A NEW FRACTAL DERIVATION

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A new fractal derive is defined, which is very easy for engineering applications to discontinuous problems, two simple examples are given to elucidate to establish governing equations with fractal derive and how to solve such equations, respectively.

Key words: *fractal, fractal derive, fractional derive*

Introduction

Fractional calculus becomes a hot topic in both mathematics and engineering. There are many definitions of fractional derivative. Hereby we write down Jumarie's definition [1]

$$ {}_0D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x (x-\xi)^{n-\alpha} [f(\xi) - f(0)] \mathrm{d}\xi \tag{1} $$

for $x \in [0, 1]$, $n - 1 \le \alpha < n$ and $n \ge 1$. Other definitions can be found in refs. [2-7]. Most fractional derivatives are very complex for engineering applications. Though the fractional equations can be solved by various methods, such as the variational iteration method [2], the homotopy perturbation method [3], and the exp-function method[4], the solution procedure is not easy enough for an engineer to master it. In order to better model an engineering problem in a discontinuous media, a new derivative is much needed.

Fractal derivative

A discontinuous media can be described by fractal dimensions. Chen *et al.* suggested a fractal derivative defined as [8]:

$$ \frac{\mathrm{d}u(x)}{\mathrm{d}x^D} = \lim_{s \to x} \frac{u(x) - u(s)}{x^D - s^D} \tag{2} $$

where D is the order of the fractal derivative.

This definition is much simpler but lack of physical understanding.

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A new fractal derivative

Now we consider a fractal media illustrated in fig.1, and assume the smallest measure is L_0, any discontinuity less than L_0 is ignored, then the distance between two points of A and B in fig.1 can be expressed using fractal geometry [7]. Hereby we introduce a new fractal derivative for engineering application:

$$\frac{\mathrm{D}u(t)}{\mathrm{D}x^{\alpha}} = \lim_{\Delta x \to L_0} \frac{u(A)-u(B)}{\text{The distance between two points}} = \frac{\mathrm{d}u}{\mathrm{d}s} = \lim_{\Delta x \to L_0} \frac{u(A)-u(B)}{kL_0^{\alpha}} \tag{3}$$

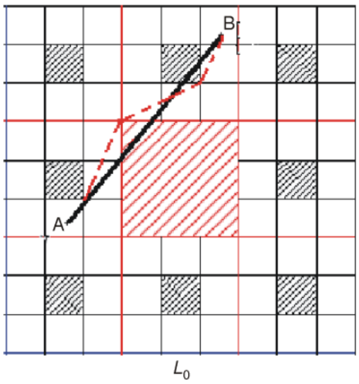

Figure 1. The distance between two points in a discontinuous spacetime

where k is a constant, α is the fractal dimension. The distance between two points in a discontinuous space can be expressed as:

$$\mathrm{d}s = kL_0^{\alpha} \tag{4}$$

Please note in the above definition Δx does tend to zero, but to the smallest measure size, L_0.

Applications

Example 1

As a simple application, we consider the Fourier's law heat conduction, which reads:

$$\frac{\partial q_h}{\partial t} = -c\frac{\mathrm{d}T}{\mathrm{d}n}, \tag{5}$$

where T is the thermal potential, q_h is heat flow.

In the discontinuous media, the Fourier's law can be simply modified as:

$$\frac{\partial q_h}{\partial t} = -c\frac{\mathrm{D}T}{\mathrm{D}n^{\alpha}} \tag{6}$$

where $\mathrm{D}T/\mathrm{D}n^{\alpha}$ is a fractal derivative defined in eq. (3).

Example 2

In this example, we consider a relaxation equation with fractal derivative:

$$\frac{\mathrm{D}u}{\mathrm{D}x^{\alpha}} + \sigma u = 0, \quad 0 < D < 1 \quad u(0) = 1 \tag{7}$$

Equation (7) can be equivalently written in the form:

$$\frac{\mathrm{d}u}{\mathrm{d}s} + \sigma u = 0 \tag{8}$$

Its solution can be easily obtained, which reads:

$$u = e^{-\sigma s} \tag{9}$$

Equation (4) can be approximately expressed in the form:

$$\mathrm{d}s = kL_0^{\alpha-1}\mathrm{d}x \,, \tag{10}$$

or

$$s = kL_0^{\alpha-1}x \tag{11}$$

We, therefore, obtain the solution of eq. (7), which is:

$$u = e^{-\sigma k L_0^{\alpha-1} x} \tag{12}$$

The solution depends upon geometrical parameter (k), measure scale (L_0) and fractal dimension.

Conclusions

The suggested fractal derivative is easy to be used for any discontinuous problems, and equations with fractal derivative can be easily solved used classical calculus.

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