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A note on idempotence-preserving maps

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Let $M_n(\mathbf{F})$ be the space of all $n \times n$ matrices over a field $\mathbf{F}$ of characteristic not 2, and let $P_n(\mathbf{F})$ be the subset of $M_n(\mathbf{F})$ consisting of all $n \times n$ idempotent matrices. We denote by $\Phi_n(\mathbf{F})$ the set of all maps from $M_n(\mathbf{F})$ to itself satisfying $A - \lambda B \in P_n(\mathbf{F})$ implies $\phi(A) - \lambda\phi(B) \in P_n(\mathbf{F})$ for every $A, B \in M_n(\mathbf{F})$ and $\lambda \in \mathbf{F}$. In this note, we prove that $\phi \in \Phi_n(\mathbf{F})$ if and only if there exist $\delta \in \{0, 1\}$ and an invertible matrix $P \in M_n(\mathbf{F})$ such that either $\phi(A) = \delta PAP^{-1}$ for every $A \in M_n(\mathbf{F})$, or $\phi(A) = \delta PA^TP^{-1}$ for every $A \in M_n(\mathbf{F})$. This improves the result of some related references.

Keywords: Field; Characteristic; Idempotence; Preserving; Homogeneous

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1. Introduction

Suppose $\mathbf{C}$ is the complex number field and $\mathbf{F}$ is an arbitrary field of characteristic not 2. Let $M_n(\mathbf{F})$ be the space of all $n \times n$ matrices over $\mathbf{F}$ and $P_n(\mathbf{F})$ be the subset of $M_n(\mathbf{F})$ consisting of all $n \times n$ idempotent matrices. The problem of characterizing linear maps preserving idempotence belongs to a large group of the so called linear preserver problems (see [3] and the references therein). The theory of linear preservers of idempotence is well-developed [1]. Recently, the first results on more difficult non-linear indempotence preserver problems have been obtained [2,4,5]. We denote by $S\Phi_n(\mathbf{F})$ the set of all maps from $M_n(\mathbf{F})$ to itself satisfying $A - \lambda B \in P_n(\mathbf{F}) \Longleftrightarrow \phi(A) - \lambda\phi(B) \in P_n(\mathbf{F})$ for every $A, B \in M_n(\mathbf{F})$ and $\lambda \in \mathbf{F}$. A map ϕ is called a strong idempotence-preserving map if $\phi \in S\Phi_n(\mathbf{F})$. Šemrl [4] showed that when $n \geq 3$, $\phi \in S\Phi_n(\mathbf{C})$ is bijective and continuous if and only if either ϕ is of the form $\phi(A) = PAP^{-1}$ for every $A \in M_n(\mathbf{C})$, or ϕ is of the form $\phi(A) = PA^TP^{-1}$ for every $A \in M_n(\mathbf{C})$, where $P \in M_n(\mathbf{C})$ is invertible and A^T denotes the transpose of A. Dolinar [2] improved the result of Šemrl by relaxing the bijectivity assumption to the surjectivity and also omitting the continuous assumption and the restriction on $n \geq 3$. Further, Zhang [5] improved Dolinar's

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result by omitting the surjectivity assumption and extended the field from complex number field to any field of characteristic not 2.

There is a natural question when thinking of possible improvements of the above mentioned characterization of maps on $M_n(\mathbf{F})$ preserving idempotence in both directions. Can we obtain the similar conclusion under the weaker assumption of idempotence-preserving in one direction only? That is, consider the set $\Phi_n(\mathbf{F})$ of all maps from $M_n(\mathbf{F})$ to itself satisfying

$$A - \lambda B \in P_n(\mathbf{F}) \Rightarrow \phi(A) - \lambda\phi(B) \in P_n(\mathbf{F}) \text{ for every } A, B \in M_n(\mathbf{F}) \text{ and } \lambda \in \mathbf{F}.$$

As we will see in the next sections we completely characterize the set $\Phi_n(\mathbf{F})$ in which every map is called an idempotence-preserving map. Namely, we will prove the following result.

Theorem 1.1 *Suppose* $\mathbf{F}$ *is any field of characteristic not* 2 *and* $\phi \in \Phi_n(\mathbf{F})$. *Then there exist* $\delta \in \{0, 1\}$ *and an invertible matrix* $P \in M_n(\mathbf{F})$ *such that either* $\phi(A) = \delta PAP^{-1}$ *for every* $A \in M_n(\mathbf{F})$, *or* $\phi(A) = \delta PA^TP^{-1}$ *for every* $A \in M_n(\mathbf{F})$.

Based on Lemma 2.1, when $n = 1$ the proof of Theorem 1.1 is very simple. Thus, we can assume that $n \geq 2$ in the rest of this article.

For any positive integer $k \leq n$, we denote $\Pi_k(\mathbf{F}) = \{X \oplus 0_{n-k} | X \in M_k(\mathbf{F})\}$, where $\oplus$ denotes the usual direct sum of matrices. Obviously, $\Pi_1(\mathbf{F}) = \{a \oplus 0_{n-1} | a \in \mathbf{F}\}$ and $\Pi_n(\mathbf{F}) = M_n(\mathbf{F})$. Notice that if $\phi \in \Phi_n(\mathbf{F})$, then both the map $A \longmapsto P\phi(A)P^{-1}$ and the map $A \longmapsto \phi(A)^T$ are also in $\Phi_n(\mathbf{F})$. Therefore, based on the inductive idea on n, the proof of Theorem 1.1 is equivalent to prove the following three propositions. The first two propositions show that Theorem 1.1 is true for $n = 2$, and the third one shows that if Theorem 1.1 is true when $n = s$ $(s \geq 2)$, then it is also true for $n = s + 1$.

Proposition 1.2 *Suppose* $\phi \in \Phi_n(\mathbf{F})$. *Then there exist* $\delta \in \{0, 1\}$ *and an invertible matrix* $T_1 \in M_n(\mathbf{F})$ *such that* $\phi(Z) = \delta T_1 Z T_1^{-1}$ *for every* $Z \in \Pi_1(\mathbf{F})$.

Proposition 1.3 *Suppose* $\delta \in \{0, 1\}$ *and* $\phi \in \Phi_n(\mathbf{F})$ *satisfying* $\phi(Z) = \delta Z$ *for every* $Z \in \Pi_1(\mathbf{F})$. *Then there exists an invertible matrix* $T_2 \in M_n(\mathbf{F})$ *such that either* $\phi(Y) = \delta T_2 Y T_2^{-1}$ *for every* $Y \in \Pi_2(\mathbf{F})$, *or* $\phi(Y) = \delta T_2 Y^T T_2^{-1}$ *for every* $Y \in \Pi_2(\mathbf{F})$.

