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Limiting behavior of the maximum of the partial sum for asymptotically negatively associated random variables under residual Cesàro alpha-integrability assumption

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A B S T R A C T

Asymptotically negative association is a special dependence structure. By relating such dependence condition to residual Cesàro alpha-integrability and to strongly residual Cesàro alpha-integrability, some L_p-convergence and complete convergence results of the maximum of the partial sum are derived, respectively. In addition, some of these conclusions are based on a new Rosenthal type inequality concerning asymptotically negatively associated random variables, which is of independent interest.

1. Introduction

The classical notion of uniform integrability of a sequence $\{X_n, n \geq 1\}$ of integrable random variables is defined through the condition $\lim_{a\to\infty}\sup_{n\geq 1} E|X_n|I(|X_n|>a)=0$. Landers and Rogge (1987) proved that the uniform integrability condition is sufficient in order that a sequence of pairwise independent random variables verifies the weak law of large numbers (WLLN). Chandra (1989) weakened the assumption of uniform integrability to Cesàro uniform integrability (CUI) and obtained L_1-convergence for pairwise independent random variables.

Chandra and Goswami (1992) improved the above-mentioned result of Landers and Rogge (1987). They showed that for a sequence of pairwise independent random variables, CUI is sufficient for the WLLN to hold and strong Cesàro uniform integrability (SCUI) is sufficient for the strong law of large numbers (SLLN) to hold. Landers and Rogge (1997) obtained a slight improvement over the results of Chandra (1989) and Chandra and Goswami (1992) for the case of non-negative random variables. They showed that, in this case, the condition of pairwise independence can be replaced by the weaker assumption of pairwise non-positive correlation.

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Chandra and Goswami (2003) introduced a new set of conditions called Cesàro alpha-integrability (CI(α)) and strong Cesàro alpha-integrability (SCI(α)) for a sequence of random variables, which are strictly weaker than CUI and SCUI, respectively. They showed that, for $\alpha < 1/2$, CI(α) is sufficient for the WLLN to hold and SCI(α) is sufficient for the SLLN to hold for a sequence of pairwise independent random variables, which are improvements over the results of Landers and Rogge (1997) and the earlier results.

Chandra and Goswami (2006) relaxed the condition of CI(α) to residual Cesàro alpha-integrability (RCI(α), see Definition 2.1 below), the condition of SCI(α) to strong residual Cesàro alpha- integrability (SRCI(α), see Definition 2.2 below), and significantly improved the results of Chandra and Goswami (2003).

Recently, Yuan et al. (2009) obtained some improvements in the area of L_p-convergence and strong law of large numbers, by relating martingale difference, pairwise negative quadrant dependence and L_p-mixingale difference to RCI(α) and SRCI(α).

In this paper, we will derive some L_p-convergence and complete convergence results of the maximum of partial sum for asymptotically negatively associated random variables when such random variables are subject to RCI(α) and SRCI(α), respectively. These results have not been established previously in the literature.

In Section 2, we discuss two class of uniform integrability, i.e. RCI(α) and SRCI(α), and a special dependence, i.e. asymptotically negative association. Simultaneously some lemmas are listed in this section which will be used in the subsequent sections. By the way, Lemma 2.3 is of independent interest.

Some new L_p-convergence results for asymptotically negatively associated random variables are obtained under RCI(α) assumption in Section 3.

Finally, in Section 4 we prove complete convergence for the above-mentioned dependence structure under SRCI(α) assumption.

Throughout this paper, S_n denotes $\sum_{i=1}^n X_i$ for a sequence $\{X_n, n \geq 1\}$ of random variables and $\|\cdot\|_p$ denotes the L_p-norm. For $p>1$, let $q:=p/(p-1)$ be the dual number of p. Moreover, $X^+ = \max(0,X)$, $X^- = \max(0,-X)$, $\ll$ represents the Vinogradov symbol O and $I(\cdot)$ is the indicator function.

2. Preliminaries

First let us specify the two special kinds of uniform integrability we are dealing with in the subsequent sections, which were introduced by Chandra and Goswami (2006).

