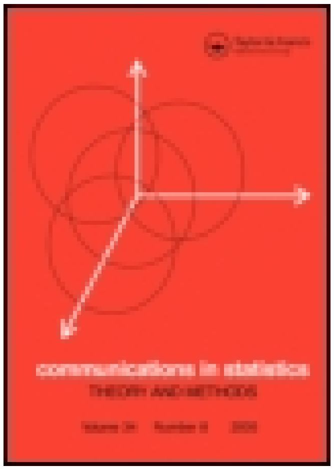

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Admissible Estimation for Linear Combination of Fixed and Random Effects in General Mixed Linear Models

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The general mixed linear model can be denoted by $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$, *where* $\boldsymbol{\beta}$ *is a vector of fixed effects,* $\boldsymbol{u}$ *is a vector of random effects, and* $\boldsymbol{e}$ *is a vector of random errors. In this article, the problem of admissibility of* $\boldsymbol{Qy}$ *and* $\boldsymbol{Qy} + \boldsymbol{q}$ *for estimating linear functions,* $\boldsymbol{\varrho} = \boldsymbol{L}'\boldsymbol{\beta} + \boldsymbol{M}'\boldsymbol{u}$, *of the fixed and random effects is considered, and the necessary and sufficient conditions for* $\boldsymbol{Qy}$ *(*resp. $\boldsymbol{Qy} + \boldsymbol{q}$*) to be admissible in the set of homogeneous (*resp. potentially inhomogeneous*) linear estimators with respect to the MSE and MSEM criteria are investigated. We provide a straightforward alternative proof to the method that was utilized by Wu (1988), Baksalary and Markiewicz (1990), and Groß and Markiewicz (1999). In addition, we derive the corresponding results on the admissibility problem under the generalized MSE criterion.*

Keywords Admissibility; General mixed linear model; (Generalized) MSE criterion; Linear combination; MSEM criterion.

Mathematics Subject Classification 62C15; 62J05.

1. Introduction

The three small-area models of nested-error regression model proposed by Battese, Harter, and Fuller (1988), random regression coefficient model considered by Dempster, Rubin, and Tsutakawa (1981), and Fay-Herriot model investigated by Fay and Herriot (1979), are all special cases (in a sense) of the general mixed linear model, denoted by

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}, \tag{1.1}$$

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where $\boldsymbol{y}$ is the n-dimensional vector of observations; $\boldsymbol{X}$ and $\boldsymbol{Z}$ are known $n \times p$ and $n \times q$ matrices, respectively, $\boldsymbol{u}$ and $\boldsymbol{e}$ are jointly distributed with means $\mathbf{0}$, and covariance matrix $\sigma^2\boldsymbol{\Gamma}$; i.e.,

$$\mathscr{E}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{e}\end{pmatrix} = \mathbf{0}, \quad \mathscr{D}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{e}\end{pmatrix} = \sigma^2\begin{pmatrix}\boldsymbol{G} & \boldsymbol{K}\\ \boldsymbol{K}' & \boldsymbol{R}\end{pmatrix} = \sigma^2\boldsymbol{\Gamma} \succcurlyeq \mathbf{0}.$$

For the three small-area models, it is customary to consider the problem of estimating (or predicting) the special case of the general linear function of the fixed effects $\boldsymbol{\beta}$ and the realized values of the random effects $\boldsymbol{u}$, say $\varrho(\boldsymbol{l}, \boldsymbol{m}) = \boldsymbol{l}'\boldsymbol{\beta} + \boldsymbol{m}'\boldsymbol{u}$, for special vectors, $\boldsymbol{l}$ and $\boldsymbol{m}$, of constants. Denote now $\boldsymbol{\Sigma} = \boldsymbol{ZGZ}' + \boldsymbol{ZK} + \boldsymbol{K}'\boldsymbol{Z}' + \boldsymbol{R}$. In the past several decades, the problem mentioned above has received considerable interest. In the case when $\boldsymbol{K} = \mathbf{0}$ and $\boldsymbol{\Sigma} \succ \mathbf{0}$, Henderson (1975) showed that the best linear unbiased (BLU) estimator (or predictor) of ϱ is expressible as $\hat{\varrho} = \boldsymbol{l}'\hat{\boldsymbol{\beta}} + \boldsymbol{m}'\boldsymbol{GZ}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})$, where $\hat{\boldsymbol{\beta}}$ is any solution to the generalized least squares (GLS) equations $\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y}$. For the general case, one can obtain the essentially unique BLU estimator of ϱ as $\varrho^* = \boldsymbol{l}'\boldsymbol{\beta}^* + \boldsymbol{m}'\boldsymbol{N}'\boldsymbol{T}^-(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}^*)$ by virtue of Rao's *unified theory of least squares* (cf. Wang, 1987), with the notations $\boldsymbol{N} = \boldsymbol{ZG} + \boldsymbol{K}'$, $\boldsymbol{T} = \boldsymbol{\Sigma} + \boldsymbol{XUX}'$, $\boldsymbol{S} = \boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{X}$, and $\boldsymbol{\beta}^* = \boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{y}$, where $\boldsymbol{U}$ is any arbitrary but fixed $p \times p$ symmetric nonnegative definite (s.n.n.d.) matrix satisfying $\mathscr{R}(\boldsymbol{T}) = \mathscr{R}(\boldsymbol{X}, \boldsymbol{\Sigma})$, or equivalently, $\mathscr{R}(\boldsymbol{X}) \subseteq \mathscr{R}(\boldsymbol{T})$. Without loss of generality, we suppose $\mathscr{R}(\boldsymbol{U}) \subseteq \mathscr{R}(\boldsymbol{X}')$ in the following. Note that one can readily justify $\mathscr{R}(\boldsymbol{N}) \subseteq \mathscr{R}(\boldsymbol{\Sigma}) \subseteq \mathscr{R}(\boldsymbol{T})$, and therefore ϱ^* is invariant with respect to the choices of involved generalized inverses since $\boldsymbol{y} \in \mathscr{R}(\boldsymbol{T})$ (with probability 1) and $\mathscr{R}(\boldsymbol{S}) = \mathscr{R}(\boldsymbol{X}')$. For more related results, one can see Harville (1976), Harville and Jeske (1992), Prasad and Rao (1990), Robinson (1991), Das, Jiang-Jiming, and Rao (2004), Groß and Markiewicz (2004), Ip, Wong, and Liu-Jinshan (2005), Heiligers and Markiewicz (1996).

