Asymptotic Analysis of Proportionally Fair Scheduling in Rayleigh Fading

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***Abstract*—This paper is concerned with the analysis of proportionally fair scheduling (PFS), and we provide an analytical approximation for the PFS throughput over Rayleigh fading channels.**

Though quite accurate, the ordinary differential equation (ODE) analysis, typically used to analyze the PFS throughput, is highly time-consuming when there are lots of users. On the other hand, due to the intricate interplay among these ODE equations, the ODE analysis generally fails to provide a closed-form approximation for estimating the PFS throughput unless with simplified models such as the linear rate model to characterize channel capacity.

Our aim is to provide a novel framework to evaluate PFS in Rayleigh fading without the above-mentioned limitations. To put our work on a firm base, we use results of stochastic approximation in the analysis and take the Gaussian approximation for capacity modeling for fading channels. Simulations validate this approach and show that our analytic result provides highly accurate estimate of the PFS throughput.

Compared to existing studies, our work advances the state of the art in three ways. First, it goes beyond the linear rate model and applies to the commonly used Shannon rate model. Second, it provides accurate estimate of the PFS throughput without the need for the time-consuming ODE analysis. Third, it provides a unified closed-form expression for estimating the PFS throughput for both the linear rate model and the Shannon rate model. It is interesting to note that our analysis provides the same result as existing studies when assuming the linear rate model. More importantly, our formula is intuitive yet easy to evaluate numerically.

***Index Terms*—Proportionally fair scheduling, Rayleigh fading, ordinary differential equation (ODE) analysis, Gaussian approximation.**

I. Introduction

THROUGHPUT and fairness are the two crucial yet conflicting performance metrics in wireless scheduling. In efforts to deal with the tradeoff between throughput and fairness, the utility-based approach has received significant attention and is widely adopted in resource scheduling [1]–[3]. The objective of a utility-based scheduling algorithm is to maximize the overall utility, where utility represents user satisfaction. Proportional fairness (PF) and max-min fairness (MMF) are the two most used fairness criteria in utility-based scheduling. It is believed that MMF is less efficient than PF [4], [5], a criterion that was introduced by Kelly to communication networks [6] and is the most known utility-based allocation. In this paper, we are concerned with the analysis of proportionally fair scheduling (PFS) in wireless networks.

Since its presence [6], the PFS algorithm has aroused considerable interest [5], [7]–[13]. To date, PFS is the most cited algorithm that provides excellent balance between throughput and fairness, and is currently implemented in 3G networks [14].

Nevertheless, due to the lack of an analytic expression for the PFS performance, the investigation of PFS has usually been performed using simulation. Until recently, one can only see limited results on PFS [7], [8], [10], [11], [15]. These studies make simplifications about signal propagation, fading, or the MAC layer to facilitate analysis. For example, [11] investigates the PFS algorithm with a modified PFS metric instead of the one used in current 3G networks [14]. To simplify the analysis, most research assumes a linear rate model for channel capacity [7], [11], [16]. With such model, channel capacity is proportional to the *signal-to-noise ratio* (SNR). The use of the linear rate model is a reasonable modeling convention [7], [11], [16]; however, when examining throughput performance, it does not seem entirely satisfactory to assume such simplification. In fact, the linear rate model is valid only for small SNR [16], and could be fairly inaccurate in typical fading environments.

To the best of our knowledge, the most useful tool in analyzing the PFS throughput over fading channels is the *ordinary differential equation* (ODE) analysis [7]. The ODE analysis uses standard results from stochastic approximation theory [17] and reveals that the throughput of PFS converges weakly to the unique equilibrium solution of a mean ODE of the related PFS problem. Though quite accurate in quantifying the PFS throughput under general rate model, the ODE analysis involves solving N ODE equations (see Section II for details) if there are N users in the networks. When there are lots of users, the ODE analysis will be very time-consuming, and only become applicable for off-line processing. This rules out the possibility of using the ODE analysis in on-line algorithms for cross-layer design. On the other hand, the ODE analysis fails to provide a closed-form expression for estimating the PFS throughput unless with simplified rate models. For example, [7] conducts the ODE analysis for PFS over Rayleigh fading channels, and only provides the analytic formula for the case of the linear rate model. Apart from this, no other closed-form

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solution has been reported in the literature. This hinders the applicability of the ODE analysis, and thus calls for further study on PFS, including its characteristics, fundamental cause, and attainable performance, in a realistic fading environment.

Our objective is to develop a framework to analyze PFS in Rayleigh fading channels, without the aforementioned limitations. We expect that our analysis would help deeper understanding of the PFS problem, apply to more general fading cases, and serve as a theoretical base for PFS-related studies.

To put our work on a solid base, we use results of the ODE analysis and stochastic approximation [7], [17]. By using the Gaussian approximation [18], [19] for capacity modeling, we derive a novel mathematical tool to quickly estimate the PFS throughput in Rayleigh fading environments. In particular, we provide closed-form expressions for evaluating the PFS throughput, without the need for the time-consuming ODE analysis.

A preliminary version of this paper has appeared in the conference proceedings [20]. The drawbacks are that we required the tracking parameter ϵ to be infinitely small and we assumed (average throughput)/(average data rate) to be constant for all users. In this paper we do not have these limitations.

We should point out that, like most prior work [7], [8], [10], [11], [15], this paper focuses on the throughput allocation of PFS for single-channel systems. For PF in multi-channel systems, one can refer to [12], [13] where Liew and Zhang analyze airtime usage among users in multi-channel scenarios and find that PF achieves equal "equivalent" airtime allocation in such systems [12]. In the future we will consider PFS in multi-channel systems, and would like to investigate this interesting property in our research.

The rest of the paper is as follows. In Section II, we review the related work. In Section III, we introduce the notation and conventions, outline the PFS algorithm, and then provide lemmas to proceed to Section IV, where we present the analytic framework together with closed-form approximation to quantify the PFS throughput. In Section V, we present simulation results to corroborate our models in Rayleigh fading environments. In Section VI, we conclude the paper. To keep the flow of exposition, all related proofs are presented in the Appendices.

II. Related Work

The concept of PF is not new. Decades ago, the model of utility maximization together with logarithmic utility in consumption of good were introduced in Microeconomics [21] for price policy. Kelly [6] applied such methodology in communication networks and derived a scheduling algorithm called PFS that maximizes the sum of user utility, wherein utility $U(\mu) = \ln(\mu)$ is a concave function of throughput μ and provides a good representative for user satisfaction of elastic services. Since then, for its excellent tradeoff between throughput and fairness, PFS has gained considerable attention in the literature and is currently implemented in 3G networks [14].

To evaluate the PFS performance for users with different fading characteristics, Holtzman [16] conducts the asymptotic analysis of PFS, with a result that the user class with more fading variability gets more throughput. In [7], Kushner and Whiting investigated the convergency of the algorithm. They state that the limiting behavior of the throughput converges to the solution of an ordinary differential equation, and find that the limit throughput is proportional to the average instantaneous rate for Rayleigh fading by assuming the instantaneous rate is proportional to the SNR which is *i.i.d.* for all users. Also, Borst [8] presents results on PFS for the scenario where the relative rate fluctuations are statistically identical, stating that each user would receive the same amount of time slots.

We note that most existing work assumes some kind of linear relationship between channel capacity and SNR and we call this a linear rate model. For example, [8] assumes that the data rate of user i with time-average rate C_j is distributed as $R_j = C_j Y_j Z$, where $Y_1, Y_2, \ldots$ are *i.i.d* copies, and Z represents a possible correlation component with unit mean, and the exponentially smoothed throughput of user j scales linearly with the time-average rate C_j, *i.e.,* $W_j = C_j V_j$, where the random variables $V_1, V_2, \ldots$ are identically distributed.

Assuming *i.i.d.* SNR and using the rate model $R = \beta \cdot SNR$ where β is a constant, [22] provides the mean throughput per time slot of a user in an N-user cellular network,

$$T_S \cdot E[\mu] = \beta \cdot T_S / N \sum_{k=1}^{N} \frac{1}{k}. \tag{1}$$

where T_S is slot duration and μ denotes throughput.

With the linear rate model, analytic result similar to (1) was independently obtained in [7]. While this rate model has its value in making the analysis tractable in prior studies, it is not always satisfactory to assume such simplified rate model when examining the PFS performance. For example, the linear rate model is valid only in small SNR region and could be fairly inaccurate for real fading scenarios [16]. Apparently, new analysis with more accurate models such as the Shannon rate model is favorable.

Among theoretical work on PFS, the ODE analysis [7] has its unique merit in that it not only applies to the linear rate model but applies to other rate models as well, which in turn provides accurate estimate of the PFS throughput in real fading scenarios. In [7], Kushner and Whiting proved that, given an N-user cellular network, under any initial condition the PFS throughputs of users converge weakly to the set of limit points of the solution of a particular set of N ODEs. Theoretically, by solving these N ODEs one can estimate the PFS throughput with high accuracy. In practice, the ODE analysis is quite time-consuming (if not time-prohibited) in typical cellular network configurations where $N > 5$, especially because the ODEs involved are nonlinear and interplay with each other in an intricate manner. Hence, this method is only applicable for off-line PFS analysis. In general, the ODE analysis fails to produce analytical expressions for the PFS throughput, unless with simplified rate models. As a result, [7] only provides for the linear rate model a closed-form result (similar to (1)) for the PFS throughput. At the time of developing this paper, there is no further work on the ODE analysis for rate models other than the linear one.