Proposition 1.4 *Suppose* $2 \leq s \leq n - 1$, $\delta \in \{0, 1\}$ *and* $\phi \in \Phi_n(\mathbf{F})$ *satisfying* $\phi(Z) = \delta Z$ *for every* $Z \in \Pi_s(\mathbf{F})$. *Then there exists an invertible matrix* $T_{s+1} \in M_n(\mathbf{F})$ *satisfying* $\phi(Y) = \delta T_{s+1} Y T_{s+1}^{-1}$ *for every* $Y \in \Pi_{s+1}(\mathbf{F})$.

It should be mentioned here that in this note, our main outline is very similar to [5]. But since we work in a condition which is weaker than [5], we must overcome more difficulties. Moreover, the technique used here allows us to remove the injectivity assumption which obtained by a strong idempotence-preserving. Clearly, a result of (not strong) linear idempotence-preserving is a natural corollary of our theorem.

We end this section by denoting a notation. Denote by E_{ij} the $n \times n$ matrix which has 1 in the (i, j) entry and is 0 elsewhere. For any positive integer $k \leq n$, let $\mathbf{F}^k$ be the set of all $k \times 1$ matrices over $\mathbf{F}$. We denote by I_k and 0_k the $k \times k$ identity matrix and zero matrix, respectively. We also write them as I and 0, respectively, when the dimensions of these matrices are clear.

2. Preliminary results

This section provides some preliminary results which will be used to prove Propositions 1.2–1.4 stated in section 1. The following Lemma 2.1 provided by Dolinar is still available for our assumption.

Lemma 2.1 [2] *If $\phi \in \Phi_n(\mathbf{F})$, then*
(*i*) $\phi(P_n(\mathbf{F})) \subseteq P_n(\mathbf{F})$;
(*ii*) *ϕ is homogeneous, i.e., $\phi(\lambda A) = \lambda\phi(A)$ for every $A \in M_n(\mathbf{F})$ and $\lambda \in \mathbf{F}$.*

Lemma 2.2 *Suppose $\delta \in \{0, 1\}$ and $\phi \in \Phi_n(\mathbf{F})$. Suppose $X, Y, Z \in M_s(\mathbf{F})$ and $W = I_t \oplus -I_{s-t}$ such that*
(*a*) $X + Y \in P_s(\mathbf{F})$;
(*b*) $X + Y + W \in P_s(\mathbf{F})$;
(*c*) $\phi(Y \oplus 0_{n-s}) = \delta Z \oplus 0_{n-s}$;
(*d*) $\phi((I_s - Y) \oplus 0_{n-s}) = \delta(I_s - Z) \oplus 0_{n-s}$;
(*e*) $\phi((Y + W) \oplus 0_{n-s}) = \delta(Z + W) \oplus 0_{n-s}$.
If we denote

$$Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in M_t(\mathbf{F})$, then we have

$$\phi(X \oplus 0_{n-s}) = \delta\begin{bmatrix} -A & u \\ v^t & I_{s-t} - D \end{bmatrix} \oplus 0_{n-s}. \tag{1}$$

Proof It follows from (*a*) that $X \oplus 0_{n-s} + Y \oplus 0_{n-s} \in P_n(\mathbf{F})$. Hence,

$$\phi(X \oplus 0_{n-s}) + \phi(Y \oplus 0_{n-s}) \in P_n(\mathbf{F}). \tag{2}$$

By (*a*) we see that $-X + (I_s - Y) = I_s - (X + Y) \in P_s(\mathbf{F})$, and hence $-X \oplus 0_{n-s} + (I_s - Y) \oplus 0_{n-s} \in P_n(\mathbf{F})$. Thus, we obtain by (ii) of Lemma 2.1 that

$$-\phi(X \oplus 0_{n-s}) + \phi((I_s - Y) \oplus 0_{n-s}) \in P_n(\mathbf{F}). \tag{3}$$

Case 1 When $\delta = 0$. Due to (*c*) and (*d*), one has $\phi(Y \oplus 0_{n-s}) = \phi((I_s - Y) \oplus 0_{n-s}) = 0$. This, together with (2) and (3), implies that $\phi(X \oplus 0_{n-s}) \in P_n(\mathbf{F})$ and $-\phi(X \oplus 0_{n-s}) \in P_n(\mathbf{F})$. Hence, $\phi(X \oplus 0_{n-s}) = 0$.

Case 2 When $\delta = 1$. It holds that $\phi(Y \oplus 0_{n-s}) = Z \oplus 0_{n-s}$ and $\phi((I_s - Y) \oplus 0_{n-s}) = (I_s - Z) \oplus 0_{n-s}$ due to (*c*) and (*d*). Let

$$\phi(X \oplus 0_{n-s}) = \begin{bmatrix} U & x_2 \\ x_3 & x_4 \end{bmatrix}$$

where $U \in M_s(\mathbf{F})$. Then (2) and (3) tell us that

$$\begin{bmatrix} U+Z & x_2 \\ x_3 & x_4 \end{bmatrix} \in P_n(\mathbf{F}) \tag{4}$$

and

$$\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} U+Z & x_2 \\ x_3 & x_4 \end{bmatrix} \in P_n(\mathbf{F}).$$

By a direct computation, one can obtain that $x_2 = 0, x_3 = 0$ and $x_4 = 0$. Furthermore, suppose that

$$U = \begin{bmatrix} \alpha & u \\ v^T & \beta \end{bmatrix}$$

where $\alpha \in M_t(\mathbf{F})$. It follows by (e) and (b) that $\phi((Y+W) \oplus 0_{n-s}) = (Z+W) \oplus 0_{n-s}$ and $X \oplus 0_{n-s} + (Y+W) \oplus 0_{n-s} \in P_n(\mathbf{F})$. Hence $\phi(X \oplus 0_{n-s}) + \phi((Y+W) \oplus 0_{n-s}) = \phi(X \oplus 0_{n-s}) + (Z+W) \oplus 0_{n-s} \in P_n(\mathbf{F})$. We deduce

$$\begin{bmatrix} \alpha + A & u + B \\ v^T + C & \beta + D \end{bmatrix} + \begin{bmatrix} I_t & 0 \\ 0 & -I_{s-t} \end{bmatrix} \in P_s(\mathbf{F}). \tag{5}$$

Note that (4) implies

$$U + Z = \begin{bmatrix} \alpha + A & u + B \\ v^T + C & \beta + D \end{bmatrix} \in P_s(\mathbf{F}).$$