In this article, we will consider an the admissibility problem estimating

$$\boldsymbol{\varrho}(\boldsymbol{L}, \boldsymbol{M}) = \boldsymbol{L}'\boldsymbol{\beta} + \boldsymbol{M}'\boldsymbol{u} = \boldsymbol{\varrho}, \quad \text{say}, \tag{1.2}$$

for given $p \times k$ and $q \times k$ matrices, $\boldsymbol{L}$ and $\boldsymbol{M}$, and investigate the sets of admissible estimators of $\boldsymbol{\varrho}$ under the mean squared error matrix (MSEM) and mean squared error (MSE) criteria, in which the former would be a generalization of a respective characterization of admissible linear estimators of linear combination of fixed effects in general linear model with respect to MSEM criterion given in Baksalary and Markiewicz (1963), whereas the latter is with respect to MSE criterion. A similar process was used in part in Baksalary and Markiewicz (1990). Notice that we call $\boldsymbol{\varrho}$ estimable (or predictable) if $\boldsymbol{L}'\boldsymbol{\beta}$ is linearly estimable under the model (1.1) or, equivalently, $\mathscr{R}(\boldsymbol{L}) \subseteq \mathscr{R}(\boldsymbol{X}')$. Further, if $\boldsymbol{\varrho}$ is estimable, then its estimator $\tilde{\boldsymbol{\varrho}}$ is said unbiased if $\mathscr{E}(\tilde{\boldsymbol{\varrho}} - \boldsymbol{\varrho}) = \mathbf{0}$.

It is well known that the admissibility is a favorable property that a good estimation or prediction, for which a number of original works appeared in the past several decades. Wang (1987) collected many useful corresponding results, in which the admissibility of $\boldsymbol{X}\hat{\boldsymbol{\beta}}$ for estimating $\boldsymbol{X}\boldsymbol{\beta}$, obtained by Rao, under the MSE criterion was mentioned. Wu (1983) investigated the necessary and sufficient conditions for $\boldsymbol{Qy}$ to be admissible for estimating $\boldsymbol{L}'\boldsymbol{\beta}$ with respect to MSEM criterion. For the similar problem under the MSE criterion, one can see Yu and

Xu (2004), and Xu and Yu (2005). In addition, Wu (1987, 1988) considered a linear model with stochastic coefficients, and derived a class of linear admissible estimators (called *predictors*) under the quadratic or matrix loss function by virtue of a respective characterization of admissible linear estimators of linear combination of fixed effects in general linear models with respect to MSE and MSEM criteria. In a way similar to this, one can see *Theorem 5 and a comment on it* in Groß and Markiewicz (1999). Here, we would like to offer another alternative straightforward proof and some new representations and then give some concluding remarks.

2. Admissibility of Linear Estimation

We will call an estimator $\tilde{\varrho}_1$ of ϱ MSEM-superior (*resp. MSE-superior*) over another estimator $\tilde{\varrho}_2$ of ϱ in the sense that $\mathrm{MSEM}(\tilde{\varrho}_2, \varrho) - \mathrm{MSEM}(\tilde{\varrho}_1, \varrho) \succcurlyeq \mathbf{0}$ (*resp.* $\mathrm{MSE}(\tilde{\varrho}_2, \varrho) - \mathrm{MSE}(\tilde{\varrho}_1, \varrho) \geq 0$) is satisfied for any pair $(\boldsymbol{\beta}, \sigma^2)$, with equality not holding for at least one point, say $(\boldsymbol{\beta}_0, \sigma_0^2)$, in which $\mathrm{MSEM}(\tilde{\varrho}, \varrho) = \mathscr{E}(\tilde{\varrho} - \varrho)(\tilde{\varrho} - \varrho)'$, $\mathrm{MSE}(\tilde{\varrho}, \varrho) = \mathscr{E}(\tilde{\varrho} - \varrho)'(\tilde{\varrho} - \varrho)$. Furthermore, an estimator $\tilde{\varrho}$ is said MSEM-admissible (*resp. MSE-admissible*) in $\mathscr{S}$ (a set of estimators for ϱ) if $\tilde{\varrho} \in \mathscr{S}$ without any other estimator $\hat{\varrho} \in \mathscr{S}$, which is MSEM-superior (*resp. MSE-superior*) over $\tilde{\varrho}$. For convenience in the following, we denote by $\mathscr{S}_{\mathrm{H}} = \{\boldsymbol{Qy} \mid \boldsymbol{Q} : k \times n\}$ and $\mathscr{S}_{\mathrm{INH}} = \{\boldsymbol{Qy} + \boldsymbol{q} \mid \boldsymbol{Qy} \in \mathscr{S}_{\mathrm{H}}, \boldsymbol{q} : k \times 1\}$ the sets of homogeneous linear estimators and (potentially) inhomogeneous linear estimators, respectively.

2.1. *Admissibility of Homogeneous Linear Estimation*

Here, we will discuss the admissibility of homogeneous linear estimator for ϱ regarding the MSEM and the MSE criteria and give the necessary and sufficient conditions for $\boldsymbol{Qy}$ to be admissible in the set $\mathscr{S}_{\mathrm{H}}$. Denote by $\mathscr{A}_{\text{H-MSEM}}$ and $\mathscr{A}_{\text{H-MSE}}$ the sets of all MSEM-admissible and MSE-admissible homogeneous linear estimators of ϱ. Clearly, $\mathscr{A}_{\text{H-MSE}} \subseteq \mathscr{A}_{\text{H-MSEM}} \subseteq \mathscr{S}_{\mathrm{H}}$. With regard to the characterizations of $\mathscr{A}_{\text{H-MSE}}$ and $\mathscr{A}_{\text{H-MSEM}}$, one can see Groß and Markiewicz (1999) in part in a sense. We mainly offer an alternative method in the present article.