In the next section, we first provide the necessary background before we go into the detailed analysis.

III. PFS Algorithm and Lemmas

We begin with the notation and assumptions used throughout the paper.

A. Notation and Conventions

For an N-user single-channel cellular network, we consider the problem where these N users want to transmit data to the BS. The rates of transmission are randomly varying due to channel fluctuations. The channel behavior is stationary and ergodic [23]; this translates into the standard assumption in the literature that each user has infinite backlog of data to transmit. We use $E[\cdot]$ and σ to denote the statistical average and standard deviation. Let R_j be the instantaneous data rate of user j and μ_j be its throughput (refer to Section IV for the formal definition). For each user j, we assume that its instantaneous data rate R_j is an independently distributed, stationary random variable with mean $E[R_j]$ and standard deviation σ_{R_j}, and its throughput μ_j is first-order wide-sense stationary with mean $E[\mu_j]$.

We consider time-division-multiple-access (TDMA) networks where time is divided into small scheduling intervals called slots and the network resource is shared amongst users via disjoint time slots. The end of slot t is called time t. In next time slot $t+1$, the instantaneous data rate of user j will be $R_j[t+1]$. Its throughput up to time t is denoted by $\mu_j[t]$. Like most prior studies, we adopt an independent Rayleigh flat fading model in the analysis: each user experiences independent Rayleigh fading, and the channel coefficient keeps constant during a slot but varies from slot to slot and from user to user. Unlike other studies on PFS, we use the well-known Shannon formula instead of the linear rate model to predict data rate (in bps/Hz), *i.e.,* $R_j = \log_2(1 + SNR_j)$ where SNR_j is user j's SNR determined via measurements based on a pilot signal. Since the time between measurement and prediction is short, fairly accurate predictions is possible.

B. Preliminaries and Lemmas

We first outline the criteria of the PFS algorithm.

Consider an N-user single-channel cellular network in an independent Rayleigh fading scenario. At each slot, the BS schedules one user for data transmission in a TDMA fashion. The selection of a user to schedule is based on a balance between the current possible rates and fairness. According to the PFS algorithm used in current 3G networks [14], PFS [6] performs this by comparing the ratio of the instantaneous data rate for each user to its throughput, which is defined as the preference metric $M_j[t+1] = R_j[t+1]/\mu_j[t]$. The user with the maximum preference metric is selected for transmission, *i.e.,* the PFS algorithm schedules at next slot the user i that maximizes in

$$\arg\max_{j \le N} \{R_j[t+1]/\mu_j[t]\}. \tag{2}$$

It is known that the PFS algorithm above maximizes the overall utility $\sum_i U(\mu_i)$ where $U(\mu_i) = \ln(\mu_i)$ is the utility function defined for elastic flows [6]. This property makes PFS very attractive and has indeed spurred the studies on network utility maximization (NUM) [1]–[3].

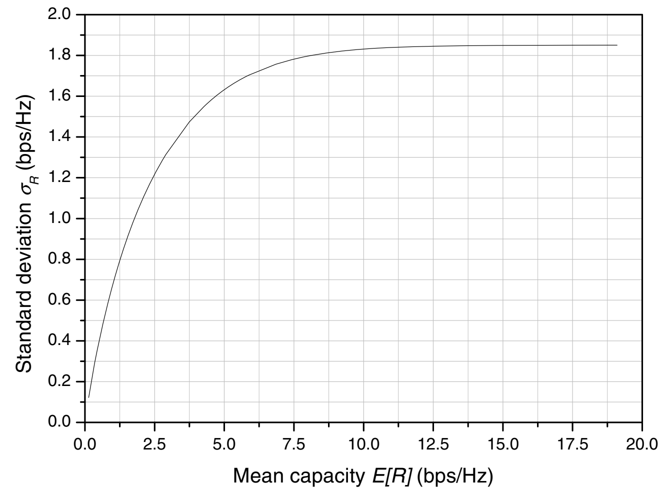

Fig. 1. σ_R vs. $E[R]$: Rayleigh fading

Next, we provide some lemmas to proceed to the analysis of PFS.

In [6], Kelly provided the following formal definition of PF and we should stick to this definition in the analysis.

Definition 1 *A vector of throughputs $x = (x_s, s \in S)$ is proportionally fair if it is feasible and if for any other feasible vector x^*, the aggregate of proportional changes is zero or negative:*

$$\sum_{s \in S} (x_s^* - x_s)/x_s \le 0. \tag{3}$$

Since the PFS preference metric is directly related to instantaneous data rate R, we investigate the characteristics of R and find that

Lemma 1 *For a Rayleigh fading channel, the standard deviation of instantaneous data rate, σ_{R_j}, is monotonically increasing, concave with regard to the statistical average $E[R_j]$.*

Proof: Refer to Appendix A. ■

For reference, we plot in Fig. 1 σ_R v.s. $E[R]$, which clearly justifies the above lemma.

We now investigate the impacts of both $E[R]$ and σ_R on average throughput $E[\mu]$. By Lemma 1, σ_R *w.r.t.* $E[R]$ is monotonically increasing concave in Rayleigh fading. Using the concavity of σ_R together with Definition 1, we have the following inequality

Lemma 2 *In a Rayleigh fading network, given $E[R_j] \le E[R_i]$ for two users i, j, we have $\sigma_{R_i}/\sigma_{R_j} \le E[\mu_i]/E[\mu_j] \le E[R_i]/E[R_j]$, under the proportional fairness criterion* (3)

Proof: Refer to Appendix B. ■

Remark 1 *According to Lemma 1, the standard deviation of channel capacity is an increasing function of the statistical average. Let say initially users i and j have the same channel quality, resulting in the same average throughput for both users. Now we improve the channel quality of user i so that it is slightly better than that of user j,* i.e., *the average* *channel capacity of user i increases from $E[R_i] = E[R_j]$ to $E[R_i] = E[R_j] + \Delta E[R_j]$, and the standard deviation increases from $\sigma_{R_i} = \sigma_{R_j}$ to $\sigma_{R_i} = \sigma_{R_j} + \Delta\sigma_{R_j}$, respectively. Obviously user i has slightly higher average throughput,* i.e., $E[\mu_{R_i}] = E[\mu_{R_j}] + \Delta E[\mu_{R_j}]$. *Using Lemma 2 we then have* $\Delta\sigma_{R_j}/\sigma_{R_j} \leq \Delta E[\mu_{R_j}]/E[\mu_{R_j}] \leq \Delta E[R_j]/E[R_j]$. *In other words, with proportional fairness, the relative increase in the average throughput of a user is bounded by the relative increases in the standard deviation and average of its link capacity.*

For stationary R_i, Kushner and Whiting [7] have provided the following two lemmas for PFS,

Lemma 3 *Assume the PFS algorithm* (2). *For any initial condition, user throughput $\mu_i[t]$ $(1 \leq i \leq N)$ converges weakly to the set of limit points of the solution of the ODE*

$$\dot{\theta}_i = \bar{h}_i(\theta) - \theta_i, 1 \leq i \leq N. \tag{4}$$

where $\bar{h}_i(\theta)$ is user i's average data rate conditional on the event $R_i/\theta_i > R_j/\theta_j, \forall j \neq i$

$$\bar{h}_i(\theta) = E[R_i | R_i/\theta_i > R_j/\theta_j, \forall j \neq i, 1 \leq j \leq N]. \tag{5}$$

Lemma 4 *Assume the PFS algorithm* (2). *The limit point of* (4)*, denoted as $\bar{\theta}_i \triangleq \theta_i(\infty)$, is unique, irrespective of the initial condition, and equals average throughput $E[\mu_i]$. So the process $\mu_i[t]$ converges to $E[\mu_i]$ as $t \to \infty$.*

These two lemmas show that $\mu_i[t]$ weakly converges to a unique asymptotically stable limit point $\bar{\theta}_i = E[\mu_i]$ of the ODE. Obviously, given an initial condition $\{\theta_i(0), i = 1, 2, \ldots, N\}$, by solving (4) one could obtain $\theta_i(1), \theta_i(2), \ldots$, and the average throughput of user i is simply $E[\mu_i] = \bar{\theta}_i = \theta_i(\infty)$.