This, together with (5), gives that $\alpha + A = 0$ and $\beta + D = I_{s-t}$. The above proof means that

$$\phi(X \oplus 0_{n-s}) = \begin{bmatrix} -A & u \\ v^T & I_{s-t} - D \end{bmatrix} \oplus 0_{n-s},$$

proving the conclusion. ∎

LEMMA 2.3 *Suppose $\phi \in \Phi_n(\mathbf{F})$. Then there exist $\delta \in \{0, 1\}$ and an invertible matrix $T_1 \in M_n(\mathbf{F})$ such that*

$$T_1 \phi(E_{ii}) T_1^{-1} = \delta E_{ii} \quad \text{for all } i \in \{1, \ldots, n\}. \tag{6}$$

Proof For any distinct $1 \le i, j \le n$, because of $E_{ii},\ E_{jj},\ E_{ii} + E_{jj} \in P_n(\mathbf{F})$, it follows from $\phi \in \Phi_n(\mathbf{F})$ and (i) of Lemma 2.1 that $\phi(E_{ii}),\ \phi(E_{jj}),\ \phi(E_{ii}) + \phi(E_{jj}) \in P_n(\mathbf{F})$.

Hence $\phi(E_{ii})\phi\big(E_{jj}\big) = \phi(E_{jj})\phi(E_{ii}) = 0$, see [1]. So by [1] we see that there exists an invertible matrix $T_0 \in M_n(\mathbf{F})$ such that

$$T_0\phi(E_{kk})T_0^{-1} = 0_{r_1} \oplus \cdots \oplus 0_{r_{k-1}} \oplus I_{r_k} \oplus 0_{r_{k+1}} \oplus \cdots \oplus 0_{r_n} \oplus 0_{n-s} \quad \text{for all } k \in \{1,\ldots,n\}, \tag{7}$$

where $r_1 + \cdots + r_n = s$ and we assume that $I_0 = 0$. Next we want to prove that $r_1 = r_2 = \cdots = r_n = 0$ or 1 and by (7) proving the conclusion.

For any distinct $1 \le i,j \le n$, we see by (7) that there is an invertible matrix $Q = Q(i,j) \in M_n(\mathbf{F})$ such that

$$Q^{-1}\phi(E_{ii})Q = I_{r_i} \oplus 0_{r_j} \oplus 0 \tag{8}$$

and

$$Q^{-1}\phi(E_{jj})Q = 0_{r_i} \oplus I_{r_j} \oplus 0. \tag{9}$$

By $\frac{1}{2}(E_{ii} + E_{ij}) + \frac{1}{2}E_{ii} \in P_n(\mathbf{F})$, we have $\frac{1}{2}\phi(E_{ii} + E_{ij}) + \frac{1}{2}\phi(E_{ii}) \in P_n(\mathbf{F})$. This, together with $\phi(E_{ii} + E_{ij}) \in P_n(\mathbf{F})$ and $\phi(E_{ii}) \in P_n(\mathbf{F})$, gives that

$$\phi(E_{ii} + E_{ij}) = -\phi(E_{ii}) + \phi(E_{ii})\phi(E_{ii} + E_{ij}) + \phi(E_{ii} + E_{ij})\phi(E_{ii}). \tag{10}$$

Let $X = Q^{-1}\phi(E_{jj} - E_{ii})Q$. By $(E_{jj} - E_{ii}) + E_{ii} \in P_n(\mathbf{F})$, $(E_{jj} - E_{ii}) + 2E_{ii} \in P_n(\mathbf{F})$ and (8), we deduce $X + (I_{r_i} \oplus 0) \in P_n(\mathbf{F})$ and $X + 2(I_{r_i} \oplus 0) \in P_n(\mathbf{F})$. So $X = -I_{r_1} \oplus X_2$ and $X_2 \in P_{n-r_1}(\mathbf{F})$. Also, it follows by (9) and

$$-(E_{jj} - E_{ii}) + E_{jj} \in P_n(\mathbf{F}), \quad -(E_{jj} - E_{ii}) + 2E_{jj} \in P_n(\mathbf{F})$$

that $-X + (0_{r_i} \oplus I_{r_j} \oplus 0) \in P_n(\mathbf{F})$ and $-X + (0_{r_i} \oplus 2I_{r_j} \oplus 0) \in P_n(\mathbf{F})$. So $X_2 = I_{r_j} \oplus X_3$ and $-X_3 \in P_{n-r_1-r_2}(\mathbf{F})$. Note that $X_2 = I_{r_j} \oplus X_3 \in P_{n-r_1}(\mathbf{F})$, so we see that $X_3 = 0$. We have shown that $X = -I_{r_i} \oplus I_{r_j} \oplus 0$, which implies $\phi(E_{jj} - E_{ii}) = \phi(E_{jj}) - \phi(E_{ii})$. This, together with $(E_{ii} + E_{ij}) + (E_{jj} - E_{ii}) \in P_n(\mathbf{F})$, gives

$$\phi(E_{ii} + E_{ij}) + \phi(E_{jj}) - \phi(E_{ii}) \in P_n(\mathbf{F}). \tag{11}$$

Thanks to $\phi(E_{ii})\phi\big(E_{jj}\big) = \phi(E_{jj})\phi(E_{ii}) = 0$, we have by (10) and (11) that

$$\phi(E_{ii} + E_{ij}) = \phi(E_{ii}) + \phi(E_{jj})\phi(E_{ii} + E_{ij}) + \phi(E_{ii} + E_{ij})\phi(E_{jj}). \tag{12}$$

By a direct computation using (10) and (12), we have

$$Q^{-1}\phi(E_{ii} + E_{ij})Q = \begin{bmatrix} I_{r_i} & U_1 \\ V_1 & 0_{r_j} \end{bmatrix} \oplus 0_{n-r_i-r_j}.$$

In a similar way, we obtain

$$Q^{-1}\phi(E_{ji}+E_{jj})Q=\begin{bmatrix}0_{r_i} & U_2\\ V_2 & I_{r_j}\end{bmatrix}\oplus 0_{n-r_i-r_j}.$$

Furthermore, by $\frac{1}{2}(E_{ii}+E_{ij})+\frac{1}{2}(E_{ji}+E_{jj})\in P_n(\mathbf{F}),$ one has

$$\frac{1}{2}\begin{bmatrix}I_{r_i} & U_1+U_2\\ V_1+V_2 & I_{r_j}\end{bmatrix}\in P_{r_i+r_j}(\mathbf{F}).$$