2.1.1. *MSEM-Admissibility of Homogeneous Linear Estimation.* Here, we follow the idea used in Xu and Yu (2005) and Yu and Xu (2004). Let us first give a lemma as follows.

Lemma 2.1. *For any given $\boldsymbol{Qy} \in \mathscr{S}_H$, there is $\boldsymbol{Q}^*\boldsymbol{y} \in \mathscr{S}_H$ such that $MSEM(\boldsymbol{Qy}, \varrho) \succcurlyeq MSEM(\boldsymbol{Q}^*\boldsymbol{y}, \varrho)$ holds for any pair $(\boldsymbol{\beta}, \sigma^2)$.*

Proof. Denoting $\boldsymbol{W} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{T}^- = \boldsymbol{XS}^-\boldsymbol{X}'\boldsymbol{T}^-$ for any fixed arbitrary generalized inverse of $\boldsymbol{T}$ gives $\boldsymbol{WX} = \boldsymbol{X}$, $\boldsymbol{W}^2 = \boldsymbol{W}$ and $\boldsymbol{WTW}' = \boldsymbol{WT} = \boldsymbol{TW}' = \boldsymbol{XS}^-\boldsymbol{X}'$. Taking

$$\boldsymbol{Q}^* = \boldsymbol{QXS}^-\boldsymbol{X}'\boldsymbol{T}^- + \boldsymbol{M}'\boldsymbol{N}'(\boldsymbol{T}^- - \boldsymbol{T}^-\boldsymbol{XS}^-\boldsymbol{X}'\boldsymbol{T}^-) = \boldsymbol{QW} + \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-(\boldsymbol{I} - \boldsymbol{W}), \quad (2.1)$$

we get $\mathrm{MSEM}(\boldsymbol{Qy}, \varrho) = \mathrm{MSEM}(\boldsymbol{Q}^*\boldsymbol{y}, \varrho) + \boldsymbol{\Lambda} + \boldsymbol{\Delta} + \boldsymbol{\Delta}'$, where $\boldsymbol{\Lambda} = \mathscr{E}(\boldsymbol{Q} - \boldsymbol{Q}^*)\boldsymbol{yy}'(\boldsymbol{Q} - \boldsymbol{Q}^*)'$ and $\boldsymbol{\Delta} = \mathscr{E}(\boldsymbol{Qy} - \boldsymbol{Q}^*\boldsymbol{y})(\boldsymbol{Q}^*\boldsymbol{y} - \varrho)'$. To finish the proof, it suffices to justify $\boldsymbol{\Delta} = \mathbf{0}$. In fact, noting that $\boldsymbol{Q} - \boldsymbol{Q}^* = (\boldsymbol{Q} - \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-)(\boldsymbol{I} - \boldsymbol{W})$, $\boldsymbol{Q}^*\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{X}$, and $\boldsymbol{T}\boldsymbol{Q}^{*\prime} - \boldsymbol{N}\boldsymbol{M} = \boldsymbol{X}\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{Q}' - \boldsymbol{W}\boldsymbol{N}\boldsymbol{M}$, we get

$$\begin{aligned}
\boldsymbol{\Delta} &= \mathscr{E}(\boldsymbol{Q} - \boldsymbol{Q}^*)(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{e})\,[(\boldsymbol{Q}^*\boldsymbol{X} - \boldsymbol{L}')\,\boldsymbol{\beta} + (\boldsymbol{Q}^*\boldsymbol{Z} - \boldsymbol{M}')\boldsymbol{u} + \boldsymbol{Q}^*\boldsymbol{e}]' \\
&= \sigma^2(\boldsymbol{Q} - \boldsymbol{Q}^*)(\boldsymbol{Z}, \boldsymbol{I})\boldsymbol{\Gamma}(\boldsymbol{Q}^*\boldsymbol{Z} - \boldsymbol{M}', \boldsymbol{Q}^*)' = \sigma^2(\boldsymbol{Q} - \boldsymbol{Q}^*)(\boldsymbol{\Sigma}\boldsymbol{Q}^{*\prime} - \boldsymbol{N}\boldsymbol{M}) \\
&= \sigma^2(\boldsymbol{Q} - \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-)(\boldsymbol{I} - \boldsymbol{W})(\boldsymbol{T}\boldsymbol{Q}^{*\prime} - \boldsymbol{N}\boldsymbol{M} - \boldsymbol{X}\boldsymbol{U}\boldsymbol{X}'\boldsymbol{Q}^{*\prime}) \\
&= \sigma^2(\boldsymbol{Q} - \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-)(\boldsymbol{I} - \boldsymbol{W})(\boldsymbol{X}\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{Q}' - \boldsymbol{W}\boldsymbol{N}\boldsymbol{M} - \boldsymbol{X}\boldsymbol{U}\boldsymbol{X}'\boldsymbol{Q}^{*\prime}) = \mathbf{0}
\end{aligned}$$

combining $\boldsymbol{W}\boldsymbol{X} = \boldsymbol{X}$ and $\boldsymbol{W}^2 = \boldsymbol{W}$. Thus, for any $(\boldsymbol{\beta}, \sigma^2)$, $\text{MSEM}(\boldsymbol{Q}\boldsymbol{y}, \boldsymbol{\varrho}) \succcurlyeq \text{MSEM}(\boldsymbol{Q}^*\boldsymbol{y}, \boldsymbol{\varrho})$, with equality holding if and only if $\boldsymbol{\Lambda} = \mathbf{0}$. $\square$