Definition 2.1. For $\alpha \in (0,\infty)$, a sequence $\{X_n, n \geq 1\}$ of random variables is said to be residually Cesàro alpha-integrable (RCI(α), in short) if

$$\sup_{n\geq 1} \frac{1}{n}\sum_{i=1}^n E|X_i| < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n E(|X_i|-i^\alpha)I(|X_i|>i^\alpha)=0. \tag{1}$$

Clearly, $\{X_n\}$ is RCI(α) for any $\alpha>0$ if $\{X_n, n\geq 1\}$ is identically distributed with $E|X_1|<\infty$, and $\{|X_n|^p, n\geq 1\}$ is RCI(α) for any $\alpha>0$ if $\{X_n, n\geq 1\}$ is stochastically dominated by a non-negative random variable X with $EX^p<\infty$ for some $p\geq 1$.

Definition 2.2. For $\alpha \in (0,\infty)$, a sequence $\{X_n, n\geq 1\}$ of random variables is said to be strongly residually Cesàro alpha-integrable (SRCI(α), in short) if

$$\sup_{n\geq 1} \frac{1}{n}\sum_{i=1}^n E|X_i| < \infty \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n} E(|X_n|-n^\alpha)I(|X_n|>n^\alpha)<\infty. \tag{2}$$

We point out that $\{|X_n|^p, n\geq 1\}$ is SRCI(α) for any $\alpha>0$, provided that $\{X_n, n\geq 1\}$ is stochastically dominated by a non-negative random variable X with $EX^{p+\delta}<\infty$ for some $p\geq 1$ and $\delta>0$.

The condition of SRCI(α) is a "strong" version of the condition of RCI(α). Moreover, for any $\alpha>0$, RCI(α) is strictly weaker than CI(α), thereby weaker than CUI, while SRCI(α) is strictly weaker than SCI(α), thereby much weaker than SCUI.

Next, we turn our attention to dependence structures for random variables. For our purpose, we have to mention a special kind of dependence, namely, negative association.

A finite family of random variables $\{X_i, 1\leq i\leq n\}$ is said to be negatively associated (NA, in short) if for every pair of disjoint subsets A and B of $\{1,2,\ldots,n\}$,

$$Cov(f(X_i, i\in A), g(X_j, j\in B)) \leq 0$$

whenever $f, g \in \mathcal{C}$ and the covariance exists. Here and in the sequel, $\mathcal{C}$ is a class of functions which are coordinatewise nondecreasing. An infinite family is NA if every finite subfamily is NA.

The notion of NA was first introduced by Block et al. (1982). Joag-Dev and Proschan (1983) showed that many well known multivariate distributions possess the NA property. The NA property has aroused wide interest because of numerous applications in reliability theory, percolation theory and multivariate statistical analysis. In the past decades, a lot of effort was dedicated to proving the limit theorems of NA random variables. We refer to Newman (1984) for the central limit theorem, Su et al. (1997) for the moment inequality and functional central limit theorem, Cai and Wu (2007) for the law of the iterated logarithm, among others.

A new kind of dependence structures called asymptotically negative association was proposed by Zhang (2000a, 2000b), which is a useful weakening of the definition of NA.

Definition 2.3. A sequence $\{X_n, n\geq 1\}$ of random variables is called asymptotically negatively associated (ANA, in short) if

$$\rho^-(r) := \sup\{\rho^-(S,T) : S,T\subset N, dist(S,T)\geq r\}\to 0 \quad \text{as} \quad r\to\infty,$$

where

$$\rho^-(S,T) := 0\vee\left\{\frac{Cov\{f(X_i : i\in S), g(X_j : j\in T)\}}{(Var\, f(X_i : i\in S))^{1/2}(Var\, g(X_j : j\in T))^{1/2}} : f,g\in\mathcal{C}\right\}.$$

It is obvious that an ANA sequence of random variables is NA if and only if $\rho^-(1)=0$. Compared to NA, ANA defines a strictly larger class of random variables (for detail examples, see Zhang (2000a). Consequently, the study of the limit theorems for ANA sequences is of much interest. Some excellent results are available, for example, Zhang (2000b) derived the central limit theorem, Wang and Lu (2006) obtained some inequalities of maximum of partial sums and weak convergence, and Wang and Zhang (2007) established the law of the iterated logarithm.