Remark 2 *For analytical tractability, most analytic work investigates the PFS problem under the linear rate model, which results in inaccurate estimate of $E[\mu]$ in real fading scenarios. By Lemmas 3 and 4, theoretically one can apply the ODE analysis to obtain the average PFS throughput $E[\mu_i]$ with high accuracy for any rate model. However, the ODE analysis requires solving N ODEs. Moreover, the nonlinear terms $\{\bar{h}_i(\theta), 1 \leq i \leq N\}$ in these ODEs depend on all N users and their expressions are typically not easily (if not impossible) obtained for a given rate model. Accordingly, Kushner and Whiting in [7] provided for mean throughput $E[\mu_i]$ the analytic expression (similar to* (1)*) only for the linear rate model wherein $\bar{h}_i(\theta)$ can be explicitly evaluated.*

Regarding the problems of the ODE analysis, in analyzing PFS we do not try to find the explicit solution to the ODEs (4) for the Shannon rate model. Instead, we use the above findings of PFS (*i.e.,* convergence property of througput μ) and results from stochastic approximation, together with advances in rate modeling for fading channels, to derive analytic expressions for $E[\mu]$ estimate under the Shannon rate model in Rayleigh fading environments. To end this section, we present the last lemma we use in the analysis

Lemma 5 *Let $Y_k(x)$ be a non-negative, monotonically non-decreasing function of x $(k = 1, 2, \ldots, N)$. If 1). $a_i, a_j, b_i, b_j, c_i, c_j$ are all positive $(\forall i, j = 1, 2, \ldots, N)$, and 2). $c_i/c_j \leq b_i/b_j \leq a_i/a_j$, $\forall a_i \geq a_j$, then $\forall x \geq 0$ and $\forall j = 1, 2, \ldots, N$, we have*

$$\prod_{\forall i \neq j} Y_i\left(\frac{b_i}{b_j} \cdot x\right) \leq \prod_{\forall a_i \geq a_j \wedge i \neq j} Y_i\left(\frac{a_i}{a_j} \cdot x\right) \prod_{\forall a_i < a_j \wedge i \neq j} Y_i\left(\frac{c_i}{c_j} \cdot x\right). \tag{6}$$

$$\prod_{\forall i \neq j} Y_i\left(\frac{b_i}{b_j} \cdot x\right) \geq \prod_{\forall a_i \geq a_j \wedge i \neq j} Y_i\left(\frac{c_i}{c_j} \cdot x\right) \prod_{\forall a_i < a_j \wedge i \neq j} Y_i\left(\frac{a_i}{a_j} \cdot x\right). \tag{7}$$

Proof: Refer to Appendix C. ■

IV. PFS: Asymptotic Analysis

We start with a definition of throughput. One general definition of the throughput for user j up to time t is the sample average

$$\mu_j[t] = \sum_{m=1}^{t} \frac{R_j[m] \cdot I_j[m]}{t}. \tag{8}$$

where $I_j[t+1]$ is the indicator function of the event that user j is scheduled to transmit in slot $t+1$, *i.e.,* $I_j[t+1] = 1$ if user j is scheduled at time $t+1$ and is zero otherwise

Let $\epsilon = 1/(t+1)$, (8) can be written in the recursive form

$$\mu_j[t+1] = \mu_j[t] + \epsilon \cdot (I_j[t+1] \cdot R_j[t+1] - \mu_j[t]). \tag{9}$$

Owing to the boundedness of the $R_j[t]$, the throughput according to (9) is bounded. Alternative definitions of throughput are also possible, which allow the use of values other than $1/(t+1)$ in the recursive representation.

The value of ϵ is chosen to balance the needs of estimating throughput with the ability to track changes in the channel characteristics. In general, ϵ should be small enough to provide an acceptable measure of the throughput. Since (9) is of the stochastic approximation form, according to stochastic approximation theory [17], we have: when the tracking parameter ϵ in (9) is small and constant, the path converges to the solution to a deterministic ordinary differential equation. On the other hand, according to [7], one would not usually want to use (9) with the variable step size $\epsilon = 1/(t+1)$ since a few bad values of the noise in the early stages can mess up the behavior of the sample path for a long time to come, and such robustness considerations require a larger discounting of past values than $1/(t+1)$ provides. Regarding these, the PFS algorithm in current 3G networks [14] uses in (9) a constant step size $\epsilon = 1/k$, where k is typically greater than 50 to provide an acceptable measure of the throughput. With (9), the throughput of user j is then updated by

$$\mu_j[t+1] = \left(1 - \frac{1}{k}\right) \mu_j[t] + I_j[t+1] \cdot \frac{R_j[t+1]}{k}. \tag{10}$$

For first-order wide-sense stationary μ_j, applying *Bayes'* theorem in (10), we have

$$\begin{aligned}&E\left[\mu_j[t]\right]=E\left[R_j[t{+}1]|I_j[t{+}1]{=}1\right]Pr\left(I_j[t{+}1]{=}1\right)\\&=Pr\left(I_j[t{+}1]{=}1\right)\cdot\int_0^\infty x\cdot f_{R_j}\left(x|I_j[t{+}1]{=}1\right)dx\\&=\int_0^\infty x\cdot f_{R_j}(x)\cdot Pr\left(I_j[t{+}1]=1|R_j[t{+}1]=x\right)dx.\end{aligned}\tag{11}$$

where $Pr(I_j[t+1]=1)$ is the probability that j will be scheduled in slot $t{+}1$, $Pr(I_j[t{+}1]{=}1|R_j[t{+}1]{=}x)$ is the conditional probability with respect to the event $R_j[t{+}1]{=}x$, and $f_{R_j}(\cdot)$ is the probability density function (pdf) of R_j. According to the PFS criterion (2), the conditional probability is given by

$$\begin{aligned}&Pr\left(I_j[t{+}1]{=}1|R_j[t{+}1]{=}x\right)\\&=Pr\left(\frac{x}{\mu_j[t]}>\frac{R_i[t{+}1]}{\mu_i[t]},\forall i{\neq}j,1{\leq}i{\leq}N\right).\end{aligned}\tag{12}$$

Let "$\rightarrow$" denote weak convergence. Slutsky's theorem [24] tells that $X_i\cdot Y_i\rightarrow X_i\cdot C$ and $X_i/Y_i\rightarrow X_i/C$, if sequences $X_i\rightarrow X$ (X is a random variable) and $Y_i\rightarrow C$ (C is a constant and $C\neq 0$). According to Lemma 4, $\mu_i[t]\rightarrow E[\mu_i]$ and $\mu_j[t]\rightarrow E[\mu_j]$ as $t\rightarrow\infty$. Since both $E[\mu_i]$ and $E[\mu_j]$ are constants, with Slutsky's theorem we have $R_i[t{+}1]\cdot\mu_j[t]/\mu_i[t]\rightarrow R_i\cdot E[\mu_j]/E[\mu_i]$ as $t\rightarrow\infty$. With the definition of weak convergence, we have

$$\begin{aligned}&\lim_{t\rightarrow\infty}Pr\left(\frac{R_i[t{+}1]\cdot\mu_j[t]}{\mu_i[t]}<x,\forall i{\neq}j,1{\leq}i{\leq}N\right)\\&=Pr\left(\frac{R_i\cdot E[\mu_j]}{E[\mu_i]}<x,\forall i{\neq}j,1{\leq}i{\leq}N\right).\end{aligned}\tag{13}$$

Taking the limit on both sides of (12) and using (13), we have

$$\begin{aligned}&\lim_{t\rightarrow\infty}Pr\left(I_j[t{+}1]{=}1|R_j[t{+}1]{=}x\right)\\&=Pr\left(\frac{R_i\cdot E[\mu_j]}{E[\mu_i]}<x,\forall i{\neq}j,1{\leq}i{\leq}N\right).\end{aligned}\tag{14}$$

Since R_i and R_j ($\forall i\neq j$) are independently distributed random variables, (14) can be rewritten as

$$\begin{aligned}&\lim_{t\rightarrow\infty}Pr\left(I_j[t{+}1]{=}1|R_j[t{+}1]{=}x\right)\\&=\prod_{\forall i\neq j}F_{R_i}\left(E[\mu_i]\cdot x/E[\mu_j]\right).\end{aligned}\tag{15}$$

where $F_{R_i}(\cdot)$ is the cumulative distribution function (cdf) of data rate R_i.

Combining (11) and (15), we have

$$\begin{aligned}E[\mu_j]&=\lim_{t\rightarrow\infty}E[\mu_j[t]]\\&=\int_0^\infty xf_{R_j}(x)\prod_{\forall i\neq j}F_{R_i}\left(E[\mu_i]\cdot x/E[\mu_j]\right)dx.\end{aligned}\tag{16}$$

To obtain an explicit expression for average throughput $E[\mu_j]$, we need to know the pdf and cdf of data rate R_j in (16). Researchers [18], [19], [25], [26] have revealed that R can be modeled by a normal distribution in various fading scenarios. This is called the Gaussian approximation for channel capacity. While most work on the Gaussian approximation is for multiple-input-multiple-output (MIMO) links, Smith *et al* [19] points out that, the Gaussian approximation is quite respectable even for single-input-single-output (SISO), Rayleigh fading links.

