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Dynamical and Geometric Phases of a Two Energy-Level Bose–Einstein Condensate Interacting with a Laser Field *

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By using of the invariant theory, we study a two energy-level Bose–Einstein condensate interacting with a time-dependent laser field, the dynamical and geometric phases are given respectively. The Aharonov–Anandan phase is also obtained under the cyclical evolution.

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Recently, much attention has been paid to investigation of Bose–Einstein condensation (BEC) in dilute and ultracold gases of neutral alkali-metal atoms using a combination of laser and evaporative cooling[1−3] due to the optical properties,[4,5] statistical properties,[6,7] phase properties,[8,9] and tunnelling effect.[10−18]

It is known that the concept of geometric phase was first introduced by Pancharatnam[19] in studying the interference of classical light in distinct states of polarization. Berry[20] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan.[21] Samuel and Bhandari[22] generalized the pure state geometric phase further by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. This led Mukunda and Simon[23] to put forward a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out.[24] Recently, the geometric phase of the mixed states has also been studied.[25]

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld[26] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized in Ref. [27] by introducing the concept of basic invariants and used to study the geometric phases in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the old theory of quantum adiabatic approximations, but also provides us with new insights into many physical phenomena. The concept of Berry's phase has developed in some different directions.[28−36] In this Letter, by using of the invariant theory, we study a two energy-level Bose–Einstein condensate interacting with a time-dependent laser field.

We consider a two energy-level Bose–Einstein condensate interacting with a time-dependent laser field. The Hamiltonian of the system reads

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{FA} + \hat{H}_{AA}, \tag{1}$$

where $\hat{H}_A$, $\hat{H}_F$, $\hat{H}_{FA}$ and $\hat{H}_{AA}$ denote the atom, the free field, the field-atom, and the atom-atom interaction Hamiltonian, respectively. We can obtain the second-quantization form in the particle-number representation (in units of $\hbar = 1$)[37]

$$\hat{H}_A = \int d^3\boldsymbol{r}\hat{\psi}^\dagger(\boldsymbol{r})\Big[\frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r})\Big]\hat{\psi}(\boldsymbol{r}) = \sum_{i=0}^{1} E_i\hat{b}_i^\dagger\hat{b}_i, \tag{2}$$

$$\hat{H}_F = \frac{1}{8\pi}\int d^3\boldsymbol{r}(\boldsymbol{B}^2 + \boldsymbol{E}^2) = \omega(t)\hat{a}^\dagger\hat{a}, \tag{3}$$

$$\begin{aligned}\hat{H}_{FA} &= \int d^3\boldsymbol{r}\hat{\psi}^\dagger(\boldsymbol{r})\Big(-\frac{q}{mc}\boldsymbol{A}\cdot\boldsymbol{p}\Big)\hat{\psi}(\boldsymbol{r})\\ &= \epsilon(t)(\hat{a} + \hat{a}^\dagger)(\hat{b}_0^\dagger\hat{b}_1 + \hat{b}_0\hat{b}_1^\dagger)\end{aligned} \tag{4}$$

$$\begin{aligned}\hat{H}_{AA} &= \frac{1}{2}\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2\psi^\dagger(\boldsymbol{r}_1)\psi^\dagger(\boldsymbol{r}_2)\\ &\quad\cdot U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\psi(\boldsymbol{r}_1)\psi(\boldsymbol{r}_2)\\ &= \sum_{i,j,k,l=0}^{1} U_{ijkl}\hat{b}_i^\dagger\hat{b}_j^\dagger\hat{b}_k\hat{b}_l\delta_{i+j,k+l},\end{aligned} \tag{5}$$

where E_0 and E_1 in $\hat{H}_A$ denote the atomic ground state energy and the excited state energy, respectively, $\hat{b}_i^\dagger$ and $\hat{b}_i$ $(i = 0, 1)$ are the creation and annihilation operators of the atomic ground and excited states. For a two energy-level atom, we can select $E_0 = 0$ and let

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$E_1 = \omega_0$. $\hat{a}^\dagger(\hat{a})$ in H_F is the creation (annihilation) operator of the laser field, and $\omega(t)$ is the frequency of laser field. Considering the Rotating-wave approximation, one has

$$\hat{H}_{FA} = \epsilon(t)(\hat{a}^\dagger \hat{b}_0^\dagger \hat{b}_1 + \hat{a}\hat{b}_1^\dagger \hat{b}_0), \tag{6}$$

$$\epsilon(t) = \frac{e}{m_e}\sqrt{\frac{2\pi}{\omega(t)V}}\int d^3\boldsymbol{r} u_0^*(\boldsymbol{r})(-i\boldsymbol{e}\cdot\nabla)u_1(\boldsymbol{r}), \tag{7}$$

where $u_i(\boldsymbol{r})$ is the eigenfunction of atomic energy, $\epsilon(t)$ denotes the interaction intensity between the atom and laser field. If let $U_{ijkl} = \Omega$ (Ω has nothing to perform with the order of i,j,k,l), one has

$$\begin{aligned}\hat{H}_{AA} =& \Omega \sum_{i,j,k,l=0}^{1} \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l \delta_{i+j,k+l} \\ =& \Omega(\hat{b}_0^\dagger \hat{b}_0^\dagger \hat{b}_0 \hat{b}_0 + \hat{b}_0^\dagger \hat{b}_1^\dagger \hat{b}_0 \hat{b}_1 + \hat{b}_1^\dagger \hat{b}_0^\dagger \hat{b}_1 \hat{b}_0 + \hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1 \hat{b}_1).\end{aligned} \tag{8}$$