By the proof of Lemma 2.1, it is easily seen that

$$\begin{aligned}
\boldsymbol{\Lambda} = \mathbf{0} &\Leftrightarrow (\boldsymbol{Q} - \boldsymbol{Q}^*)(\sigma^2\boldsymbol{\Sigma} + \boldsymbol{X}\boldsymbol{\beta}\boldsymbol{\beta}'\boldsymbol{X}')(\boldsymbol{Q} - \boldsymbol{Q}^*)' = \mathbf{0} \Leftrightarrow (\boldsymbol{Q} - \boldsymbol{Q}^*)\boldsymbol{\Sigma}(\boldsymbol{Q} - \boldsymbol{Q}^*)' = \mathbf{0} \\
&\Leftrightarrow (\boldsymbol{Q} - \boldsymbol{Q}^*)(\boldsymbol{\Sigma} + \boldsymbol{X}\boldsymbol{U}\boldsymbol{X}')(\boldsymbol{Q} - \boldsymbol{Q}^*)' = \mathbf{0} \Leftrightarrow (\boldsymbol{Q} - \boldsymbol{Q}^*)\boldsymbol{T} = \mathbf{0} \\
&\Leftrightarrow (\boldsymbol{Q} - \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-)\,(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{T} = \mathbf{0} \Leftrightarrow (\boldsymbol{Q} - \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-)(\boldsymbol{T} - \boldsymbol{X}\boldsymbol{S}^-\boldsymbol{X}') = \mathbf{0} \\
&\Leftrightarrow P_{\boldsymbol{T}^{+\frac{1}{2}}\boldsymbol{X}}\left(\boldsymbol{T}^{\frac{1}{2}}\,(\boldsymbol{Q}' - \boldsymbol{T}^-\boldsymbol{N}\boldsymbol{M})\right) = \boldsymbol{T}^{\frac{1}{2}}(\boldsymbol{Q}' - \boldsymbol{T}^-\boldsymbol{N}\boldsymbol{M}), \\
&\Leftrightarrow \mathscr{R}\left(\boldsymbol{T}^{\frac{1}{2}}\,(\boldsymbol{Q}' - \boldsymbol{T}^-\boldsymbol{N}\boldsymbol{M})\right) \subset \mathscr{R}\left(\boldsymbol{T}^{+\frac{1}{2}}\boldsymbol{X}\right) \\
&\Leftrightarrow \mathscr{R}(\boldsymbol{T}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M}) \subset \mathscr{R}(\boldsymbol{X}) \Leftrightarrow \mathscr{R}(\boldsymbol{\Sigma}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M}) \subset \mathscr{R}(\boldsymbol{X}).
\end{aligned}$$

in view of the inherent equality $(\boldsymbol{Q} - \boldsymbol{Q}^*)\boldsymbol{X} = \mathbf{0}$. That is, $\boldsymbol{\Lambda} = \mathbf{0} \Leftrightarrow \mathscr{R}(\boldsymbol{\Sigma}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M}) \subset \mathscr{R}(\boldsymbol{X})$. Let $\mathscr{Q} = \{\boldsymbol{Q} \,|\, \mathscr{R}(\boldsymbol{\Sigma}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M}) \subseteq \mathscr{R}(\boldsymbol{X})\}$ below. We restate the result in a lemma version as follows.

Lemma 2.2. *Assume that $\boldsymbol{Q}^*$ is defined as* (2.1) (*the same below*), *$\boldsymbol{Q}\boldsymbol{y}$ is a given estimator* (*or predictor*) *in $\mathscr{S}_H$. Then $MSEM(\boldsymbol{Q}\boldsymbol{y}, \boldsymbol{\varrho}) = MSEM(\boldsymbol{Q}^*\boldsymbol{y}, \boldsymbol{\varrho})$ holds for any $(\boldsymbol{\beta}, \sigma^2)$ if and only if $\boldsymbol{Q} \in \mathscr{Q}$.*

From Lemmas 2.1 and 2.2, it follows that $\boldsymbol{Q}\boldsymbol{y} \in \mathscr{S}_{\text{H}} \backslash \mathscr{A}_{\text{H-MSEM}}$ if $\mathscr{R}(\boldsymbol{\Sigma}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M}) \subset \mathscr{R}(\boldsymbol{X})$ is not satisfied. Consequently, we need only to consider the subset of $\mathscr{S}_{\text{H}}$, say,

$$\mathscr{S}_{\text{H}}^* = \{\boldsymbol{Q}\boldsymbol{y} \in \mathscr{S}_{\text{H}} \,|\, \mathscr{R}(\boldsymbol{\Sigma}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M}) \subset \mathscr{R}(\boldsymbol{X})\}.$$

Let us deduce the MSEM of $\boldsymbol{Q}\boldsymbol{y}$ with respect to $\boldsymbol{\varrho}$ for a given $\boldsymbol{Q}\boldsymbol{y} \in \mathscr{S}_{\text{H}}^*$. Actually, noticing that $\mathscr{R}(\boldsymbol{N}) \subset \mathscr{R}(\boldsymbol{T})$, which combining $\boldsymbol{Q}\boldsymbol{T} = \boldsymbol{Q}^*\boldsymbol{T} = \boldsymbol{Q}\boldsymbol{X}\boldsymbol{S}^-\boldsymbol{X}' + \boldsymbol{M}'\boldsymbol{N}'(\boldsymbol{T}^- - \boldsymbol{T}^-\boldsymbol{X}\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-)\boldsymbol{T}$ yields $\boldsymbol{Q}\boldsymbol{N} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{T}^-\boldsymbol{N} = \boldsymbol{Q}^*\boldsymbol{T}\boldsymbol{T}^-\boldsymbol{N} = \boldsymbol{Q}^*\boldsymbol{N}$, we obtain