Using (16), Definition 1 and Lemmas 1-5 together with the Gaussian approximation for link capacity in Rayleigh fading channels, we are able to provide our main results in the following

Theorem 1 *With the Shannon rate model, for user j ($\forall j=1,2,\ldots,N$) in an independent Rayleigh fading N-user cellular network, its average PFS throughput is estimated by*

$$E[\mu_j]{=}\frac{E[R_j]}{N}{+}\sigma_{R_j}\cdot\int_{-m_j}^\infty y\cdot\rho(y)\cdot\phi(y)^{N-1}dy.\tag{17}$$

where $m_j=E[R_j]/\sigma_{R_j}$, $\rho(\cdot)$ and $\phi(\cdot)$ are the pdf and cdf of zero mean, unit variance standard normal distribution, respectively. $E[R_j]$ and σ_{R_j} are the mean and standard deviation of R_j

$$E[R_j]=\int_0^\infty\log_2\left(1+\overline{SNR}_j\cdot\lambda\right)\times e^{-\lambda}d\lambda.\tag{18}$$

$$\sigma^2_{R_j}=\int_0^\infty\left(\log_2\left(1{+}\overline{SNR}_j\cdot\lambda\right)\right)^2\times e^{-\lambda}d\lambda-\left(E[R_j]\right)^2.\tag{19}$$

where $\overline{SNR}_j$ is the average SNR of user j.

Proof: Refer to Appendix D. ∎

From (18) and (19), m_j depends solely on $\overline{SNR}_j$. Define $A(N,\overline{SNR})$ the PFS gain under average SNR,

$$A(N,\overline{SNR}){=}N\cdot\int_{-m_j}^\infty y\cdot\rho(y)\cdot\phi(y)^{N-1}dy.\tag{20}$$

We can then rewrite (17) in the form

$$E[\mu_j]{=}\frac{1}{N}\cdot\left\{E[R_j]{+}\sigma_{R_j}\cdot A(N,\overline{SNR})\right\}.\tag{21}$$

With (18) and (19), we plot in Fig. 1 σ_{R_j} *w.r.t.* $E[R_j]$, which justifies that $m_j\rightarrow\infty$ when $\overline{SNR}_j\rightarrow\infty$. We remove m_j in (20) by letting $\overline{SNR}_j\rightarrow\infty$ and define $A(N)$ the PFS gain under infinite large average SNR,

$$A(N){=}A(N,\infty){=}\ N\cdot\int_{-\infty}^\infty y\cdot\rho(y)\cdot\phi(y)^{N-1}dy.\tag{22}$$

Using (20) and (22), we plot in Fig. 2 $A(N)$ and $A(N,\overline{SNR})$ with respect to network size ($N=1\sim 50$) for various $\overline{SNR}$.

It is clearly that $A(N)$ and $A(N,\overline{SNR})$ are indistinguishable for any $\overline{SNR}$ when $N>5$. We thus have

Corollary 1 *With the Shannon rate model, for user j ($\forall j=1,2,\ldots,N$) in an independent Rayleigh fading N-user cellular network, its average PFS throughput is estimated by*

$$E[\mu_j]{=}\frac{1}{N}\cdot\left\{E[R_j]{+}\sigma_{R_j}\cdot A(N)\right\}.\tag{23}$$

where $E[R_j]$, σ_{R_j} and $A(N)$ are given by (18)*,* (19)*, and* (22)*, respectively.*

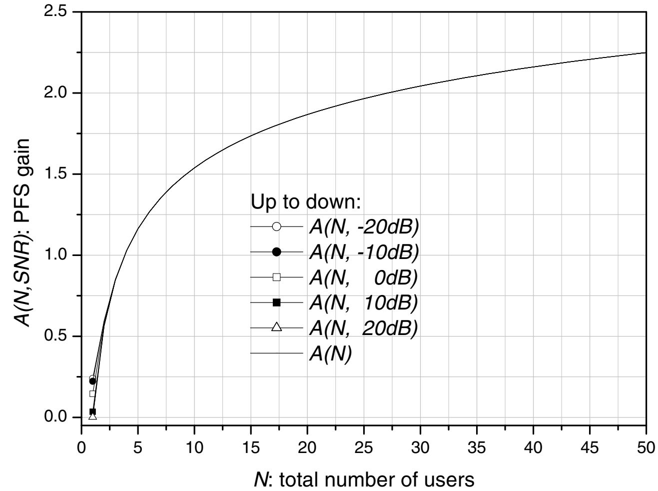

Fig. 2. PFS gain vs. network size

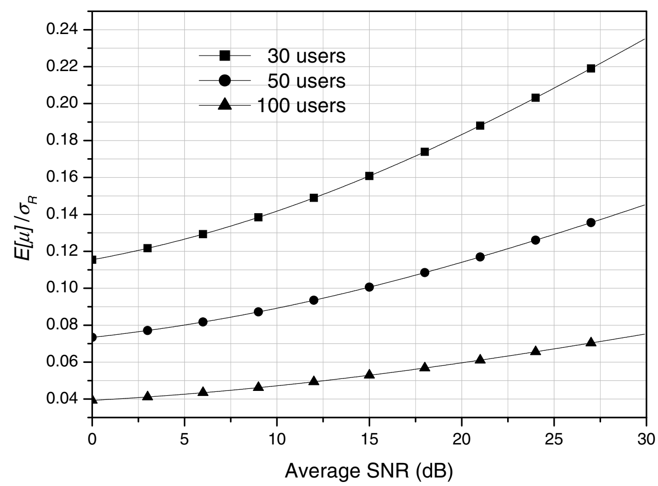

Fig. 3. $E[\mu]/\sigma_R$ vs. $\overline{SNR}$

Remark 3 *By* (23)*, we have*

$$\frac{E[\mu_j]}{\sigma_{R_j}} = \frac{1}{N}\cdot\{m_j + A(N)\}\,, \frac{E[\mu_j]}{E[R_j]} = \frac{1}{N}\cdot\left\{1+\frac{1}{m_j}\cdot A(N)\right\}. \tag{24}$$

Since $E[R]$ is a monotonically increasing function of $\overline{SNR}$. From Figs. 3 and 4, one can easily verify that $E[\mu]/\sigma_R$ w.r.t. *$E[R]$ is monotonically increasing, and $E[\mu]/E[R]$* w.r.t. *$E[R]$ is monotonically decreasing. This agrees with the statements of Lemma 2.*

Remark 4 *Though we consider Rayleigh fading channels in the analysis, we would like to point out that our results may also apply to other fading environments. Indeed, in our analysis, we rely on the following two assumptions: 1). the Gaussian approximation of link capacity in a fading channel, and 2). the increasing concavity of σ_R* w.r.t *$E[R]$.*

Researchers have revealed that the Gaussian approximation holds for various scenarios such as Rayleigh flat fading [19], Rayleigh frequency-selective fading [26], Rayleigh semicorrelated flat fading [18], [25], and Rician fading [18] etc. *Specifically, for the Rician fading case, the SNR of user j is a noncentral chi-square distribution [27] with two degrees and a noncentrality parameter ν^2, where ν is the ratio of signal strength in dominant component over the scattered one. The average and standard deviation of user j's capacity are then given by*

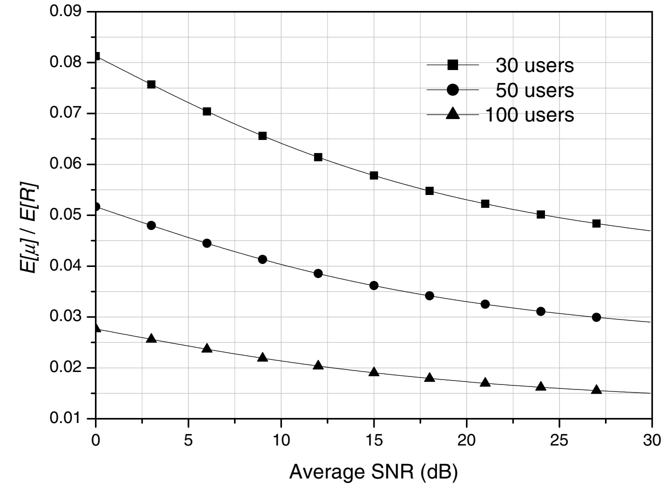

Fig. 4. $E[\mu]/E[R]$ vs. $\overline{SNR}$

$$E[R_j] = \int_0^\infty I_0\left(\nu\sqrt{\lambda}\right)\cdot\frac{e^{-\left(\lambda+\nu^2\right)/2}}{2}\cdot\log_2\left(1+\frac{\overline{SNR}_j\cdot\lambda}{2+\nu^2}\right)d\lambda. \tag{25}$$

$$\sigma^2_{R_j} = \int_0^\infty\left(\log_2\left(1+\frac{\overline{SNR}_j\times\lambda}{2+\nu^2}\right)\right)^2 d\lambda \times I_0\left(\nu\sqrt{\lambda}\right)\cdot\frac{e^{-\left(\lambda+\nu^2\right)/2}}{2} - \left(E[R_j]\right)^2. \tag{26}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind.

Using (25) *and* (26)*, one can verify the increasing concavity of σ_R* w.r.t *$E[R]$ for Rician fading channels. Since the Gaussian approximation and increasing concavity are both valid for the Rician fading case, our results apply to such scenarios directly. Specifically, Lemma 1, Lemma 2, and Theorem 1 are all true for Rician fading channels, with $E[R_j]$ and σ_{R_j} replaced by* (25) *and* (26)*, respectively. In the future, we would like to investigate the application of our results in other scenarios such as log-normal fading.*

Theorem 1 (or Corollary 1) provides a closed-form approximation for the PFS throughput with the Shannon rate model. Interestingly, our analysis also applies to the linear rate model case. To be specific, we have

Theorem 2 *With the linear rate model $R_j = \beta\cdot SNR_j$, for user j ($\forall j = 1, 2, \ldots, N$) in an independent Rayleigh fading N-user cellular network, its average PFS throughput is estimated by*

$$E[\mu_j] = \frac{E[R_j]}{N}\cdot\sum_{k=1}^{N}\frac{1}{k}. \tag{27}$$

where $E[R_j] = \beta\cdot\overline{SNR}_j$, $\overline{SNR}_j$ is the average SNR of user j.