This tells us that $I_{r_i}=(U_1+U_2)(V_1+V_2)$ and $I_{r_j}=(V_1+V_2)(U_1+U_2)$. For (U_1+U_2), we see that its row rank is r_i and its column rank is r_j. Thus, $r_i=r_j$. By the arbitrariness of i,j, we have $r_1=r_2=\cdots=r_n=\delta$. But since $\phi(E_{ii})\phi\big(E_{jj}\big)=\phi(E_{jj})\phi(E_{ii})=0$, it is clear that $\delta=0$ or 1. This completes the proof. ■

Lemma 2.4 *Suppose* $\delta\in\{0,1\}$ *and* $\phi\in\Phi_n(\mathbf{F})$. *For* $1\le s\le n$, *we denote by* $V_s(\mathbf{F})$ *the set* $M_s(\mathbf{F})$ *or* $D_s(\mathbf{F})=\big\{\mathrm{diag}(d_1,\ldots,d_s): d_i\in\mathbf{F}, i=1,\ldots,s\big\}$. *If* ϕ *satisfies* (*a*) $\phi(E_{ii})=\delta E_{ii}$ *for all* $i\in\{1,\ldots,n\}$ *and* (*b*) $\phi(A\oplus 0_{n-s})=\delta A\oplus 0_{n-s}$ *for all* $A\in V_s(\mathbf{F})$, *then*

$$\phi(A\oplus\mu\oplus 0_{n-s-1})=\delta(A\oplus\mu\oplus 0_{n-s-1})\quad \textit{for all } A\in V_s(\mathbf{F}) \textit{ and } \mu\in\mathbf{F}.$$

Proof Take $\Delta_s(\mathbf{F})$ as a maximal linear independent set of $V_s(\mathbf{F})\cap P_s(\mathbf{F})$. It is clear to see that $\mathrm{Span}(\Delta_s(\mathbf{F}))=V_s(\mathbf{F})$ and $\mathrm{Card}\Delta_s(\mathbf{F})\le s^2$. By the hypothesis (*b*), we can assume that $\mu\ne 0$.

Case 1 When $\delta=0$. We first give the following claim.

Claim 1 *Suppose that* $\phi(B\oplus\mu\oplus 0_{n-s-1})=0$ *for some* $B\in V_s(F)$, *then we have* $\phi((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})=0$ *for any* $\lambda\in\mathbf{F}$ *and* $P\in\Delta_s(\mathbf{F})$.

Proof of Claim 1 We can assume without loss of generality that $\lambda\ne 0$. By (*b*) and

$$\mu^{-1}((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})-\mu^{-1}((B+\lambda P)\oplus 0_{n-s})=E_{s+1,s+1}\in P_n(\mathbf{F}),$$

one has

$$\mu^{-1}\phi((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})\in P_n(\mathbf{F}). \tag{13}$$

Also, by $\lambda^{-1}((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})-\lambda^{-1}(B\oplus\mu\oplus 0_{n-s-1})=P\oplus 0_{n-s}\in P_n(\mathbf{F})$, we have

$$\lambda^{-1}\phi((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})\in P_n(\mathbf{F}). \tag{14}$$

If $\lambda\ne\mu$, then (13) and (14) yield that $\phi((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})=0$, proving Claim 1 for $\lambda\ne\mu$. So, by $\mu\ne-\mu$, we see that $\phi((B-\mu P)\oplus\mu\oplus 0_{n-s-1})=0$. Note that

$$\frac{1}{2}\mu^{-1}((B+\mu P)\oplus\mu\oplus 0_{n-s-1})-\frac{1}{2}\mu^{-1}((B-\mu P)\oplus\mu\oplus 0_{n-s-1})\in P_n(\mathbf{F}). \tag{15}$$

One has

$$\frac{1}{2}\mu^{-1}\phi((B+\mu P)\oplus\mu\oplus 0_{n-s-1})\in P_n(\mathbf{F}).$$

This, together with (13) for $\lambda=\mu$, yields that $\phi((B+\mu P)\oplus\mu\oplus 0_{n-s-1})=0$. This means that the claim holds for $\lambda=\mu$. The proof of Claim 1 is completed.

The condition (a) implies that

$$0=\phi\big(\mu E_{s+1,s+1}\big)=\phi(0_s\oplus\mu\oplus 0_{n-s-1}). \tag{16}$$

Note that every $A\in V_s(\mathbf{F})$ can be written as $A=\sum_{i=1}^{t}\lambda_i P_i$ where $\lambda_i\in\mathbf{F}$ and $P_i\in\Delta_s(\mathbf{F})$, $i=1,\ldots,t$ and $t\le s^2$. This, together with (16) and Claim 1, gives that $\phi(A\oplus\mu\oplus 0_{n-s-1})=0$.

Case 2 When $\delta=1$. By (b) we have $\phi(A\oplus 0_{n-s})=A\oplus 0_{n-s}$ for all $A\in V_s(\mathbf{F})$. Since $\mu^{-1}(A\oplus\mu\oplus 0_{n-s-1})-\mu^{-1}A\oplus 0_{n-s}\in P_n(\mathbf{F})$ and $\mu^{-1}(A\oplus\mu\oplus 0_{n-s-1})+\big((I_s-\mu^{-1}A)\oplus 0_{n-s}\big)\in P_n(\mathbf{F})$, we obtain

$$\mu^{-1}\phi(A\oplus\mu\oplus 0_{n-s-1})-\mu^{-1}A\oplus 0_{n-s}\in P_n(\mathbf{F}) \tag{17}$$

and

$$\mu^{-1}\phi(A\oplus\mu\oplus 0_{n-s-1})+\big((I_s-\mu^{-1}A)\oplus 0_{n-s}\big)\in P_n(\mathbf{F}). \tag{18}$$

Thanks to (17) and (18), one can assume that $\mu^{-1}\phi(A\oplus\mu\oplus 0_{n-s-1})-\mu^{-1}A\oplus 0_{n-s}=0_s\oplus U(A,\mu)$, where $U(A,\mu)\in P_{n-s}(\mathbf{F})$. So we have

$$\phi(A\oplus\mu\oplus 0_{n-s-1})=A\oplus\mu U(A,\mu). \tag{19}$$

We state another claim as follows.

Claim 2 We can assume without loss of generality that $\mu\neq 0$. Suppose that there is $B\in V_s(F)$ such that $U(B,\mu)=1\oplus 0_{n-s-1}$ for all nonzero $\mu\in\mathbf{F}$, then we have $U(B+\lambda P,\mu)=1\oplus 0_{n-s-1}$ for any $\lambda,\mu\in\mathbf{F},\mu\neq 0$ and $P\in\Delta_s(\mathbf{F})$.