From the definition of an ANA sequence, we have

Lemma 2.1. *Nondecreasing or nonincreasing functions defined on disjoint subsets of an ANA sequence $\{X_n, n\geq 1\}$ with mixing coefficients $\rho^-(s)$ is also ANA with mixing coefficients not greater than $\rho^-(s)$.*

Wang and Lu (2006) proved the following Rosenthal type inequality.

Lemma 2.2. *For a integer $N\geq 1$, real numbers $p\geq 2$ and $0\leq r<(1/(6p))^{p/2}$, if $\{X_i, i\geq 1\}$ is a sequence of random variables with $\rho^-(N)\leq r$, with $EX_i=0$ and $E|X_i|^p<\infty$ for every $i\geq 1$, then for all $n\geq 1$, there is a positive constant $D=D(p,N,r)$ such that*

$$E\max_{1\leq i\leq n}\left|\sum_{j=1}^{i}X_j\right|^p\leq D\left(\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right).$$

However, the Rosenthal type inequality is not available as yet, for ANA random variables, when $1<p<2$. Inspired by the proof of Theorem 2.1 of Yuan and An (2009), we fill here this void, which is of independent interest and is also the main tool for studying the limit results in the subsequent sections.

Lemma 2.3. *Let $\{X_n, n\geq 1\}$ be an ANA sequence of zero mean random variables with mixing coefficients $\rho^-(s)$, then for all $n\geq N\geq 1$ and $1<p<2$,*

$$E\left(\max_{1\leq i\leq n}|S_i|^p\right)\leq N^{p-1}\left[2^{3-p}p\sum_{i=1}^{n}E|X_i|^p+2(6p)^p(\rho^-(N))^{2(p-1)}\left(\sum_{i=1}^{n}(E|X_i|^p)^{1/p}\right)^p\right]. \tag{3}$$

Proof. Without loss of generality, we can assume that $E|X_i|^p<\infty$ for each $i\geq 1$, for otherwise the right-hand side of (3) is infinity and there is nothing to prove.

First, we assume that $N=1$. Set

$$U_i=\max(X_i, X_i+X_{i+1},\ldots,X_i+\cdots+X_n),\ 1\leq i\leq n.$$

Clearly, $U_i=\max(X_i, X_i+U_{i+1})$. We obtain for $1\leq i\leq n-1$

$$E|U_i|^p\leq E(|X_i|^pI(U_{i+1}\leq 0))+E(|X_i+U_{i+1}|^pI(U_{i+1}>0))\leq 2^{2-p}E|X_i|^p+E|U_{i+1}|^p+pE(X_i|U_{i+1}|^{p-1}I(U_{i+1}>0)).$$

In the second inequality, we have used an elementary inequality

$$|x+y|^p\leq 2^{2-p}|x|^p+|y|^p+px|y|^{p-1}\operatorname{sgn} y$$

for $1<p<2$. It is easy to show that $g(X_{i+1},\ldots,X_n) := |U_{i+1}|^{p-1}I(U_{i+1}>0)\in\mathcal{C}$. Thus, by Lemma 3.1 of Zhang (2000a), we have for $1\leq i\leq n-1$

$$\|U_i\|_p^p\leq 2^{2-p}\|X_i\|_p^p+\|U_{i+1}\|_p^p+6p(\rho^-(1))^{2/q}\|X_i\|_p\|U_{i+1}\|_p^{p/q}.$$

Let

$$\xi_i^p=\begin{cases}2^{2-p}\|X_i\|_p^p+\xi_{i+1}^p+6p(\rho^-(1))^{2/q}\|X_i\|_p\xi_{i+1}^{p/q}, & 1\leq i\leq n-1,\\ 2^{2-p}\|X_n\|_p^p, & i=n.\end{cases}$$

It is easy to show that $\|U_i\|_p \le \xi_i$, and that $\{\xi_i\}$ is nonincreasing, hence

$$\begin{cases} \xi_i^p \le 2^{2-p}\|X_i\|_p^p + \xi_{i+1}^p + 6p(\rho^-(1))^{2/q}\|X_i\|_p \xi_1^{p/q}, & 1 \le i \le n-1, \\ \xi_n^p = 2^{2-p}\|X_n\|_p^p. \end{cases}$$