Proof: Refer to Appendix E. ∎

Remark 5 *We would like to point out that* (27) *provides the same result independently obtained in other studies [7], [22]* *for the linear rate model. The difference is that our analysis only requires Rayleigh fading to be independent, while prior research [7], [22] requires Rayleigh fading to be independent and identically distributed (*i.i.d.*) for all users.*

Since $\sigma_{R_j} = E[R_j]$ in the linear rate model for Rayleigh fading, Theorem 1 and Theorem 2 can merge,

Theorem 3 *For user j ($\forall j = 1, 2, \ldots, N$) in an independent Rayleigh fading N-user cellular network, its average PFS throughput is estimated by*

$$E[\mu_j] = \frac{1}{N} \cdot \left\{ E[R_j] + \sigma_{R_j} \cdot A(N) \right\}. \tag{28}$$

where the PFS gain $A(N)$ is given by

$$A(N) = \begin{cases} N \cdot \int_{-\infty}^{\infty} y \cdot \rho(y) \cdot \phi(y)^{N-1} dy, & \textit{Shannon rate model} \\ \sum_{k=2}^{N} \frac{1}{k}, & \textit{linear rate model} \end{cases}. \tag{29}$$

Theorem 3 is our result for the PFS throughput in Rayleigh fading scenarios, and applies to both the Shannon rate model and the linear rate model. Built upon stochastic approximation theory and recent results on rate modeling, Theorem 3 provides a unified, mathematically elegant solution to evaluate the performance of PFS over Rayleigh fading channels.

Since $E[\mu_{R_j}] = E[R_j]/N$ with simple TDMA (*i.e.,* round-robin scheduling), formula (28) has a very clear physical meaning: the first item in the right-hand side (RHS) represents the average throughput from simple TDMA scheduling, while the second item represents the average throughput from fading variability. This translates into the fact that users with more fading variability (*i.e.,* larger σ_R) get more average throughput, a property first observed in [16] for PFS.

Equation (28) also tells that, given the data rate statistics of a Rayleigh fading channel, the PFS throughput is solely determined by $A(N)$. We plot in Fig. 5 the PFS gain *w.r.t.* network size for both rate models. We can see that, while both rate models can be used in the analysis when $N \leq 5$, one should stick to the Shannon rate model for $N > 5$ wherein the linear rate model would overestimate the PFS gain.

Note that, according to (29), numerical calculation is still required to get the PFS gain $A(N)$ for the Shannon rate model. By using logarithmic fitting, a more convenient form can be obtained for $A(N)$ in this case. Specifically, we have the following approximation for $A(N)$ in the Shannon rate model

$$A(N) \approx 0.67464 + 0.39702 \ln(N - 0.83526). \tag{30}$$

Obviously, (30) is of closed form and can be evaluated easily. Fig. 5 also plots $A(N)$ using (30), and it shows that the two $A(N)$ curves from the Shannon rate model and logarithmic fitting are virtually indistinguishable. Indeed, the above fitting provide provides high accuracy, with a relative error of less than 1.1% for $N = 10 \sim 200$.

V. Simulations

In the following, we present two simulation scenarios. We adopt the following setup used in the CDMA 1xEV-DO system [14]: stationary Rayleigh fading with constant and white external noise; 1.67 ms slot duration; the doppler frequency

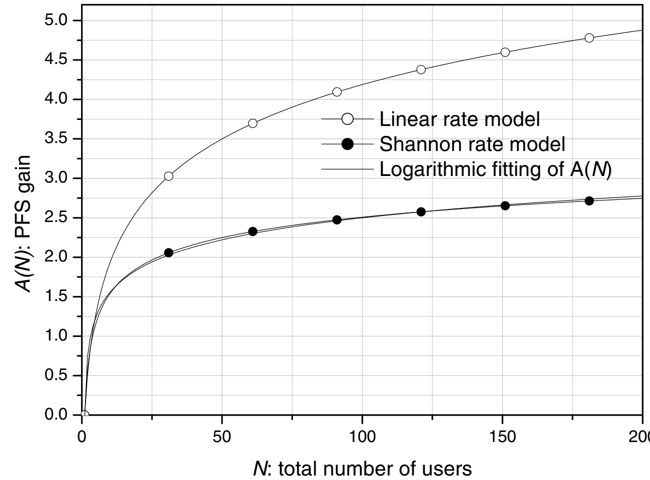

Fig. 5. PFS gain vs. network size

spread is 60 Hz, this corresponds to ~ 7 ms coherence time and will actually simulate a block fading channel, *i.e.,* the channel changes to another implementation every four slots. In the first scenario, we evaluate the convergence of PFS with different tracking parameter ϵ for a two-user network. In the second scenario, we evaluate our analytic results for a more realistic setup ($N = 50$). For both scenarios, the data rate of user j is generated by $R_j = W \cdot \log_2[1 + \overline{SNR}_j \cdot |h_j|^2]$, where W is the bandwidth, and the channel gain h_j for user j is a normalized complex Gaussian random variable.

A. Scenario 1: two-user network

Like [7], we consider a network of two users with their mean rates 572 and 128 bits/slot, respectively. These rates correspond to very low average SNRs of -5.3 dB and -12.5 dB with a bandwidth $W = 1$ MHz.

To track changes in the channel statistics while provide a acceptable measure of throughput, the tracking parameter ϵ should be small enough. In the first example, we set $\epsilon = 0.001$ while in the second example it is 0.00025. The simulation runs for 20000 slots and the results are shown in Fig. 6. We can see that a smaller value of ϵ enables better estimate of throughput, while a larger value of ϵ is able to track changes in the channel characteristics. Fig. 6 also shows that the algorithm converges after about 3000 slots with $\epsilon = 0.001$, and 12000 slots with $\epsilon = 0.00025$. This indicates that the convergence time of the PFS algorithm is in a order of $1/\epsilon$ slots. In both examples, the simulation results match well with our analysis. Note that for the same setup, the user throughputs in equilibrium of the ODE [7] are 429 and 96 bits/slot with the linear rate model. Not surprisingly, these values are very close to our analysis results shown in Fig. 6, since the average SNRs of users are very low in the setting and it does not matter using either the linear rate model or the Shannon rate model.

B. Scenario 2: 50-user network

As shown in Fig. 7, in this scenario 50 users are located in an area of 1.0×1.0 km^2. Users are numbered $n_1 \sim n_{50}$, from up to down and left to right. The BS is located at the center of this area.