Proof of Claim 2 We can assume without loss of generality that $\lambda\neq 0$. Because of

$$\lambda^{-1}((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})-\lambda^{-1}(B\oplus\mu\oplus 0_{n-s-1})\in P_n(\mathbf{F}),$$

one has

$$\lambda^{-1}\mu U(B+\lambda P,\mu)-\lambda^{-1}\mu U(B,\mu)\in P_{n-s}(\mathbf{F}). \tag{20}$$

Also, by $\lambda^{-1}((B+\lambda P)\oplus\mu\oplus 0_{n-s-1})-\lambda^{-1}(B\oplus(\mu-\lambda)\oplus 0_{n-s-1})\in P_n(\mathbf{F})$, we have

$$\lambda^{-1}\mu U(B+\lambda P,\mu)-\lambda^{-1}(\mu-\lambda)U(B,\mu-\lambda)\in P_{n-s}(\mathbf{F}). \tag{21}$$

By the hypothesis, we know that $U(B,\mu)=U(B,\mu-\lambda)=1\oplus 0_{n-s-1}$. This, together with (20) and (21), yields that

$$U(B+\lambda P,\mu)=1\oplus W(B+\lambda P,\mu),$$

where $W(B+\lambda P,\mu)\in P_{n-s-1}(\mathbf{F})$. On the other hand, it follows from (20) that $\lambda^{-1}\mu W(B+\lambda P,\mu)\in P_{n-s-1}(\mathbf{F})$. Hence, if $\lambda\neq\mu$ then we have $W(B+\lambda P,\mu)=0$ and so that $U(B+\lambda P,\mu)=1\oplus 0_{n-s-1}$. Since $\mu\neq-\mu$, one obtain that $W(B-\mu P,\mu)=0_{n-s-1}$. We want to prove that $U(B+\mu P,\mu)=1\oplus 0_{n-s-1}$ and proving the claim. Due to (15), we obtain

$$\frac{1}{2}W(B+\mu P,\mu)-\frac{1}{2}W(B-\mu P,\mu)=\frac{1}{2}W(B+\mu P,\mu)\in P_{n-s-1}(\mathbf{F}).$$

This, together with $W(B+\mu P,\mu)\in P_{n-s-1}(\mathbf{F})$, gives that $W(B+\mu P,\mu)=0$. The proof of Claim 2 is completed.

The condition (a) implies that $\mu E_{s+1,s+1}=\phi\big(\mu E_{s+1,s+1}\big)=\phi(0_s\oplus\mu\oplus 0_{n-s-1})$, so we have by (19) that

$$U(0_s,\mu)=1\oplus 0_{n-s-1}. \tag{22}$$

Note that every $A\in V_s(\mathbf{F})$ can be written as $A=\sum_{i=1}^{t}\lambda_iP_i$ where $\lambda_i\in\mathbf{F}$ and $P_i\in\Delta_s(\mathbf{F})$, $i=1,\ldots,t$ and $t\leq s^2$. This, together with (22), Claim 2 and (19), gives $\phi(A\oplus\mu\oplus 0_{n-s-1})=A\oplus\mu\oplus 0_{n-s-1}$. ■

LEMMA 2.5 *Let s be a positive integer $\leq n-1$ and $\phi\in\Phi_n(\mathbf{F})$. Suppose that there exists a nonzero scalar $w\in\mathbf{F}$ satisfying*

$$\begin{cases}\phi\left(\begin{bmatrix}A & x\\ 0 & z\end{bmatrix}\oplus 0\right)=\begin{bmatrix}A & wx\\ 0 & z\end{bmatrix}\oplus 0\\ \phi\left(\begin{bmatrix}A & 0\\ x^T & z\end{bmatrix}\oplus 0\right)=\begin{bmatrix}A & 0\\ w^{-1}x^T & z\end{bmatrix}\oplus 0\end{cases}\quad \text{for all } A\in M_s(\mathbf{F}),\ x\in\mathbf{F}^s,\ z\in\mathbf{F}. \tag{23}$$

Then there exists an invertible matrix $T_{s+1}\in M_n(\mathbf{F})$ satisfying $\phi(V)=T_{s+1}VT_{s+1}^{-1}$ for every $V\in\Pi_{s+1}(\mathbf{F})$.

Proof The conclusion is proved in [5] by using the fact that ϕ is an injection. Here we have to renewedly prove it, since in our case the map ϕ is not injective.

For any $V\in\Pi_{s+1}(\mathbf{F})$, let

$$V=\begin{bmatrix}B & \alpha\\ \beta^T & \mu\end{bmatrix}\oplus 0_{n-s-1}$$

where $B \in M_s(\mathbf{F})$, $\alpha, \beta \in \mathbf{F}^s$ and $\mu \in \mathbf{F}$. We will prove that

$$\phi(V) = \begin{bmatrix} B & w\alpha \\ w^{-1}\beta^T & \mu \end{bmatrix} \oplus 0_{n-s-1},$$

and so $\phi(V) = T_{s+1} V T_{s+1}^{-1}$ where $T_{s+1} = I_s \oplus w^{-1} \oplus I_{n-s-1}$.

Without loss of generality, we can assume that $\alpha, \beta \in \mathbf{F}^s \setminus \{0\}$. It is easy to check that matrices

$$X = \begin{bmatrix} B & \alpha \\ \beta^T & \mu \end{bmatrix}, Y = \begin{bmatrix} -B & -\alpha \\ 0 & 1-\mu \end{bmatrix}, Z = \begin{bmatrix} -B & -w\alpha \\ 0 & 1-\mu \end{bmatrix}, W = I_s \oplus -1$$

satisfy the conditions $(a) - (e)$ of Lemma 2.2. It follows from Lemma 2.2 that

$$\phi(V) = \phi(X \oplus 0_{n-s-1}) = \begin{bmatrix} B & u \\ v^T & \mu \end{bmatrix} \oplus 0_{n-s-1} \text{ for some } u, v \in \mathbf{F}^s. \tag{24}$$

Note that $V + Y \oplus 0_{n-s-1} \in P_n(\mathbf{F})$, so we have by (24) that $\begin{bmatrix} 0_s & u - w\alpha \\ v^T & 1 \end{bmatrix} \in P_{s+1}(\mathbf{F})$. Thus,

$$(u - w\alpha)v^T = 0_s. \tag{25}$$

As $\beta \neq 0,$ we can find $\gamma \in \mathbf{F}^s \setminus \{0\}$ such that $\beta^T\gamma = 1$. Furthermore, one can see that

$$\frac{1}{2}\begin{bmatrix} B & \alpha \\ \beta^T & \mu \end{bmatrix} \oplus 0_{n-s-1} + \frac{1}{2}\begin{bmatrix} \gamma\beta^T - B & \gamma - \alpha \\ 0 & 1-\mu \end{bmatrix} \oplus 0_{n-s-1} \in P_n(\mathbf{F}).$$

By a simple computation, we obtain

$$\frac{1}{2}\begin{bmatrix} \gamma\beta^T & u + w(\gamma - \alpha) \\ v^T & 1 \end{bmatrix} \in P_{s+1}(\mathbf{F}),$$

and so that $\frac{1}{4}\gamma\beta^T + \frac{1}{4}(u + w(\gamma - \alpha))v^T = \frac{1}{2}\gamma\beta^T$. Indeed, we see that

$$(u + w(\gamma - \alpha))v^T = \gamma\beta^T.$$

This, together with (25), yields $w\gamma v^T = \gamma\beta^T$. Note that $\beta^T\gamma = 1,$ one has $v^T = w^{-1}\beta^T$. Similarly, we have $u = w\alpha$. The proof is completed. ■

For every nonzero elements $y \in \mathbf{F}^n,$ we denote $S_y = \{P \in P_n(\mathbf{F}) : Py \neq y\}$.