Then the Hamiltonian of the system can be rewritten by

$$\begin{aligned}\hat{H} =& \omega_0 \hat{b}^\dagger \hat{b} + \omega(t)\hat{a}^\dagger \hat{a} + \epsilon(t)(\hat{a}^\dagger \hat{b}_0^\dagger \hat{b} + \hat{a}\hat{b}_0 \hat{b}^\dagger) \\ &+ \Omega(\hat{b}_0^\dagger \hat{b}_0^\dagger \hat{b}_0 \hat{b}_0 + \hat{b}_0^\dagger \hat{b}_1^\dagger \hat{b}_0 \hat{b}_1 + \hat{b}_1^\dagger \hat{b}_0^\dagger \hat{b}_1 \hat{b}_0 + \hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1 \hat{b}_1).\end{aligned} \tag{9}$$

In this study, we consider the case that the laser field is weaker, and adopt the Bogoliubov approximation. Suppose that the atomic number of the Bose–Einstein condensate initially being larger, the slowly change of the atomic number in the ground state can be omitted in the process of the atom-field interaction, then operators $\hat{b}_0$ and $\hat{b}_0^\dagger$ in Eq. (9) can be replaced by $\sqrt{N}_0 e^{i\gamma}$ and $\sqrt{N}_0 e^{-i\gamma}$, omitting the term $\hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1 \hat{b}_1$; and letting $\hat{b}_1 = b$, $\hat{b}_1^\dagger = \hat{b}^\dagger$, one has

$$\begin{aligned}\hat{H} =& (\omega_0 + 2N_0\Omega)\hat{b}^\dagger \hat{b} + \omega(t)\hat{a}^\dagger \hat{a} \\ &+ \epsilon(t)\sqrt{N}_0(\hat{a}^\dagger \hat{b} e^{i\gamma} + \hat{a}\hat{b}^\dagger e^{-i\gamma}) + N_0^2\Omega.\end{aligned} \tag{10}$$

For self-consistency, we first illustrate the Lewis–Riesenfeld (L-R) invariant theory.[38]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{11}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n |\lambda_n, t\rangle, \tag{12}$$

where $\dfrac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial |\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{13}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of Eq. (13) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$ for the non-degenerate state, i.e.

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{14}$$

which shows that $|\lambda_n, t\rangle_s$ ($n = 1, 2, \ldots$) forms a complete set of the solutions to Eq. (13). Then the general solution of the Schrödinger equation (13) can be written as

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{15}$$

where

$$\delta_n(t) = \int_0^t dt' \langle \lambda_n, t'| i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{16}$$

and $C_n = \langle \lambda_n, 0|\psi(0)\rangle_s$.

In order to obtain the exact solutions to Eq. (13), we can define operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ as follows:

$$\hat{K}_+ = \hat{a}^\dagger \hat{b}, \quad \hat{K}_- = \hat{b}^\dagger \hat{a}, \quad \hat{K}_0 = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}, \tag{17}$$

which hold the commutation relations

$$[\hat{K}_0, \hat{K}_\pm] = \pm 2\hat{K}_\pm, \quad [\hat{K}_+, \hat{K}_-] = \hat{K}_0. \tag{18}$$

It is easy to prove that operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ together with the Hamiltonian $\hat{H}$ construct a quasi-algebra.

Then we can obtain the L-R invariant as follows:

$$\hat{I}(t) = \cos\theta \hat{K}_0 - e^{-i\varphi}\sin\theta \hat{K}_+ - e^{i\varphi}\sin\theta \hat{K}_-, \tag{19}$$

where θ and φ are determined by Eq. (11), and they satisfy the relations

$$\dot{\theta} = 2\epsilon(t)\sqrt{N}_0 \sin(\varphi + \gamma), \tag{20}$$

$$\begin{aligned}&\dot{\theta}\cos\theta\sin\varphi - 2\epsilon(t)\sqrt{N}_0\cos\theta\cos\gamma \\ &\quad + [\omega_0 + 2N_0\Omega + \dot{\varphi} - \omega(t)]\sin\theta\cos\varphi = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\dot{\theta}\cos\theta\cos\varphi - 2\epsilon(t)\sqrt{N}_0\cos\theta\sin\gamma \\ &\quad - [\omega_0 + 2N_0\Omega + \dot{\varphi} - \omega(t)]\sin\theta\cos\varphi = 0,\end{aligned} \tag{22}$$

where the overdot denotes the time derivative.