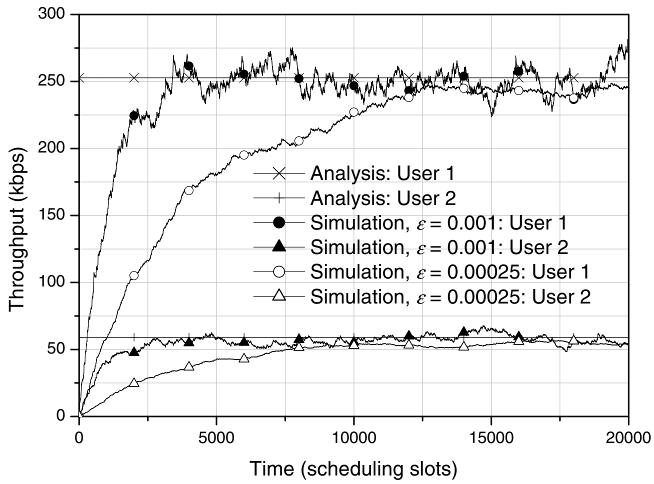

Fig. 6. Convergence of PFS

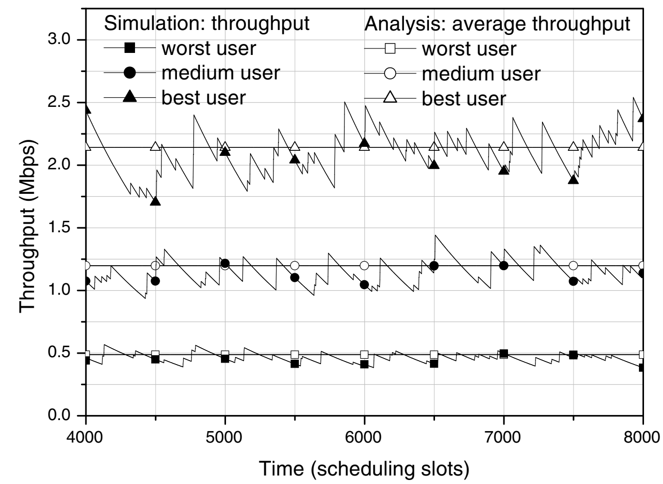

Fig. 8. PFS throughputs of the best, medium and worse users

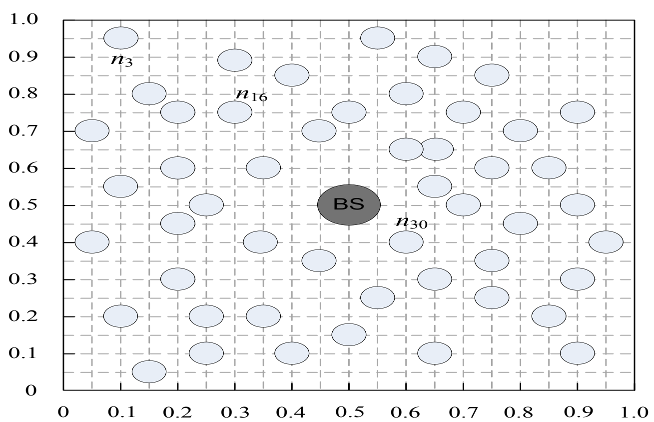

Fig. 7. 50 users in an area of 1.0×1.0 km^2

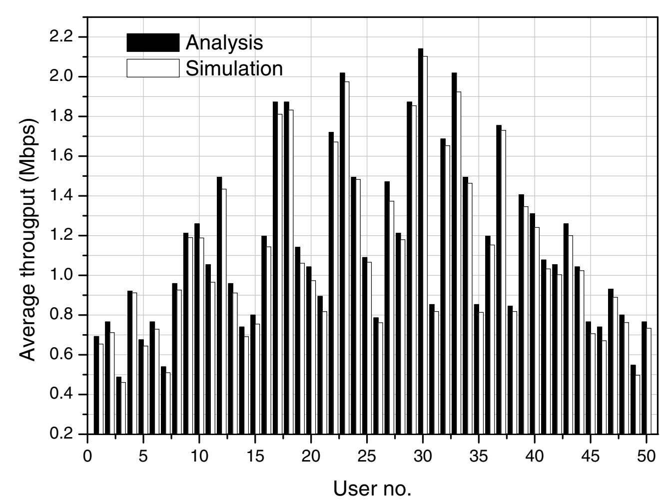

Fig. 9. Average PFS throughput: Simulation and Analysis

We assume that the bandwidth is $W=10$ MHz. The average SNR (in dB) of user j is a distance-related constant determined by the path loss model, $\overline{SNR}_j = \overline{SNR}_{d_0} - 10\cdot\alpha\cdot\log_{10}[d_j/d_0]$, where α is the path exponent, d_j is the distance between user j and the BS, and $\overline{SNR}_{d_0}$ is the average SNR at the reference distance d_0. In the simulation, we have $\epsilon=0.001$, $\alpha=3.5$, $d_0=$ 100 m, and $\overline{SNR}_{d_0}=28$ dB. With the above setting, the average SNR of user varies widely from 0.71 dB to 22.7 dB, which is representative for real fading environments. The simulation runs for 8000 slots.

With the topology shown in Fig. 7, n_3 and n_{30} are the worst and the best users, respectively. Fig. 8 illustrates the PFS throughputs for n_3, n_{30}, and a medium user n_{16}. According to the simulation setting, we can calculate and have $\overline{SNR}_3 = 0.71$ dB, $\overline{SNR}_{16} = 10.3$ dB, and $\overline{SNR}_{30} = 22.7$ dB. By Theorem 3, these average SNRs correspond to theoretical throughputs of 0.49 Mbps, 1.2 Mbps, and 2.14 Mbps, respectively. A direct comparison between simulation and analysis is more clearly illustrated in Fig. 9 where we plot the average throughput for all users. The throughput gap between the simulation and analysis could be further decreased with smaller tracking parameter ϵ, at the price of longer convergence time.

To quantitatively evaluate the accuracy of our analysis, we perform 200 simulation runs with different seeds and for each simulation, we record for the average throughput the difference between the analysis approximation and simulation, which is plotted in Fig. 10 in terms of relative error. It reveals that that the probability that the relative error is less than 10.0% is 98.3%. Another way for measuring this gap is to use interval estimate. From statistics, it is known that 95% confidence interval is $\pm Z_c\cdot\sigma/\sqrt{n}$, where $Z_c = 1.96$ is the z-score associated with the 95% confidence level, σ is normalized the standard deviation of the relative error between the analysis and simulation, and n is the number of simulation ($n=200$ in our case). This 95% confidence interval is used to estimate the gap between our analysis and simulation. Our experiments show that the gap is about $\pm6.3\%$. Obviously both methods validate the adequacy of our analysis approximation.

These results strongly suggest that our asymptotical analysis provides accurate estimate of the PFS throughput over Rayleigh fading channels.

VI. Conclusions

This paper considers the PFS problem and has derived a theoretical framework to facilitate studies on PFS.

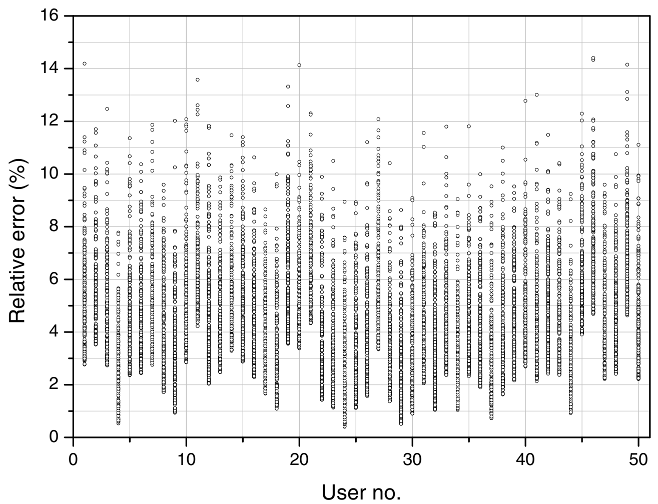

Fig. 10. Relative error between Simulation and Analysis

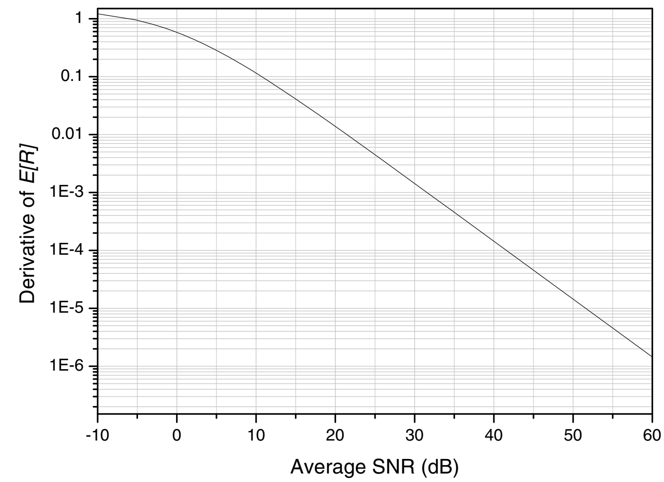

Fig. 11. $\frac{dE[R]}{d\overline{SNR}}$ *w.r.t.* $\overline{SNR}$

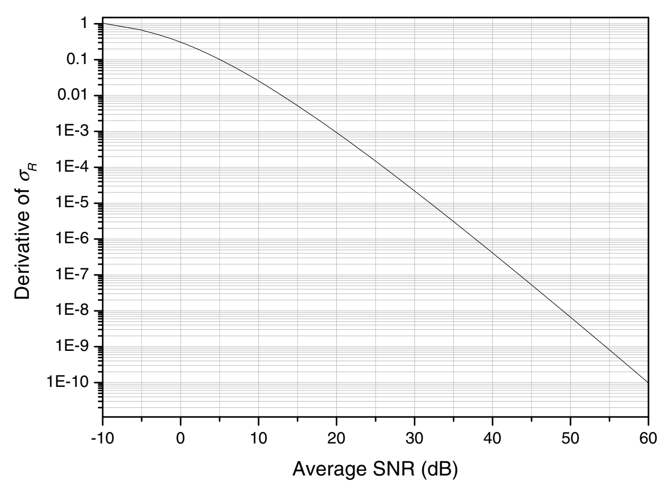

Fig. 12. $\frac{d\sigma_R}{d\overline{SNR}}$ *w.r.t.* $\overline{SNR}$

Built upon stochastic approximation theory, our analysis provides accurate estimate of the PFS throughput in realistic fading channels.

Unlike prior research which either uses the linear rate model or turns to the off-line ODE analysis, the closed-form formula presented here has the practical and theoretical interest: it holds for both the linear rate model and the Shannon rate model, while ruling out the need for highly time-consuming ODE analysis.

Being mathematically graceful, simple and accurate, our results and findings provide guideline and analytical support on system design, simulation-based modeling and performance analysis of the PFS algorithm in the context of cross-layer design.

Though this work is promising, there are still lots of challenges we did not address in this paper. For example, throughout this paper, we focused on PFS in single-channel cellular networks. In addition, channels are assumed to be Rayleigh flat fading (note that our results apply to Rican fading as well). Moreover, the analysis uses continuous data rate and each user has infinite backlog of data for transmission. In real application, user buffers could be empty at some slots and a user might transmit at only one of a discrete set of data rates. In future work, we will explore these issues and would like to extend PFS to the ad-hoc and/or multi-channel networks, in more general fading scenarios.