Lemma 2.6 [5] *Let $G \in M_n(\mathbf{F})$, and $\beta \in \mathbf{F}^n$ be nonzero. Then there exist nonzero scalars $a_1, \ldots, a_q \in \mathbf{F}$ and $G_1, \ldots, G_q \in S_\beta$ such that $G = \sum_{u=1}^{q} a_u G_u$.*

3. The proof of Proposition 1.4

Depending on the value of δ, the proof of Proposition 1.4 is divided into the following two cases.

Case 1 When $\delta = 0$. For any $\lambda \in \mathbf{F} \setminus \{0\}, A \in M_s(\mathbf{F}), x \in \mathbf{F}^s$ and $z \in \mathbf{F}$, by Lemma 2.4 and

$$\lambda\left(\begin{bmatrix} A & x \\ 0 & z \end{bmatrix} \oplus 0_{n-s-1}\right) + \lambda\left(\begin{bmatrix} -A & 0 \\ 0 & \lambda^{-1} - z \end{bmatrix} \oplus 0_{n-s-1}\right) \in P_n(\mathbf{F})$$

we get

$$\lambda\phi\left(\begin{bmatrix} A & x \\ 0 & z \end{bmatrix} \oplus 0_{n-s-1}\right) \in P_n(\mathbf{F}) \text{ for all } \lambda \in \mathbf{F}.$$

It follows by the arbitrariness of λ that

$$\phi\left(\begin{bmatrix} A & x \\ 0 & z \end{bmatrix} \oplus 0_{n-s-1}\right) = 0 \text{ for all } A \in M_s(\mathbf{F}), x \in \mathbf{F}^s \text{ and } z \in \mathbf{F}. \tag{26}$$

Since for any $\lambda \in \mathbf{F} \setminus \{0\}$ one has

$$\lambda\left(\begin{bmatrix} A & x \\ y^T & z \end{bmatrix} \oplus 0_{n-s-1}\right) + \lambda\left(\begin{bmatrix} -A & -x \\ 0 & \lambda^{-1} - z \end{bmatrix} \oplus 0_{n-s-1}\right) \in P_n(\mathbf{F}),$$

hence by (26) we get

$$\lambda\phi\left(\begin{bmatrix} A & x \\ y^T & z \end{bmatrix} \oplus 0_{n-s-1}\right) \in P_n(\mathbf{F}) \text{ for all } \lambda \in \mathbf{F}.$$

Note that λ is arbitrary, so we have that

$$\phi\left(\begin{bmatrix} A & x \\ y^T & z \end{bmatrix} \oplus 0_{n-s-1}\right) = 0 \text{ for all } \begin{bmatrix} A & x \\ y^T & z \end{bmatrix} \in M_{s+1}(\mathbf{F}).$$

Case 2 When $\delta = 1$. We divide the proof into the following two steps.
Step 1

$$\phi\left(\begin{bmatrix} A & x \\ 0 & z \end{bmatrix} \oplus 0_{n-s-1}\right) = \begin{bmatrix} A & f(x)x \\ 0 & z \end{bmatrix} \oplus 0_{n-s-1} \text{ for all } A \in M_s(\mathbf{F}), x \in \mathbf{F}^s \setminus \{0\},\ z \in \mathbf{F},$$

where f is a map from $\mathbf{F}^s \setminus \{0\}$ to $\mathbf{F}$ satisfying $f(cx) = f(x)$ for all $x \in \mathbf{F}^s \setminus \{0\},\ c \in \mathbf{F} \setminus \{0\}$.

In fact, using Lemmas 2.4 and 2.2 for $X_1 = \left[\begin{smallmatrix} A & x \\ 0 & 1 \end{smallmatrix}\right], Y_1 = Z_1 = \left[\begin{smallmatrix} -A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $W_1 = I_s \oplus -1$, we have

$$\phi\left(\begin{bmatrix} A & x \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1}\right) = \begin{bmatrix} A & u(A,x) \\ v(A,x)^T & 1 \end{bmatrix} \oplus 0_{n-s-1}, \tag{27}$$

where $u(A,x), v(A,x) \in \mathbf{F}^s$. As $x \in \mathbf{F}^s \backslash \{0\}$, there exists an invertible matrix $P \in M_s(\mathbf{F})$ such that $x = P\begin{bmatrix} 1 & 0_{1\times(s-1)} \end{bmatrix}^T$. Let

$$H_1 = P(1 \oplus 0_{s-1})P^{-1}, \; H_2 = P\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \oplus 0_{s-2}\right)P^{-1}.$$

Then

$$\left(\begin{bmatrix} A & x \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1}\right) - \left(\begin{bmatrix} A - H_i & 0 \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1}\right) \in P_n(\mathbf{F}), \; i = 1, 2.$$

Using $\phi \in \Phi_n(\mathbf{F})$, (27) and Lemma 2.4, one can obtain that

$$\begin{bmatrix} H_i & u(A,x) \\ v(A,x)^T & 0 \end{bmatrix} \in P_{s+1}(\mathbf{F}), i = 1, 2,$$

and hence $v(A,x) = 0$ and $u(A,x) = \delta_1(A,x)x$ for some $\delta_1(A,x) \in \mathbf{F}$. This, together with Lemma 2.6 and in the same discuss of [5], implies that

$$\phi\left(\begin{bmatrix} A & x \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1}\right) = \begin{bmatrix} A & f(x)x \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1}, \quad \text{for all } x \in \mathbf{F}^s \backslash \{0\}, \; A \in M_s(\mathbf{F}), \tag{28}$$

where f is a map from $\mathbf{F}^s \backslash \{0\}$ to $\mathbf{F}$.