$$\begin{aligned}
&\text{MSEM}(\boldsymbol{Q}\boldsymbol{y}, \boldsymbol{\varrho}) \\
&\quad = \mathscr{E}[(\boldsymbol{Q}\boldsymbol{X} - \boldsymbol{L}')\boldsymbol{\beta} + (\boldsymbol{Q}\boldsymbol{Z} - \boldsymbol{M}')\boldsymbol{u} + \boldsymbol{Q}\boldsymbol{e}][(\boldsymbol{Q}\boldsymbol{X} - \boldsymbol{L}')\boldsymbol{\beta} + (\boldsymbol{Q}\boldsymbol{Z} - \boldsymbol{M}')\boldsymbol{u} + \boldsymbol{Q}\boldsymbol{e}]' \\
&\quad = (\boldsymbol{Q}\boldsymbol{X} - \boldsymbol{L}')\boldsymbol{\beta}\boldsymbol{\beta}'(\boldsymbol{Q}\boldsymbol{X} - \boldsymbol{L}')' + \sigma^2(\boldsymbol{Q}\boldsymbol{\Sigma}\boldsymbol{Q}' - \boldsymbol{Q}\boldsymbol{N}\boldsymbol{M} - \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{Q}' + \boldsymbol{M}'\boldsymbol{G}\boldsymbol{M}) \\
&\quad = \sigma^2(\boldsymbol{Q}^*\boldsymbol{T}\boldsymbol{Q}^{*\prime} - \boldsymbol{Q}\boldsymbol{X}\boldsymbol{U}\boldsymbol{X}'\boldsymbol{Q}' - \boldsymbol{Q}^*\boldsymbol{N}\boldsymbol{M} - \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{Q}^{*\prime} + \boldsymbol{M}'\boldsymbol{G}\boldsymbol{M}) + \boldsymbol{J}\boldsymbol{\beta}\boldsymbol{\beta}'\boldsymbol{J}',
\end{aligned}$$

with the notation

$$J = QX - L',$$

the same below. Further, we have $Q^*TQ^{*\prime} = QXS^-X'Q' + M'N'(T^- - T^-XS^-X'T^-)NM$ and $Q^*NM = QXS^-X'T^-NM + M'N'(T^- - T^-XS^-X'T^-)NM$, and thus it follows that $\mathrm{MSEM}(Qy, \varrho) = \Omega(Q) + H$, where $H = \sigma^2[M'GM - M'N'(T^- - T^-XS^-X'T^-)NM]$, and $\Omega(Q) = \sigma^2[QX(S^- - U)X'Q' - QXS^-X'T^-NM - M'N'T^-XS^-X'Q'] + J\beta\beta'J'$. In the following, we give a lemma concerning the comparison of two estimators in $\mathcal{S}_{\mathrm{H}}^*$.

Lemma 2.3. *Assume that $Py, Qy \in \mathcal{S}_H^*$. If there does not exist any scalar $\alpha \in [-1, 1]$ such that $PX - L' = \alpha(QX - L') = \alpha J$, then Py is not MSEM-superior over Qy.*

Proof. Provided that Py is MSEM-superior over Qy, i.e., $\mathrm{MSEM}(Qy, \varrho) - \mathrm{MSEM}(Py, \varrho) \succcurlyeq 0$, or, equivalently, $\Omega(Q) \succcurlyeq \Omega(P)$ for all (β, σ^2) and $\Omega(Q) \neq \Omega(P)$ for some (β_0, σ_0^2), from the fact (see Appendix) that for symmetric matrices A and C, $A + Bxx'B' \preccurlyeq C + Dxx'D'$ holds for any x if and only if $A \preccurlyeq C$ and $B = \alpha D$ for some $\alpha \in [-1, 1]$, it follows that our supposition contradicts the conditions given in Lemma 2.3.

From the above discussions, we give the following theorems concerning $\mathcal{A}_{\text{H-MSEM}}$.

Theorem 2.1. *Assume ϱ is not estimable under model* (1.1). *Then $\mathcal{A}_{H\text{-}MSEM} = \mathcal{S}_H^*$.*

Proof. Clearly, $\mathcal{A}_{\text{H-MSEM}} \subseteq \mathcal{S}_{\mathrm{H}}^*$. On the other hand, we assume Py and Qy are two fixed arbitrary estimators of ϱ, with $Py \in \mathcal{S}_{\mathrm{H}}$ and $Qy \in \mathcal{S}_{\mathrm{H}}^*$. Without loss of generality, we suppose $Py \in \mathcal{S}_{\mathrm{H}}^*$ due to the above discussions.

- If $PX = QX$, then $\Omega(P) = \Omega(Q)$, and thereafter $\mathrm{MSEM}(Py, \varrho) = \mathrm{MSEM}(Qy, \varrho)$.
- If $PX \neq QX$, then together with ϱ not estimable, or, equivalently, $\mathcal{R}(L) \subset \mathcal{R}(X')$, one can readily verify that there does not exist $\alpha \in [-1, 1]$ such that $PX - L' = \alpha(QX - L')$. *In fact*, $PX - L' = QX - L'$ contradicts $PX \neq QX$, and $PX - L' = \alpha(QX - L')$ for some $\alpha \in [-1, 1)$ implies $L = X'(P - \alpha Q)'/(1 - \alpha)$, which contradicts $\mathcal{R}(L) \subset \mathcal{R}(X')$.

The above analysis (the latter) combining Lemma 2.3 would mean that $Qy \in \mathcal{S}_{\mathrm{H}}^*$ is MSEM-admissible in the set $\mathcal{S}_{\mathrm{H}}$, i.e., $Qy \in \mathcal{A}_{\text{H-MSEM}}$, which completes the proof. □

Theorem 2.2. *Assume that ϱ is estimable under the model* (1.1), *and $Qy \in \mathcal{S}_H$. Then $Qy \in \mathcal{A}_{H\text{-}MSEM}$ if and only if $Qy \in \mathcal{S}_H^*$ and either $QX = L'$, or $QX \neq L'$ but for any $\alpha \in (0, 1)$, $\tau(\alpha, Q) \not\succcurlyeq 0$, where $\tau(\alpha, Q) = \alpha J(S^- - U)J' + QX(S^- - U)X'Q' - L'(S^- - U)L - JS^-X'T^-NM - M'N'T^-XS^-J'$.*