Appendix A
Proof of Lemma 1

In a Rayleigh fading channel, the SNR of user j is an exponentially distributed variable. Using the Shannon capacity formula, we have the mean and variance of user j's instantaneous data rate

$$E[R_j] = \int_0^\infty \log_2\left(1 + \overline{SNR}_j \cdot \lambda\right) \times e^{-\lambda} d\lambda. \tag{31}$$

$$\sigma_{R_j}^2 = \int_0^\infty \left(\log_2\left(1 + \overline{SNR}_j \cdot \lambda\right)\right)^2 \times e^{-\lambda} d\lambda - \left(E[R_j]\right)^2. \tag{32}$$

where $\overline{SNR}_j$ is the average SNR of user j.

With (31) and (32), one can obtain the expressions for $\frac{d\sigma_R}{dE[R]}$ and $\frac{d}{dE[R]}\left(\frac{d\sigma_R}{dE[R]}\right)$ after tedious manipulation. The details of mathematical reasoning are not necessary for the development of the proof. Instead, with Figs. 11, 12 and 13, we know $E'[R] = \frac{dE[R]}{d\overline{SNR}} > 0$, $\sigma'_R = \frac{d\sigma_R}{d\overline{SNR}} > 0$, and $\frac{d}{d\overline{SNR}}\left(\frac{d\sigma_R}{d\overline{SNR}} / \frac{dE[R]}{d\overline{SNR}}\right) < 0$. By the chain rule for derivatives, we have $\frac{d\sigma_R}{dE[R]} = \frac{d\sigma_R}{d\overline{SNR}} / \frac{dE[R]}{d\overline{SNR}}$ and $\frac{d}{dE[R]}\left(\frac{d\sigma_R}{dE[R]}\right) = \frac{d}{d\overline{SNR}}\left(\frac{d\sigma_R}{d\overline{SNR}} / \frac{dE[R]}{d\overline{SNR}}\right) / \frac{dE[R]}{d\overline{SNR}}$. It is then easy to verify that $\frac{d\sigma_R}{dE[R]} > 0$ and $\frac{d}{dE[R]}\left(\frac{d\sigma_R}{dE[R]}\right) < 0$. With the properties of the first and second derivative tests, we conclude that σ_R *w.r.t.* $E[R]$ is increasing, concave.

Appendix B
Proof of Lemma 2

For a well-designed scheduling algorithm in fading environments, one can readily verify that larger average data rate $E[R]$ (*i.e.,* better average channel quality) produces larger average throughput $E[\mu^*]$. On the other hand, Holtzman [16] has shown that users with more fading variability get more average throughput, *i.e.,* larger σ_R produces larger $E[\mu^*]$. In wireless networks, it is easy to justify that both channel fluctuation (*i.e.,* σ_R) and average channel quality (*i.e.,* $E[R]$) contribute

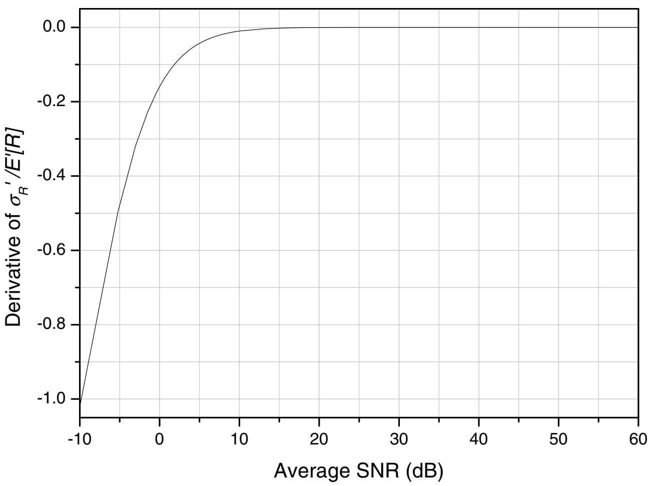

Fig. 13. $\frac{d}{d\overline{SNR}}\left(\frac{d\sigma_R}{d\overline{SNR}}/\frac{dE[R]}{d\overline{SNR}}\right)$ *w.r.t.* $\overline{SNR}$

to average throughput $E[\mu^*]$. Without lost of generality, a very small increase in average throughput can be written as $\Delta E[\mu^*]=f^*\cdot E[R]\cdot\Delta\overline{SNR}+g^*\cdot\sigma_R\cdot\Delta\overline{SNR}$ where $f^*>0$, $g^*>0$ represent the weights of $E[R]$ and σ_R, respectively.

In an N-user cellular network, let say users i, j in PFS are provided average throughputs of $E[\mu_i]$ and $E[\mu_j]$. We assume $E[\mu_i]\geq E[\mu_j]$.

Now we consider a particular scheduling algorithm S^*: it simply schedules all other users ($k=1,2,\ldots,N, k\neq i,j$) in the same slots allocated for them in PFS, and it allocates for user i some slots which previously belonged to user j in PFS. In such way, S^* will allocate an increased average throughput of to user i, and a decreased average throughput to user j, while keeping all other users' average throughputs unchanged. Let say in S^* the average throughputs for users i, j are $E[\mu_i]+\Delta E[\mu_i^*]$ and $E[\mu_j]-\Delta E[\mu_j^*]$, respectively. Obviously S^* is different from PFS. With the definition of proportional fairness (Definition 1) and after straightforward algebraic manipulation, we have

$$\frac{\Delta E[\mu_j^*]}{E[\mu_j]}\geq\frac{\Delta E[\mu_i^*]}{E[\mu_i]}. \tag{33}$$

Lemma 1 shows that $\frac{\sigma_{R_i}}{E[R_i]}\leq\frac{\sigma_{R_j}}{E[R_j]}$. Since $\frac{\Delta E[\mu_i^*]}{\Delta E[\mu_j^*]}=\frac{f^*\cdot E[R_i]+g^*\sigma_{R_i}}{f^*\cdot E[R_j]+g^*\sigma_{R_j}}$, we have

$$\frac{\Delta E[\mu_i^*]}{\Delta E[\mu_j^*]}\geq\frac{\sigma_{R_i}}{\sigma_{R_j}}. \tag{34}$$

Combining (33) and (34), we obtain

$$\frac{E[\mu_i]}{E[\mu_j]}\geq\frac{\sigma_{R_i}}{\sigma_{R_j}}. \tag{35}$$

Similarly, if S^* allocates a decreased average throughput of $E[\mu_i]-\Delta E[\mu_i^*]$ to user i, and an increased average throughput of $E[\mu_j]+\Delta E[\mu_j^*]$ to user j, while keeping all other users' average throughputs unchanged. Following the same steps as above, we will have

$$\frac{E[\mu_i]}{E[\mu_j]}\leq\frac{E[R_i]}{E[R_j]}. \tag{36}$$

Putting together (35) and (36) completes the proof.

Appendix C
Proof of Lemma 5

Proof: Let $B=\prod_{\forall i\neq j}Y_i(b_i x/b_j)$. Since $c_i/c_j\leq b_i/b_j\leq a_i/a_j$ ($\forall a_i>a_j$), for non-negative, monotonically increasing $Y_k(\cdot)$ ($\forall k=1,2,\ldots,N$), we have

$$\begin{aligned}B^2\leq&\prod_{\forall a_i\geq a_j\wedge i\neq j}Y_i\left(\frac{a_i}{a_j}\cdot x\right)\prod_{\forall a_i<a_j\wedge i\neq j}Y_i\left(\frac{b_i}{b_j}\cdot x\right)\\&\times\prod_{\forall a_i\geq a_j\wedge i\neq j}Y_i\left(\frac{b_i}{b_j}\cdot x\right)\prod_{\forall a_i<a_j\wedge i\neq j}Y_i\left(\frac{c_i}{c_j}\cdot x\right)\\=&B\times\prod_{\forall a_i\geq a_j\wedge i\neq j}Y_i\left(\frac{a_i}{a_j}\cdot x\right)\prod_{\forall a_i<a_j\wedge i\neq j}Y_i\left(\frac{c_i}{c_j}\cdot x\right).\end{aligned}\tag{37}$$

$$\begin{aligned}B^2\geq&\prod_{\forall a_i\geq a_j\wedge i\neq j}Y_i\left(\frac{c_i}{c_j}\cdot x\right)\prod_{\forall a_i<a_j\wedge i\neq j}Y_i\left(\frac{b_i}{b_j}\cdot x\right)\\&\times\prod_{\forall a_i\geq a_j\wedge i\neq j}Y_i\left(\frac{b_i}{b_j}\cdot x\right)\prod_{\forall a_i<a_j\wedge i\neq j}Y_i\left(\frac{a_i}{a_j}\cdot x\right)\\=&B\times\prod_{\forall a_i\geq a_j\wedge i\neq j}Y_i\left(\frac{c_i}{c_j}\cdot x\right)\prod_{\forall a_i<a_j\wedge i\neq j}Y_i\left(\frac{a_i}{a_j}\cdot x\right).\end{aligned}\tag{38}$$