Substituting sequentially, we conclude that

$$\xi_1^p \le 2^{2-p}\sum_{i=1}^n \|X_i\|_p^p + 6p(\rho^-(1))^{2/q}\xi_1^{p/q}\sum_{i=1}^{n-1}\|X_i\|_p \le 2^{2-p}\sum_{i=1}^n \|X_i\|_p^p + q^{-1}\xi_1^p + p^{-1}\left(6p(\rho^-(1))^{2/q}\sum_{i=1}^{n-1}\|X_i\|_p\right)^p. \tag{4}$$

Here, in the last inequality of (4), we have used an elementary inequality $a^\alpha b^\beta \le \alpha a + \beta b$ for non-negative numbers a, b, α, β with $\alpha+\beta=1$.

Now we are in a good position to complete the proof of (3). From (4), we have

$$\xi_1^p \le 2^{2-p}p\sum_{i=1}^n \|X_i\|_p^p + \left(6p(\rho^-(1))^{2/q}\sum_{i=1}^{n-1}\|X_i\|_p\right)^p.$$

Recall that $\|U_1\|_p \le \xi_1$, and therefore we have

$$E|\max_{1\le i\le n} S_i|^p \le 2^{2-p}p\sum_{i=1}^n \|X_i\|_p^p + (6p)^p(\rho^-(1))^{2(p-1)}\left(\sum_{i=1}^{n-1}\|X_i\|_p\right)^p. \tag{5}$$

By Lemma 2.1, $\{-X_n, n\ge 1\}$ is also an ANA sequence of zero mean random variables with the same mixing coefficients $\rho^-(s)$. In the same way, we have also

$$E|\max_{1\le i\le n}(-S_i)|^p \le 2^{2-p}p\sum_{i=1}^n \|X_i\|_p^p + (6p)^p(\rho^-(1))^{2(p-1)}\left(\sum_{i=1}^{n-1}\|X_i\|_p\right)^p. \tag{6}$$

Clearly,

$$\max_{1\le i\le n}|S_i|^p \le |\max_{1\le i\le n} S_i|^p + |\max_{1\le i\le n}(-S_i)|^p,$$

from which, together with (5) and (6), (3) follows immediately for the case $N=1$.

Let $N\ge 2$ be the integer mentioned in Lemma 2.3 such that $N\le n$. We consider now N sequences random variables $\{Y_{ij}, i\ge 0\}1\le j\le N$, defined by $Y_{ij}=X_{iN+j}$.

Notice that for each $1\le j\le N$, the first interlaced mixing coefficient $\rho_Y^-(1)$ for the sequence $\{Y_{ij}, i\ge 0\}$ is smaller than $\rho^-(N)$.

It is easy to see that

$$\max_{1\le m\le n}|S_m|^p \le N^{p-1}\sum_{j=1}^N \max_{1\le k\le [n/N]}\left|\sum_{i=0}^k X_{iN+j}\right|^p.$$

By (1), which has been proved for this case,

$$\begin{aligned} E\max_{1\le m\le n}|S_m|^p &\le N^{p-1}\sum_{j=1}^N E\max_{1\le k\le [n/N]}\left|\sum_{i=0}^k X_{iN+j}\right|^p \le N^{p-1}\sum_{j=1}^N\left[2^{3-p}p\sum_{i=0}^{[n/N]}|X_{iN+j}|^p + 2(6p)^p(\rho^-(N))^{2(p-1)}\left(\sum_{i=0}^{[n/N]}(E|X_{iN+j}|^p)^{1/p}\right)^p\right] \\ &\le N^{p-1}\left[2^{3-p}p\sum_{i=1}^n |X_i|^p + 2(6p)^p(\rho^-(N))^{2(p-1)}\left(\sum_{i=1}^n (E|X_i|^p)^{1/p}\right)^p\right] \end{aligned}$$

as desired. The proof is complete. □

Lemma 2.3 ought to be compared with the following result, which is due to Zhang (2000a), Lemma 3.3.