Proof of Necessity. Let $Qy \in \mathcal{A}_{\text{H-MSEM}}$. It is clear that $Qy \in \mathcal{S}_{\mathrm{H}}^*$. If $QX \neq L'$ and there is some $\alpha_0 \in (0, 1)$ such that $\tau(\alpha_0, Q) \succcurlyeq 0$, taking $P = \alpha_0 Q + (1 - \alpha_0)[L'S^-X'T^- + M'N'(T^- - T^-XS^-X'T^-)]$, noting that ϱ is estimable, or, equivalently, $\mathcal{R}(L) \subset \mathcal{R}(X') = \mathcal{R}(S)$, it follows that $Py \in \mathcal{S}_{\mathrm{H}}^*$ and $PX - L' = \alpha_0(QX - L') = \alpha_0 J$, and therefore

$$\mathrm{MSEM}(Qy, \varrho) - \mathrm{MSEM}(Py, \varrho) = (1 - \alpha_0)\sigma^2\tau(\alpha_0, Q) + (1 - \alpha_0^2)J\beta\beta'J' \succcurlyeq 0$$

holds for any $(\boldsymbol{\beta}, \sigma^2)$. The left side of the above expression is non-zero for at least some pair $(\boldsymbol{\beta}_0, \sigma_0^2)$, which contradicts $\boldsymbol{Qy} \in \mathscr{A}_{\text{H-MSEM}}$, since $\boldsymbol{QX} \neq \boldsymbol{L}'$.

Proof of Sufficiency. To show the sufficiency, it suffices to show that $\boldsymbol{Py}$ is impossible MSEM-superior over $\boldsymbol{Qy}$ for any $\boldsymbol{Py} \in \mathscr{S}_{\text{H}}^*$, following from Lemmas 2.1 and 2.2. Actually,

- For the case of $\boldsymbol{QX} = \boldsymbol{L}'$. If $\boldsymbol{PX} = \boldsymbol{L}'$, then $\text{MSEM}(\boldsymbol{Py}, \varrho) = \text{MSEM}(\boldsymbol{Qy}, \varrho)$; If $\boldsymbol{PX} \neq \boldsymbol{L}'$, then by Lemma 2.3, $\boldsymbol{Py}$ is not MSEM-superior over $\boldsymbol{Qy}$.
- For the case of $\boldsymbol{QX} \neq \boldsymbol{L}'$. If $\boldsymbol{PX} = \boldsymbol{L}'$, then by direct operations, one can conclude that $\{\text{MSEM}(\boldsymbol{Qy}, \varrho) - \text{MSEM}(\boldsymbol{Py}, \varrho)\}_{\boldsymbol{\beta}=\boldsymbol{0}} = \sigma^2\boldsymbol{\tau}(0, \boldsymbol{Q})$ is not s.n.n.d.; If $\boldsymbol{PX} \neq \boldsymbol{L}'$, it suffices to show that $\boldsymbol{Py}$ is not MSEM-superior over $\boldsymbol{Qy}$ for any $\boldsymbol{P}$ satisfying $\boldsymbol{PX} - \boldsymbol{L}' = \alpha(\boldsymbol{QX} - \boldsymbol{L}')$ for some $\alpha \in [-1, 0) \cup (0, 1]$, following from Lemma 2.3 (since $\boldsymbol{PX} \neq \boldsymbol{L}'$). Actually, as one can see $\{\text{MSEM}(\boldsymbol{Qy}, \varrho) - \text{MSEM}(\boldsymbol{Py}, \varrho)\}_{\boldsymbol{\beta}=\boldsymbol{0}} = (1-\alpha)\sigma^2\boldsymbol{\tau}(\alpha, \boldsymbol{Q})$:
  - If $-1 \leq \alpha < 0$, then $\boldsymbol{\tau}(\alpha, \boldsymbol{Q}) \preccurlyeq \boldsymbol{\tau}(0, \boldsymbol{Q})$ and therefore $(1-\alpha)\sigma^2\boldsymbol{\tau}(\alpha, \boldsymbol{Q}) \not\succeq \boldsymbol{0}$;
  - If $\alpha = 1$, then $\boldsymbol{PX} = \boldsymbol{QX}$, it follows that $\text{MSEM}(\boldsymbol{Qy}, \varrho) = \text{MSEM}(\boldsymbol{Py}, \varrho)$;
  - If $0 < \alpha < 1$, it is clear that $[\text{MSEM}(\boldsymbol{Qy}, \varrho) - \text{MSEM}(\boldsymbol{Py}, \varrho)]_{\boldsymbol{\beta}=\boldsymbol{0}} \not\succeq \boldsymbol{0}$.

Therefore, $\boldsymbol{Py}$ is not MSEM-superior over $\boldsymbol{Qy}$ for any $\boldsymbol{Py} \in \mathscr{S}_{\text{H}}^*$ with $\boldsymbol{Q}$ satisfying conditions given in context. This conclusion would imply $\boldsymbol{Qy} \in \mathscr{A}_{\text{H-MSEM}}$. □

Corollary 2.1. *Assume that* ϱ *is estimable under the mixed linear model* (1.1). *Then* $\varrho^* \in \mathscr{A}_{\textit{H-MSEM}}$, *where* $\varrho^* = \{\boldsymbol{L}'\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^- + \boldsymbol{M}'\boldsymbol{N}'(\boldsymbol{T}^- - \boldsymbol{T}^-\boldsymbol{X}\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-)\}\boldsymbol{y}$ *is the BLU estimator of* ϱ.

The following corollary concerns the case $\boldsymbol{M} = \boldsymbol{0}$; e.g., cf. Wu (1986).

Corollary 2.2. *Assume* $\boldsymbol{L}'\boldsymbol{\beta}$ *is linearly estimable under the mixed linear model* (1.1). *Then* $\boldsymbol{Qy}$ *is MSEM-admissible for* $\boldsymbol{L}'\boldsymbol{\beta}$ *if and only if the following conditions hold simultaneously*:

- $\boldsymbol{QT} = \boldsymbol{QXS}^-\boldsymbol{X}'$;
- *Either* $\boldsymbol{QX} = \boldsymbol{L}'$, *or* $\boldsymbol{\tau}(\alpha, \boldsymbol{Q})|_{\boldsymbol{M}=\boldsymbol{0}} \succcurlyeq \boldsymbol{0}$ *does not hold for all* $\alpha \in (0, 1)$ *when* $\boldsymbol{QX} \neq \boldsymbol{L}'$.