This completes the proof. ■

Appendix D
Proof of Theorem 1

By Lemma 2, we have $\frac{\sigma_{R_i}}{\sigma_{R_j}}\leq\frac{E[\mu_i]}{E[\mu_j]}\leq\frac{E[R_i]}{E[R_j]}$ if $E[R_j]\leq E[R_i]$, and $\frac{\sigma_{R_i}}{\sigma_{R_j}}\geq\frac{E[\mu_i]}{E[\mu_j]}\geq\frac{E[R_i]}{E[R_j]}$ if $E[R_j]\geq E[R_i]$. Let $a_l=E[R_l]$, $b_l=\mu_{R_l}$, and $c_l=\sigma_l$ ($l=i,j$). Since $F_{R_i}(x)$ is non-negative, non-decreasing with respect to x, by applying Lemma 5 we have

$$\begin{aligned}\prod_{\forall i\neq j}F_{R_i}\left(\frac{E[\mu_i]}{E[\mu_j]}\cdot x\right)\leq&\prod_{\forall E[R_i]\geq E[R_j]\wedge i\neq j}F_{R_i}\left(\frac{E[R_i]}{E[R_j]}\cdot x\right)\\&\times\prod_{\forall E[R_i]<E[R_j]\wedge i\neq j}F_{R_i}\left(\frac{\sigma_{R_i}}{\sigma_{R_j}}\cdot x\right)\end{aligned}\tag{39}$$

$$\begin{aligned}\prod_{\forall i\neq j}F_{R_i}\left(\frac{E[\mu_i]}{E[\mu_j]}\cdot x\right)\geq&\prod_{\forall E[R_i]<E[R_j]\wedge i\neq j}F_{R_i}\left(\frac{E[R_i]}{E[R_j]}\cdot x\right)\\&\times\prod_{\forall E[R_i]\geq E[R_j]\wedge i\neq j}F_{R_i}\left(\frac{\sigma_{R_i}}{\sigma_{R_j}}\cdot x\right).\end{aligned}\tag{40}$$

Define $m_j=E[R_j]/\sigma_{R_j}$. Substituting (39) and (40) in (16) and after some algebra, we have

$$\begin{aligned}E[\mu_j]\leq\sigma_{R_j}\cdot&\int_{-m_j}^{\infty}\left(y\cdot\sigma_{R_j}+E[R_j]\right)\cdot f_{R_j}\left(y\cdot\sigma_{R_j}+E[R_j]\right)\\&\times\prod_{\forall E[R_i]\geq E[R_j]\wedge i\neq j}F_{R_i}\left(y\cdot\frac{E[R_i]}{E[R_j]}\cdot\sigma_{R_j}+E[R_i]\right)\\&\times\prod_{\forall E[R_i]<E[R_j]\wedge i\neq j}F_{R_i}\left(y\cdot\sigma_{R_i}+\frac{\sigma_{R_i}}{\sigma_{R_j}}\cdot E[R_j]\right)dy.\end{aligned}\tag{41}$$

$$E[\mu_j] \geq \sigma_{R_j}\cdot\int_{-m_j}^{\infty}\left(y\cdot\sigma_{R_j}+E[R_j]\right)\cdot f_{R_j}\left(y\cdot\sigma_{R_j}+E[R_j]\right) \times \prod_{\forall E[R_i]<E[R_j]\wedge i\neq j} F_{R_i}\left(y\cdot\frac{E[R_i]}{E[R_j]}\cdot\sigma_{R_j}+E[R_i]\right) \times \prod_{\forall E[R_i]\geq E[R_j]\wedge i\neq j} F_{R_i}\left(y\cdot\sigma_{R_i}+\frac{\sigma_{R_i}}{\sigma_{R_j}}\cdot E[R_j]\right)dy. \quad (42)$$

With Lemma 2, we have $\frac{\sigma_{R_i}}{\sigma_{R_j}} \leq \frac{E[R_i]}{E[R_j]}$ if $E[R_j] \leq E[R_i]$, and $\frac{\sigma_{R_i}}{\sigma_{R_j}} \geq \frac{E[R_i]}{E[R_j]}$ if $E[R_j] \geq E[R_i]$. Since $F_{R_i}(x)$ *w.r.t.* x is monotonically non-decreasing, it is easy to prove that the following expression lies between the bounds given by (41) and (42),

$$\sigma_{R_j}\cdot\int_{-m_j}^{\infty}\left(y\cdot\sigma_{R_j}+E[R_j]\right)f_{R_j}\left(y\cdot\sigma_{R_j}+E[R_j]\right) \times \prod_{\forall i\neq j} F_{R_i}\left(y\cdot\sigma_{R_i}+E[R_i]\right)dy. \quad (43)$$

We can then use (43) to estimate $E[\mu_j]$. Next we apply results on rate modeling to further simplify (43).

Smith *et al.* [19] have revealed that data rate R over Rayleigh fading channels can be accurately modeled by a normal distribution, *i.e.,* approximately $R \sim \mathcal{N}(E[R], \sigma_R^2)$. When assuming the Shannon rate model for R, $E[R]$ and σ_R are determined by (31) and (32), respectively. For normally distributed R_i

$$f_{R_i}(x)=\frac{1}{\sigma_{R_i}}\cdot\rho\left(\frac{x-E[R_i]}{\sigma_{R_i}}\right), F_{R_i}(x)=\phi\left(\frac{x-E[R_i]}{\sigma_{R_i}}\right). \quad (44)$$

Substituting (44) into (43) yields

$$E[\mu_j] = \frac{E[R_j]}{N}\cdot\left(1-\left(\phi(-m_j)\right)^N\right) + \sigma_{R_j}\cdot\int_{-m_j}^{\infty} y\cdot\rho(y)\cdot\phi(y)^{N-1}dy. \quad (45)$$

Since $(\phi(-m_j))^N \ll 1$, (45) further reduces to

$$E[\mu_j]=\frac{E[R_j]}{N}+\sigma_{R_j}\cdot\int_{-m_j}^{\infty} y\cdot\rho(y)\cdot\phi(y)^{N-1}dy. \quad (46)$$

This completes the proof.

APPENDIX E
PROOF OF THEOREM 2

In Appendix D (the proof of Theorem 1), we have provided an intermediate result for the estimated mean throughput with any rate model,

$$E[\mu_j] = \sigma_{R_j}\cdot\int_{-m_j}^{\infty}\left(y\cdot\sigma_{R_j}+E[R_j]\right)\cdot f_{R_j}\left(y\cdot\sigma_{R_j}+E[R_j]\right) \times \prod_{\forall i\neq j} F_{R_i}\left(y\cdot\sigma_{R_i}+E[R_i]\right)dy. \quad (47)$$

When we assume the linear rate model for R over independent Rayleigh fading channels, *i.e.,* $R_j = \beta\cdot SNR_j$ where β is a positive constant and SNR_j is an exponentially distributed random variable with the mean $\overline{SNR}_j$,

$$f_{R_j}(x) = \begin{cases} \frac{1}{\beta\cdot\overline{SNR}_j}\cdot e^{-x/(\beta\cdot\overline{SNR}_j)}, & x \geq 0 \\ 0, & x < 0 \end{cases}. \quad (48)$$

$$F_{R_j}(x) = \begin{cases} 1-e^{-x/(\beta\cdot\overline{SNR}_j)}, & x \geq 0 \\ 0, & x < 0 \end{cases}. \quad (49)$$

Since R_j is also an exponentially distributed random variable, we have

$$E[R_j] = \sigma_{R_j} = \beta\cdot\overline{SNR}_j. \quad (50)$$

Using (50) and applying variable substitution, (47) can be rewritten as

$$E[\mu_j] = \int_0^{\infty} x\cdot f_{R_j}(x) \times \prod_{\forall i\neq j} F_{R_i}\left(\frac{\sigma_{R_i}}{\sigma_{R_j}}\cdot x\right)dx. \quad (51)$$

With (48), (49), and (50), by applying variable substitution in (51), we have

$$\begin{aligned} E[\mu_j] &= \int_0^{\infty}(\beta\cdot\overline{SNR}_j)\cdot y\cdot e^{-y}\cdot\left(1-e^{-y}\right)^{N-1}dy \\ &= \beta\cdot\overline{SNR}_j\cdot\int_0^1\{-\ln(1-x)\}\cdot x^{N-1}dx \\ &= \beta\cdot\overline{SNR}_j\cdot\int_0^1\left(\sum_{k=1}^{\infty}\frac{x^{N+k-1}}{k}\right)dx \\ &= \beta\cdot\overline{SNR}_j\cdot\sum_{k=1}^{\infty}\frac{1}{k\cdot(k+N)} \\ &= \frac{\beta\cdot\overline{SNR}_j}{N}\cdot\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+N}\right) \\ &= \frac{\beta\cdot\overline{SNR}_j}{N}\cdot\sum_{k=1}^{N}\frac{1}{k} = \frac{E[R_j]}{N}\cdot\sum_{k=1}^{N}\frac{1}{k}. \end{aligned} \quad (52)$$

where the *Maclaurin* series expansion of $\ln(1-x)$ [28] is used to obtain the third equality.

This completes the proof.

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