Lemma 2.4. *Let $\{X_i, i\ge 1\}$ be a sequence of zero mean random variables with mixing coefficients $\rho^-(s)$, then for any $1<p<2$, there is a positive constant $D=D(p,\rho^-(\cdot))$ such that*

$$E\left|\sum_{i=1}^n X_i\right|^p \le D\sum_{i=1}^n E|X_i|^p$$

for all $n\ge 1$.

Finally, we give a lemma which supplies us with the analytical part in the proofs of theorems in the subsequent sections.

Lemma 2.5. *For sequences $\{a_n, n\ge 1\}$ and $\{b_n, n\ge 1\}$ of non-negative real numbers, if*

$$\sup_{n\ge 1} n^{-1}\sum_{i=1}^n a_i<\infty \quad \text{and} \quad \sum_{n=1}^\infty b_n<\infty,$$

then

$$\sum_{i=1}^n a_ib_i\le\left(\sup_{m\ge 1} m^{-1}\sum_{i=1}^m a_i\right)\sum_{i=1}^n b_i$$

for every $n\ge 1$.

Proof. Let $\{a_i', 1\le i\le n\}$ and $\{b_i', 1\le i\le n\}$ respectively be the rearrangements of $\{a_i, 1\le i\le n\}$ and $\{b_i, 1\le i\le n\}$ satisfying $a_1'\ge a_2'\ge\cdots\ge a_n'$ and $b_1'\ge b_2'\ge\cdots\ge b_n'$. Then $\sum_{i=1}^n a_ib_i\le\sum_{i=1}^n a_i'b_i'$. So without loss of generality, one can assume that $\{a_i, 1\le i\le n\}$ and $\{b_i, 1\le i\le n\}$ are nonincreasing. By applying Remark 3(i) in Landers and Rogge (1997), the rest of the proof can completed if we note the monotonicity of $\{a_n\}$ and $\{b_n\}$. □

3. Residual Cesàro alpha-integrability and L_p-convergence of the maximum of the partial sum

Let $p>1$ and let $h(x)$ be a strictly positive function defined on $(1,+\infty)$. In this section, we discuss L_p-convergence of the form of $n^{-h(p)}\max_{1\le i\le n}|S_i-ES_i|$ for an ANA sequence $\{X_n, n\ge 1\}$ of random variables, provided that $\{|X_n|^p, n\ge 1\}$ is RCI(α) for appropriate.

It ought to be mentioned that there are many papers dealing with the partial sum, but not the maximum of the partial sum. Therefore, to the best of our knowledge, our results presented in this section and next section have not been established previously in the literature.

Our first result is dealing with the case $1<p<2$.

Theorem 3.1. *Let $1<p<2$ and let integer $N\ge 1$. Suppose that $\{X_n, n\ge 1\}$ is an ANA sequence of random variables with mixing coefficients $\rho^-(s)$ such that $\rho^-(N)<(1/(6p))^{p/2}$. If $\{|X_n|^p, n\ge 1\}$ is RCI(α) for some $\alpha\in(0,1/(2-p))$, then*

$$n^{-1}\max_{1\le i\le n}|S_i-ES_i|\to 0 \text{ in } L_p.$$

Proof. Let

$$Y_n=-n^\alpha I(X_n<n^\alpha)+X_nI(|X_n|\le n^\alpha)+n^\alpha I(X_n>n^\alpha),\quad n\ge 1$$

and define, for each $n\ge 1$,

$$Z_n=X_n-Y_n, S_n^{(1)}=\sum_{i=1}^n Y_i,\quad \text{and}\quad S_n^{(2)}=\sum_{i=1}^n Z_i.$$

It is easy to see that $|Y_n|=\min\{|X_n|,n^\alpha\}$, $|Z_n|=(|X_n|-n^\alpha)I(|X_n|>n^\alpha)$ and

$$|Z_n|^p\le(|X_n|^p-n^\alpha)I(|X_n|^p>n^\alpha) \tag{7}$$

for all $p>1$. Note that, for each $n\ge 1$, Y_n and Z_n are monotone transformations of the initial variable X_n. This implies that an ANA assumption is preserved by this construction in view of Lemma 2.1. Precisely, $\{Y_n-EY_n, n\ge 1\}$ and $\{Z_n-EZ_n, n\ge 1\}$ are also ANA sequences of zero mean random variables with mixing coefficients not greater than $\rho^-(s)$.