Next, using Lemma 2 for $X_2 = \left[\begin{smallmatrix} A & x \\ 0 & 0 \end{smallmatrix}\right], Y_2 = Z_2 = \left[\begin{smallmatrix} -A & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $W_2 = I_s \oplus -1$, we have

$$\phi\left(\begin{bmatrix} A & x \\ 0 & 0 \end{bmatrix} \oplus 0_{n-s-1}\right) = \begin{bmatrix} A & u \\ v^T & 0 \end{bmatrix} \oplus 0_{n-s-1}, \tag{29}$$

where $u, v \in \mathbf{F}^s$. Noting

$$\begin{bmatrix} I_s + A & x \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1} - \begin{bmatrix} A & x \\ 0 & 0 \end{bmatrix} \oplus 0_{n-s-1} \in P_n(\mathbf{F}),$$

by (28) and (29), one has

$$\begin{bmatrix} I_s & f(x)x - u \\ -v^T & 1 \end{bmatrix} \in P_{s+1}(\mathbf{F}).$$

So we have $u = f(x)x$ and $v^T = 0$. Thus (29) become the following

$$\phi\left(\begin{bmatrix} A & x \\ 0 & 0 \end{bmatrix} \oplus 0_{n-s-1}\right) = \begin{bmatrix} A & f(x)x \\ 0 & 0 \end{bmatrix} \oplus 0_{n-s-1}. \tag{30}$$

For any nonzero scalar $c \in \mathbf{F}$, using $\phi \in \Phi_n(\mathbf{F})$ and

$$\left(\begin{bmatrix} I_s + cI_s & cx \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1}\right) - c\left(\begin{bmatrix} I_s & x \\ 0 & 0 \end{bmatrix} \oplus 0_{n-s-1}\right) \in P_n(\mathbf{F}) \text{ for all } c \in \mathbf{F},$$

we have

$$\phi\left(\begin{bmatrix} I_s + cI_s & cx \\ 0 & 1 \end{bmatrix} \oplus 0_{n-s-1}\right) - c\phi\left(\begin{bmatrix} I_s & x \\ 0 & 0 \end{bmatrix} \oplus 0_{n-s-1}\right) \in P_n(\mathbf{F}).$$

This, together with (28) and (30), gives $f(cx) = f(x)$. Since ϕ is homogeneous, we can complete the proof of Step 1.

Step 2 Using Lemma 2.5 we show in the same way as in [5] that there exists an invertible matrix $T_{s+1} \in M_n(\mathbf{F})$ satisfying $\phi(V) = T_{s+1} V T_{s+1}^{-1}$ for every $V \in \Pi_{s+1}(\mathbf{F})$.

4. The proofs of Propositions 1.2 and 1.3

Proof of Proposition 1.2 By Lemmas 2.1 and 2.3, the conclusion can be easily obtained. ■

Proof of Proposition 1.3 The proof for $\delta = 0$ is similar to the proof of Proposition 1.4. So we assume that $\delta = 1$. By Lemma 2.3 , we can assume that

$$\phi(E_{ii}) = E_{ii}, \quad i = 1, \ldots, n. \tag{31}$$

This, together with Lemma 2.4, one has

$$\phi(a_{11}E_{11} + a_{22}E_{22}) = a_{11}E_{11} + a_{22}E_{22}. \tag{32}$$

Because of $E_{11} \pm E_{12} \in P_n(\mathbf{F})$ and $E_{22} + E_{12} \in P_n(\mathbf{F})$, we have

$$\phi(E_{11}) \pm \phi(E_{12}) \in P_n(\mathbf{F}),\ \phi(E_{22}) + \phi(E_{12}) \in P_n(\mathbf{F}).$$

This, together with (31), gives that

$$\phi(E_{12}) = u_1 E_{12} + v_1 E_{21} \text{ for some } u_1, v_1 \in \mathbf{F} \text{ with } u_1 v_1 = 0. \tag{33}$$

Similarly, we have

$$\phi(E_{21}) = u_2 E_{12} + v_2 E_{21} \text{ for some } u_2, v_2 \in \mathbf{F} \text{ with } u_2 v_2 = 0. \tag{34}$$

Using Lemma 2.2 for $X_1 = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$, $Y_1 = Z_1 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $W_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, we have

$$\phi\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \oplus 0_{n-2}\right) = \begin{bmatrix} 1 & u_3 \\ v_3 & 1 \end{bmatrix} \oplus 0_{n-2} \text{ for some } u_3, v_3 \in \mathbf{F}. \tag{35}$$

Since

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \oplus 0_{n-2} - bE_{12} \in P_n(\mathbf{F}),$$

we have by (33) and (35) that

$$\begin{bmatrix} 1 & u_3 - bu_1 \\ v_3 - bv_1 & 1 \end{bmatrix} \in P_2(\mathbf{F}).$$

Hence, $u_3 = bu_1$ and $v_3 = bv_1$. Indeed, we have

$$\phi\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \oplus 0_{n-2}\right) = \begin{bmatrix} 1 & bu_1 \\ bv_1 & 1 \end{bmatrix} \oplus 0_{n-2} \quad \text{for all } b \in \mathbf{F}. \tag{36}$$

Note that

$$\frac{1}{2}E_{21} + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus 0_{n-2} \in P_n(\mathbf{F}),$$

so we have by $\phi \in \Phi_n(\mathbf{F})$ and (34), (36) that

$$\frac{1}{2}\begin{bmatrix} 1 & u_1 + u_2 \\ v_1 + v_2 & 1 \end{bmatrix} \in P_2(\mathbf{F}).$$

Thus, $(u_1 + u_2)(v_1 + v_2) = 1$. This tells us that either $u_1 \neq 0$, $v_2 = u_1^{-1}, u_2 = v_1 = 0$ or $v_1 \neq 0$, $u_2 = v_1^{-1}, u_1 = v_2 = 0$. That is, there is $w \in \mathbf{F}\backslash\{0\}$ such that $\phi(E_{12}) = wE_{12}, \phi(E_{21}) = w^{-1}E_{21}$ or $\phi(E_{12}) = wE_{21}, \phi(E_{21}) = w^{-1}E_{12}$.

