moreover, if σ and τ are invertible with respect to the convolution product, then C is a Hopf algebra.*

In this case,

$$k \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow k \qquad (*)$$

is an exact sequence of Hopf algebras.

Conversely, let (*) *be an exact sequence of Hopf algebras and assume that in addition it is cleft. Then there exist* $\rightharpoonup, \sigma, \rho, \tau$ *satisfying the given conditions, such that* $C \simeq A^{\tau}\#_{\sigma} B$.

PROPOSITION 3.2. Let (H, R) *be a finite dimensional almost-triangular Hopf algebra,* $B = R^1_B\langle H^*, R^2_B\rangle$ *be the subspace generated by the first tensorand of* $R_B = R^{21}R$ and $N = H/HB^+$. *Then as a Hopf algebra* $H \simeq B^{\tau}\#_{\sigma} N$ *for some compatible* σ, τ, *with the trivial action of* N *on* B.

Proof. Since $B \subseteq C(H)$, the action induced from any section is trivial. The result is then a direct consequence of [2, Theorem 3.1.17, Proposition 3.1.12]. □

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