Specially, if $\boldsymbol{M} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} \succ \boldsymbol{0}$, we get the conclusion obtained by Wu (1983) by taking $\boldsymbol{U} = \boldsymbol{0}$, which will be restated in a corollary below.

Corollary 2.3. *Assume* $\boldsymbol{L}'\boldsymbol{\beta}$ *is linearly estimable under the mixed linear model* (1.1) *with* $\boldsymbol{\Sigma} \succ \boldsymbol{0}$. *Then* $\boldsymbol{Qy}$ *is MSEM-admissible for* $\boldsymbol{L}'\boldsymbol{\beta}$ *if and only if the following conditions hold simultaneously*:

- $\boldsymbol{Q\Sigma} = \boldsymbol{QX}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^-\boldsymbol{X}'$;
- *Either* $\boldsymbol{QX} = \boldsymbol{L}'$, *or* $\boldsymbol{QX}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^-\boldsymbol{J}' + \boldsymbol{J}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Q}' - (1-\alpha)\boldsymbol{J}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^-\boldsymbol{J}' \succeq \boldsymbol{0}$ *does not hold for all* $\alpha \in (0, 1)$ *when* $\boldsymbol{QX} \neq \boldsymbol{L}'$.

2.1.2. *MSE-Admissibility of Homogeneous Linear Estimation.* Being similar to Lemmas 2.1 and 2.2, we give Lemma 2.4 concerning $\mathscr{S}_{\text{H}}^*$ as follows.

Lemma 2.4. *For $\boldsymbol{Qy} \in \mathcal{S}_H$, let $\boldsymbol{Q}^*$ be defined as* (2.1). *Then $MSE(\boldsymbol{Qy}, \boldsymbol{\varrho}) \geq MSE(\boldsymbol{Q}^*\boldsymbol{y}, \boldsymbol{\varrho})$, with equality holding for all $(\boldsymbol{\beta}, \sigma^2)$ iff either $(\boldsymbol{Q} - \boldsymbol{Q}^*)T = \boldsymbol{0}$ or $\mathcal{R}(\boldsymbol{TQ}' - \boldsymbol{NM}) \subseteq \mathcal{R}(\boldsymbol{X})$ is satisfied.*

Since $\mathrm{MSE}(\boldsymbol{Qy}, \boldsymbol{\varrho}) = \mathrm{tr}\{\mathrm{MSEM}(\boldsymbol{Qy}, \boldsymbol{\varrho})\}$, it follows that Lemma 2.4 holds clearly. With notations $\omega(\boldsymbol{Q}) = \mathrm{tr}\{\boldsymbol{\Omega}(\boldsymbol{Q})\}$ and $h = \mathrm{tr}(\boldsymbol{H})$, we give the following theorem concerning the set of MSE-admissible homogeneous linear estimators, of $\boldsymbol{\varrho}$.

Theorem 2.3. *Assume $\boldsymbol{\varrho}$ is estimable under model* (1.1), *and $\boldsymbol{Qy} \in \mathcal{S}_H$. Then $\boldsymbol{Qy} \in \mathcal{A}_{H\text{-}MSE}$ if and only if $\boldsymbol{Qy} \in \mathcal{S}_H^*$ and the following conditions hold simultaneously*:

(1) $\boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}' \preccurlyeq \boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L} + \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-\boldsymbol{XS}^-\boldsymbol{J}'$ $(= \boldsymbol{F}$, *say*);
(2) *Either $\boldsymbol{QX} = \boldsymbol{L}'$, or $\boldsymbol{QX} \neq \boldsymbol{L}'$ but $\mathcal{R}(\boldsymbol{J}) = \mathcal{R}(\boldsymbol{D})$, where the matrix $\boldsymbol{D}$ refers to*

$$\boldsymbol{D} = \boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{J}' + \boldsymbol{F} - \boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}' = \boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-\boldsymbol{XS}^-\boldsymbol{J}' - \boldsymbol{L}'(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{J}'.$$

Proof of Necessity. Let us first show the necessity. Assume $\boldsymbol{Qy} \in \mathcal{A}_{\text{H-MSE}}$, then it is clear that $\boldsymbol{Qy} \in \mathcal{S}_{\text{H}}^*$. For any k-dimensional vector $\boldsymbol{d}$, and any fixed arbitrary scalar $\alpha \in [0, 1]$, denoting $\boldsymbol{q} = \boldsymbol{Q}'\boldsymbol{d}$, $\boldsymbol{m} = \boldsymbol{Md}$, $\boldsymbol{l} = \boldsymbol{Ld}$, and taking $\boldsymbol{p}'_{(\alpha)} = \alpha\boldsymbol{q}' + (1 - \alpha)[\boldsymbol{l}'\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^- + \boldsymbol{m}'\boldsymbol{N}'(\boldsymbol{T}^- - \boldsymbol{T}^-\boldsymbol{XS}^-\boldsymbol{X}'\boldsymbol{T}^-)]$, with the notation $f(\alpha; \boldsymbol{\beta}, \sigma^2) = \omega(\boldsymbol{p}'_{(\alpha)})$, we obtain

$$\begin{aligned} &f(\alpha; \boldsymbol{\beta}, \sigma^2) \\ &\quad = \alpha^2\boldsymbol{\beta}'(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')'(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')\boldsymbol{\beta} + \sigma^2[\alpha^2\boldsymbol{q}'\boldsymbol{X}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{q} + (1-\alpha)^2\boldsymbol{l}'(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{l} \\ &\qquad + 2\alpha(1-\alpha)\boldsymbol{q}'\boldsymbol{X}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{l} - 2\alpha\boldsymbol{q}'\boldsymbol{XS}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{Nm} - 2(1-\alpha)\boldsymbol{l}'\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{Nm}]. \end{aligned}$$