Case 1 Suppose $\phi(E_{12}) = wE_{12}$ and $\phi(E_{21}) = w^{-1}E_{21}$ hold. By (36) and the homogeneous property of ϕ, we have

$$\phi(aE_{11} + bE_{12} + aE_{22}) = aE_{11} + bwE_{12} + aE_{22} \text{ for all } a, b \in \mathbf{F}. \tag{37}$$

Similarly, one can see that

$$\phi(aE_{11}+bE_{21}+aE_{22}) = aE_{11}+bw^{-1}E_{21}+aE_{22} \text{ for all } a,b \in \mathbf{F}. \tag{38}$$

Suppose

$$\phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-2}\right) = \begin{bmatrix} U & x_2 \\ x_3 & x_4 \end{bmatrix}$$

where $U \in M_2(\mathbf{F})$. It follows by $(E_{12}+E_{21})+(\frac{1}{2}E_{11}+\frac{1}{2}E_{22}-\frac{3}{4}E_{12}) \in P_n(\mathbf{F})$ that

$$\begin{bmatrix} U+\begin{bmatrix} \frac{1}{2} & -\frac{3}{4}w \\ 0 & \frac{1}{2} \end{bmatrix} & x_2 \\ x_3 & x_4 \end{bmatrix} \in P_n(\mathbf{F}). \tag{39}$$

Again, by $-(E_{12}+E_{21})+(E_{11}+E_{22}-(\frac{1}{2}E_{11}+\frac{1}{2}E_{22}-\frac{3}{4}E_{12})) \in P_n(\mathbf{F})$ and (37), we deduce

$$I_2 \oplus 0_{n-2} - \begin{bmatrix} U+\begin{bmatrix} \frac{1}{2} & -\left(\frac{3}{4}\right)w \\ 0 & \frac{1}{2} \end{bmatrix} & x_2 \\ x_3 & x_4 \end{bmatrix} \in P_n(\mathbf{F}). \tag{40}$$

This, together with (39), gives that $x_2=0, x_3=0$ and $x_4=0$. Let $U=[\begin{smallmatrix} \alpha & \beta \\ \gamma & \tau \end{smallmatrix}]$ where $\alpha,\beta,\gamma,\tau \in \mathbf{F}$. Then by (40), one can obtain that

$$\begin{bmatrix} \alpha+\left(\frac{1}{2}\right) & \beta-\left(\frac{3}{4}\right)w \\ \gamma & \tau+\left(\frac{1}{2}\right) \end{bmatrix} \in P_2(\mathbf{F}). \tag{41}$$

Similarly, we have by (38) and $(E_{12}+E_{21})+(\frac{1}{2}E_{11}+\frac{1}{2}E_{22}-(\frac{3}{4})E_{21}) \in P_n(\mathbf{F})$ that

$$\begin{bmatrix} \alpha+\left(\frac{1}{2}\right) & \beta \\ \gamma-\left(\frac{3}{4}\right)w^{-1} & \tau+\left(\frac{1}{2}\right) \end{bmatrix} \in P_2(\mathbf{F}). \tag{42}$$

It follows from $\frac{1}{2}(E_{12}+E_{21})+\frac{1}{2}(E_{11}+E_{22})\in P_n(\mathbf{F})$ and (37) that

$$\frac{1}{2}\begin{bmatrix}\alpha+1 & \beta\\ \gamma & \tau+1\end{bmatrix}\in P_2(\mathbf{F}). \tag{43}$$

Combining (41) with (42) and (43), one can conclude that

$$\phi(E_{12}+E_{21})=wE_{12}+w^{-1}E_{21}. \tag{44}$$

Using Lemma 2.2 for $X_2=\left[\begin{smallmatrix}1 & b\\ 0 & 0\end{smallmatrix}\right]$, $Y_2=Z_2=\left[\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right]$ and $W_2=\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$, we have

$$\phi\left(\begin{bmatrix}1 & b\\ 0 & 0\end{bmatrix}\oplus 0_{n-2}\right)=\begin{bmatrix}1 & u_4\\ v_4 & 0\end{bmatrix}\oplus 0_{n-2}\text{ for some }u_4,v_4\in\mathbf{F}. \tag{45}$$

Note that $E_{11}+bE_{12}-cE_{12}\in P_n(\mathbf{F})$ for every $c\in\mathbf{F}$ and that $(E_{11}+bE_{12})-b(E_{12}+E_{21})\in P_n(\mathbf{F})$. We see by (44) and (45) that $u_4=bw$ and $v_4=0$. Furthermore, by (45) and the homogeneous property of ϕ, one has

$$\phi\left(\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}\oplus 0_{n-2}\right)=\begin{bmatrix}a & bw\\ 0 & 0\end{bmatrix}\oplus 0_{n-2}\text{ for all }a,b\in\mathbf{F}. \tag{46}$$

Using Lemma 2.2 for $X_3=\left[\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right]$, $Y_3=Z_3=\left[\begin{smallmatrix}-a & 0\\ 0 & 0\end{smallmatrix}\right]$ and $W_3=\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$, we can assume that

$$\phi\left(\begin{bmatrix}a & b\\ 0 & 1\end{bmatrix}\oplus 0_{n-2}\right)=\begin{bmatrix}a & u_5\\ v_5 & 1\end{bmatrix}\oplus 0_{n-2}\text{ for some }u_5,v_5\in\mathbf{F}. \tag{47}$$

Note that

$$\begin{bmatrix}a & b\\ 0 & 1\end{bmatrix}\oplus 0_{n-2}+\begin{bmatrix}1-a & -b\\ 0 & 0\end{bmatrix}\oplus 0_{n-2}\in P_n(\mathbf{F}).$$

This, together with (46) and (47), implies that $u_5=bw$ and $v_5=0$. Furthermore, by (47) and the homogeneous property of ϕ, one has

$$\phi\left(\begin{bmatrix}a & b\\ 0 & d\end{bmatrix}\oplus 0_{n-2}\right)=\begin{bmatrix}a & bw\\ 0 & d\end{bmatrix}\oplus 0_{n-2}\quad\text{for all }a,b,d\in\mathbf{F}. \tag{48}$$

Similarly, we have

$$\phi\left(\begin{bmatrix}a & 0\\ c & d\end{bmatrix}\oplus 0_{n-2}\right)=\begin{bmatrix}a & 0\\ cw^{-1} & d\end{bmatrix}\oplus 0_{n-2}\quad\text{for all }a,c,d\in\mathbf{F}. \tag{49}$$

In view with (48), (49) and Lemma 2.5, there exists an invertible matrix $T_2 \in M_n(\mathbf{F})$ such that $\phi(A) = T_2 A T_2^{-1}$ for every $A \in \Pi_2(\mathbf{F})$.

Case 2 Suppose $\phi(E_{12}) = wE_{21}$ and $\phi(E_{21}) = w^{-1}E_{12}$ hold. By a similar argument to Case 1, there exists an invertible matrix $T_2 \in M_n(\mathbf{F})$ such that $\phi(A) = T_2 A^T T_2^{-1}$ for every $A \in \Pi_2(\mathbf{F})$.

The proof is completed. ■

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