For our purpose, it suffices to prove

$$n^{-1}\max_{1\le i\le n}|S_i^{(1)}-ES_i^{(1)}|\to 0 \text{ in } L_2 \tag{8}$$

and

$$n^{-1}\max_{1\le i\le n}|S_i^{(2)}-ES_i^{(2)}|\to 0 \text{ in } L_p. \tag{9}$$

Using relation (3) of Lemma 2.3, the Hölder inequality, relation (7) and the second condition of the RCI(α) property (1) of the sequence $\{|X_n|^p, n\ge 1\}$, we obtain

$$\begin{aligned} n^{-p}E\left(\max_{1\le i\le n}|S_i^{(2)}-ES_i^{(2)}|^p\right) &\ll n^{-p}\sum_{i=1}^n E|Z_i-EZ_i|^p+n^{-p}\left(\sum_{i=1}^n(E|Z_i-EZ_i|^p)^{1/p}\right)^p\ll n^{-p}\sum_{i=1}^n E|Z_i-EZ_i|^p+n^{-1}\sum_{i=1}^n E|Z_i-EZ_i|^p\\ &\ll n^{-1}\sum_{i=1}^n E|Z_i|^p\le n^{-1}\sum_{i=1}^n E[(|X_i|^p-i^\alpha)I(|X_i|^p>i^\alpha)]\to 0. \end{aligned}$$

This proves (9). To verify (8), using Lemma 2.2, we have

$$n^{-2}E\left(\max_{1\le i\le n}|S_i^{(1)}-ES_i^{(1)}|^2\right)\ll n^{-2}\sum_{i=1}^n E(Y_i-EY_i)^2\le n^{-2}\sum_{i=1}^n EY_i^2\le n^{-2+(2-p)\alpha}\sum_{i=1}^n E|X_i|^p\le n^{-1+(2-p)\alpha}\cdot\sup_{n\ge 1}\left(n^{-1}\sum_{i=1}^n E|X_i|^p\right).$$

Using the first condition of the RCI(α) property (1) of the sequence $\{|X_n|^p, n\geq 1\}$, the last expression above clearly goes to 0 as $n\to\infty$, because $1<p<2$ and $\alpha<1/(2-p)$, thus completing the proof. □

Remark. Let $1<p<2$ and let integer $N\geq 1$. Suppose that $\{X_n, n\geq 1\}$ is an ANA sequence of random variables with mixing coefficients $\rho^-(s)$. If $\{|X_n|^p, n\geq 1\}$ is RCI(α) for some $\alpha\in(0,1/p)$, then $n^{-1/p}(S_n-ES_n)\to 0$ in L_p.

Compared with Theorem 3.1, this result, whose proof can be completed by using Lemma 2.4, drops the maximum of the partial sum at the price of enlarging $1/n$ into $1/n^{1/p}$.

Next we consider the case $p\geq 2$.

Theorem 3.2. *Let $p\geq 2$ and let integer $N\geq 1$. Suppose that $\{X_n, n\geq 1\}$ is an ANA sequence of random variables with mixing coefficients $\rho^-(s)$ such that $\rho^-(N)<(1/(6p))^{p/2}$. If $\{X_n, n\geq 1\}$ satisfies*

$$\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}E|X_i|^p<\infty,$$

then for any $\delta>1/2$

$$n^{-\delta}\max_{1\leq i\leq n}|S_i-ES_i|\to 0 \text{ in } L_p.$$

Proof. By Lemma 2.2 and the Hölder inequality,

$$E\left(n^{-\delta}\max_{1\leq i\leq n}|S_i-ES_i|\right)^p \ll n^{-p\delta}\sum_{i=1}^{n}E|X_i|^p+n^{-p\delta}\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\leq n^{-p\delta}\sum_{i=1}^{n}E|X_i|^p+n^{-p\delta+(p/2)-1}\sum_{i=1}^{n}(EX_i^2)^{p/2}\ll n^{-p\delta+(p/2)-1}\sum_{i=1}^{n}E|X_i|^p$$
$$\leq n^{-p(\delta-1/2)}\cdot\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}(EX_i^2)^{p/2}\to 0$$

as desired. □

4. Strongly residual Cesàro alpha-integrability and complete convergence of the maximum of the partial sum

A sequence of random variables $\{X_n, n\geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty}P(|X_n-a|>\varepsilon)<\infty.$$