We can construct the unitary transformation

$$\hat{V}(t) = \exp[\sigma \hat{K}_+ - \sigma^* \hat{K}_-], \tag{23}$$

where $\sigma = \dfrac{\theta}{2}e^{-i\varphi}$ and $\sigma^* = \dfrac{\theta}{2}e^{i\varphi}$. The invariant $\hat{I}(t)$ can be transformed into a new time-independent operator $\hat{I}_V$:

$$\hat{I}_V = \hat{V}^\dagger(t)\hat{I}(t)\hat{V}(t) = \hat{K}_0. \tag{24}$$

Correspondingly, we can obtain the eigenvalue equation of operator $\hat{I}(t)$

$$\hat{I}_V|m\rangle_{\hat{a}}|n\rangle_{\hat{b}} = (m - n)|m\rangle_{\hat{a}}|n\rangle_{\hat{b}}, \tag{25}$$

In terms of the unitary transformation $\hat{V}(t)$ and the Baker–Campbell–Hausdoff formula[38]

$$\hat{V}^\dagger(t)\frac{\partial \hat{V}(t)}{\partial t} = \frac{\partial \hat{L}}{\partial t} + \frac{1}{2!}\Big[\frac{\partial \hat{L}}{\partial t}, \hat{L}\Big] + \frac{1}{3!}\Big[\Big[\frac{\partial \hat{L}}{\partial t}, \hat{L}\Big], \hat{L}\Big]$$

$$+\frac{1}{4!}\Big[\Big[\Big[\frac{\partial\hat{L}}{\partial t},\hat{L}\Big],\hat{L}\Big],\hat{L}\Big]+\cdots, \tag{26}$$

where $\hat{V}(t)=\exp[\hat{L}(t)]$. One has

$$\begin{aligned}\hat{H}_V(t)=&\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}\\ =&\Big[(\omega_0+2N_0\Omega)\sin^2\frac{\theta}{2}+\omega(t)\cos^2\frac{\theta}{2}\\ &-\epsilon(t)\sqrt{N_0}\sin\theta\cos(\varphi+\gamma)\\ &+\frac{\dot{\varphi}}{2}(1-\cos\theta)\Big]\hat{a}^\dagger\hat{a}\\ &+\Big[(\omega_0+2N_0\Omega)\cos^2\frac{\theta}{2}+\omega(t)\sin^2\frac{\theta}{2}\\ &+\epsilon(t)\sqrt{N_0}\sin\theta\cos(\varphi+\gamma)\\ &-\frac{\dot{\varphi}}{2}(1-\cos\theta)\Big]\hat{b}^\dagger\hat{b}.\end{aligned} \tag{27}$$

It is easy to find that $\hat{H}(t)$ differs from $\hat{I}_V$ only by a time-dependent c-number factor. Thus we can obtain the general solution of the time-dependent Schrödinger equation (13),

$$|\Psi(t)\rangle_s=\sum_n\sum_m C_{nm}\exp[i\delta_{nm}(t)]\hat{V}(t)|m\rangle_{\hat{a}}|n\rangle_{\hat{b}}, \tag{28}$$

with the coefficients $C_{nm}=< n, m, t=0|\Psi(0)\rangle_s$. The phase $\delta_{nm}(t)=\delta^d_{nm}(t)+\delta^g_{nm}(t)$ includes the dynamical phase

$$\begin{aligned}\delta^d_{nm}(t)=&-m\int_{t_0}^{t}\Big[(\omega_0+2N_0\Omega)\sin^2\frac{\theta}{2}+\omega(t)\cos^2\frac{\theta}{2}\\ &-\epsilon(t)\sqrt{N_0}\sin\theta\cos(\varphi+\gamma)\Big]dt'-\int_{t_0}^{t}N_0^2\Omega dt'\\ &-n\int_{t_0}^{t}\Big[(\omega_0+2N_0\Omega)\cos^2\frac{\theta}{2}\\ &+\omega(t)\sin^2\frac{\theta}{2}+\epsilon(t)\sqrt{N_0}\sin\theta\cos(\varphi+\gamma)\Big]dt',\end{aligned} \tag{29}$$

and the geometric phase

$$\delta^g_{nm}(t)=\int_{t_0}^{t}(n-m)\frac{\dot{\varphi}}{2}(1-\cos\theta)dt'. \tag{30}$$

Particularly, under the cyclical evolution the geometric phase becomes

$$\delta^g_{nm}(t)=\frac{1}{2}\oint(n-m)(1-\cos\theta)d\varphi, \tag{31}$$

which is the well-known geometric Aharonov–Anandan phase.

It is pointed out that the dynamics and Berry phases of two-species Bose–Einstein condensations have been studied recently by using the $SU(2)$ coherent state.[39] The model used in Ref. [39] is similar to that in this study, the result of geometric phase is the same, i.e. $\delta^g_{nm}(t)\sim\oint(1-\cos\theta)d\varphi$.

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