In this case we write $X_n\to a$ completely. This notion was given by Hsu and Robbins (1947). Note that the complete convergence implies the almost sure convergence in view of the Borel–Cantelli Lemma.

The condition of SRCI(α) is a "strong" version of the condition of RCI(α). In this section, we will show that each of the theorems in the previous section has a corresponding "strong" analogue in the sense of complete convergence.

Theorem 4.1. *Let $1<p<2$ and let integer $N\geq 1$. Suppose that $\{X_n, n\geq 1\}$ is an ANA sequence of random variables with mixing coefficients $\rho^-(s)$ such that $\rho^-(N)<(1/(6p))^{p/2}$. If $\{|X_n|^p, n\geq 1\}$ is SRCI(α) for some $\alpha\in(0,1/(2-p))$, then*

$$n^{-1}\max_{1\leq i\leq n}|S_i-ES_i|\to 0 \text{ completely}.$$

Proof. For each $n\geq 1$, let $m=m_n$ be the integer such that $2^{m-1}<n\leq 2^m$. Observe that

$$n^{-1}\max_{1\leq i\leq n}|S_i-ES_i|\leq n^{-1}\max_{1\leq i\leq 2^m}|S_i-ES_i|\leq (2^{m-1})^{-1}\max_{1\leq i\leq 2^m}|S_i-ES_i|=2\cdot 2^{-m}\max_{1\leq i\leq 2^m}|S_i-ES_i|. \tag{10}$$

Hence it suffices to prove

$$2^{-m}\max_{1\leq i\leq 2^m}|S_i-ES_i|\to 0 \text{ completely}.$$

Let Y_n, Z_n, $S_n^{(1)}$ and $S_n^{(2)}$ be defined as in the proof of Theorem 3.1. We first prove that $2^{-m}\max_{1\leq i\leq 2^m}|S_i^{(2)}-ES_i^{(2)}|\to 0$ completely, namely

$$2^{-m}\max_{1\leq i\leq 2^m}\left|\sum_{k=1}^{i}(Z_k-EZ_k)\right|\to 0 \text{ completely}. \tag{11}$$

Using Lemma 2.3, the Hölder inequality, relation (7) and the second condition of the SRCI(α) property (2) of the sequence $\{|X_n|^p\}$, we have

$$\sum_{m=0}^{\infty}E\left(2^{-m}\max_{1\leq i\leq 2^m}\left|\sum_{k=1}^{i}(Z_k-EZ_k)\right|\right)^p \ll \sum_{m=0}^{\infty}2^{-mp}\sum_{i=1}^{2^m}E|Z_i|^p+\sum_{m=0}^{\infty}2^{-mp}\left(\sum_{i=1}^{2^m}(E|Z_i|^p)^{1/p}\right)^p \ll \sum_{m=0}^{\infty}2^{-mp}\left(\sum_{i=1}^{2^m}(E|Z_i|^p)^{1/p}\right)^p$$

$$\le \sum_{m=0}^{\infty} 2^{-m} \sum_{i=1}^{2^m} E|Z_i|^p = \sum_{i=1}^{\infty} E|Z_i|^p \sum_{\{m:2^m \ge i\}} 2^{-m} \le \sum_{i=1}^{\infty} i^{-1} E|Z_i|^p \le \sum_{i=1}^{\infty} i^{-1} E[(|X_i|^p - i^{\alpha}) I(|X_i|^p > i^{\alpha})] < \infty,$$

which yields (11).

Next we show that $2^{-m}\max_{1 \le i \le 2^m} |S_i^{(1)} - ES_i^{(1)}| \to 0$ completely, namely

$$2^{-m} \max_{1 \le i \le 2^m} \left| \sum_{k=1}^{i} (Y_k - EY_k) \right| \to 0 \text{ completely}. \tag{12}$$

By Lemma 2.2 and the Hölder inequality,

$$\sum_{m=0}^{\infty} E\left(2^{-m} \max_{1 \le i \le 2^m} \sum_{k=1}^{i} (Y_k - EY_k)\right)^2 \ll \sum_{m=0}^{\infty} 2^{-2m} \sum_{i=1}^{2^m} EY_i^2 \le \sum_{m=0}^{\infty} 2^{-2m} \sum_{i=1}^{2^m} i^{(2-p)\alpha} E|X_i|^p.$$

In view of the first condition of the SRCI(α) property (2) of the sequence $\{|X_n|^p\}$ and Lemma 2.5, we conclude that

$$\sum_{m=0}^{\infty} E\left(2^{-m} \max_{1 \le i \le 2^m} \sum_{k=1}^{i} (Y_k - EY_k)\right)^2 \ll \sum_{m=0}^{\infty} 2^{-2m} \sum_{i=1}^{2^m} i^{(2-p)\alpha} \le \sum_{i=1}^{\infty} i^{(2-p)\alpha} \sum_{\{m:2^m \ge i\}} 2^{-2m} \le \sum_{i=1}^{\infty} i^{-2+(2-p)\alpha}.$$

The last series above converges since $\alpha \in (0, 1/(2-p))$ implies $-2+(2-p)\alpha < -1$ and therefore (12) holds. This completes the proof. □

For the case $p \ge 2$, we have

Theorem 4.2. *Let $p \ge 2$ and let integer $N \ge 1$. Suppose that $\{X_n, n \ge 1\}$ is an ANA sequence of random variables with mixing coefficients $\rho^-(s)$ such that $\rho^-(N) < (1/(6p))^{p/2}$. If $\{X_n, n \ge 1\}$ satisfies*

$$\sup_{n \ge 1} \frac{1}{n} \sum_{i=1}^{n} E|X_i|^p < \infty,$$

then for any $\delta > 1/2$

$$n^{-\delta} \max_{1 \le i \le n} |S_i - ES_i| \to 0 \text{ completely}.$$

Proof. Let m_n, $n \ge 1$ be defined as in the proof of Theorem 4.1. Proceeding as in the proof of (10), we see that it suffices to show that

$$2^{-m\delta} \max_{1 \le i \le 2^m} |S_i - ES_i| \to 0 \text{ completely}. \tag{13}$$

Indeed, by Lemma 2.2 and the Hölder inequality,

$$\sum_{m=0}^{\infty} E\left(2^{-m\delta} \max_{1 \le i \le 2^m} |S_i - ES_i|\right)^p \ll \sum_{m=0}^{\infty} 2^{-mp\delta} \sum_{i=1}^{2^m} E|X_i|^p + \sum_{m=0}^{\infty} 2^{-mp\delta} \left(\sum_{i=1}^{2^m} EX_i^2\right)^{p/2} \ll \sum_{m=0}^{\infty} 2^{-mp\delta} \left(\sum_{i=1}^{2^m} EX_i^2\right)^{p/2}$$
$$\ll \sum_{m=0}^{\infty} 2^{-mp\delta - m + mp/2} \sum_{i=1}^{2^m} E|X_i|^p \ll \sum_{i=1}^{\infty} E|X_i|^p \sum_{\{m:2^m \ge i\}} 2^{-mp\delta - m + mp/2} \le \sum_{i=1}^{\infty} i^{-p\delta - 1 + p/2} E|X_i|^p.$$

In view of Lemma 2.5,

$$\sum_{i=1}^{\infty} i^{-p\delta - 1 + p/2} E|X_i|^p \le \sup_{n \ge} \frac{1}{n} \sum_{i=1}^{n} E|X_i|^p \cdot \sum_{n=1}^{\infty} n^{-(p\delta + 1 - (p/2))} \ll \sum_{n=1}^{\infty} n^{-(p\delta + 1 - (p/2))}.$$

The last series converges because $p\delta + 1 - p/2 > 1$. Therefore (13) holds. □

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