It follows that $\partial^2 f(\alpha; \boldsymbol{\beta}, \sigma^2)/\partial\alpha^2 = 2\boldsymbol{\beta}'(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')'(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')\boldsymbol{\beta} + 2\sigma^2(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')(\boldsymbol{S}^- - \boldsymbol{U})(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')' \geq 0$ for any given pair $(\boldsymbol{\beta}, \sigma^2)$. Note that $\omega(\boldsymbol{q}') = f(1; \boldsymbol{\beta}, \sigma^2)$. Following from the fact (the proof is similar to Theorem 7.2 shown in Wang (1987) and thus omitted here) that $\boldsymbol{Qy} \in \mathcal{A}_{\text{H-MSE}}$ implies $\boldsymbol{q}'\boldsymbol{y}$ is admissible for $\boldsymbol{d}'\boldsymbol{\varrho}$, and observing that $(1 - \alpha^2)\boldsymbol{\beta}'(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')'(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')\boldsymbol{\beta} \geq 0$ holds for any $\alpha \in [0, 1)$, we get $h(\alpha) \leq 0$, where the symbol $h(\alpha)$ stands for

$$\begin{aligned} h(\alpha) &= \{f(1; \boldsymbol{\beta}, \sigma^2) - f(\alpha; \boldsymbol{\beta}, \sigma^2) - (1-\alpha^2)\,\boldsymbol{\beta}'(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')'\,(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')\,\boldsymbol{\beta}\}/[(1-\alpha)\,\sigma^2] \\ &= (1+\alpha)\boldsymbol{q}'\boldsymbol{X}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}' \\ &\quad \times \boldsymbol{q} - (1-\alpha)\boldsymbol{l}'(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{l} - 2\,\alpha\boldsymbol{q}'\boldsymbol{X}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{l} - 2\,(\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{Nm}. \end{aligned}$$

Let α now tend to 1^-, then we get $\boldsymbol{q}'\boldsymbol{X}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{q} \leq (\boldsymbol{q}'\boldsymbol{X} - \boldsymbol{l}')\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{Nm} + \boldsymbol{q}'\boldsymbol{X}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{l}$, or equivalently,

$$\boldsymbol{d}'\boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}'\boldsymbol{d} \leq \boldsymbol{d}'\boldsymbol{Fd}. \tag{2.2}$$

To establish (1), it is sufficient to show that $\boldsymbol{F}' = \boldsymbol{F}$. If not, then $\boldsymbol{D}' \neq \boldsymbol{D}$, which combining the algebraic fact stated in Wang (1987) (Lemma 7.2, p. 205) yields that there is an orthogonal matrix $\boldsymbol{H}$ such that $\mathrm{tr}(\boldsymbol{HD}) > \mathrm{tr}(\boldsymbol{D})$. Taking $\boldsymbol{P} = \boldsymbol{HQ} + (\boldsymbol{I} - \boldsymbol{H})[\boldsymbol{L}'\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^- + \boldsymbol{M}'\boldsymbol{N}'(\boldsymbol{T}^- - \boldsymbol{T}^-\boldsymbol{XS}^-\boldsymbol{X}'\boldsymbol{T}^-)]$, it follows that $\boldsymbol{Py} \in \mathcal{S}_{\text{H}}^*$ and $\boldsymbol{PX} - \boldsymbol{L}' = \boldsymbol{H}(\boldsymbol{QX} - \boldsymbol{L}')$. Further, by direct operations, we obtain

$$\omega(\boldsymbol{P}) = \boldsymbol{\beta}'(\boldsymbol{PX} - \boldsymbol{L}')'(\boldsymbol{PX} - \boldsymbol{L}')\boldsymbol{\beta} + \sigma^2\mathrm{tr}\{\boldsymbol{PX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{P}' - 2\boldsymbol{PXS}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{NM}\}$$

$$= \beta' J' J \beta + \sigma^2 \mathrm{tr}\{QX(S^- - U)X'Q' - 2QXS^- X' T^- NM\} + 2\sigma^2 \mathrm{tr}\{(I - H)D\}$$
$$= \omega(Q) + 2\sigma^2 \mathrm{tr}\{D - HD\} < \omega(Q).$$

This fact contradicts $Qy \in \mathcal{A}_{\text{H-MSE}}$, and therefore $F' = F$, which combining (2.2) holding for any d yields (1); Now, it is clear that $\mathcal{R}(D) \subseteq \mathcal{R}(J)$ since $D' = D$. If $d'D = 0$, then $d'Dd = 0$; that is,

$$d'J(S^- - U)J'd + [d'Fd - d'QX(S^- - U)X'Q'd] = 0. \tag{2.3}$$

Note that the first term of the left side of (2.3) is nonnegative. Together with Eqs. (2.2) and (2.3), it follows that $d'J(S^- - U)J'd = 0$ and the following equality hold simultaneously,

$$d'QX(S^- - U)X'Q'd = d'Fd = d'QX(S^- - U)Ld + d'M'N'T^- XS^- J'd \tag{2.4}$$

Further, $d'J(S^+ - U) = 0$, since we have assumed $\mathcal{R}(U) \subset \mathcal{R}(X') = \mathcal{R}(S)$ in Sec. 1. Thus, one can conclude that $S^+ - U \succcurlyeq 0$ (for the general case, either $X(S^+ - U)X' \succcurlyeq 0$ or $S^+ - P_{X'}UP_{X'} \succcurlyeq 0$ holds), which combining Eq. (2.4) yields $d'JS^- X'T^- NMd = 0$ and

$$d'QX(S^- - U)X'Q'd = d'L'(S^- - U)X'Q'd = d'L'(S^- - U)Ld.$$

Consequently,

$$\begin{aligned}\mathrm{MSE}(q'y, \varrho) - \mathrm{MSE}(p'_{(0)}y, \varrho) = f(1; \beta, \sigma^2) - f(0; \beta, \sigma^2) &= \beta'(q'X - l') \\ &\times (q'X - l')'\beta + \sigma^2\{q'X(S^- - U)X'q \\ &+ 2l'S^- X'T^- Nm - 2q'XS^- X'T^- Nm - l'(S^- - U)l\} \\ = \beta'(q'X - l')(q'X - l')'\beta.\end{aligned}$$

Therefore, $q'X = l'$, or equivalently, $d'QX = d'L$, since $q'y$ is admissible for $d'\varrho$. It follows that the implication $d'D = 0 \Rightarrow d'J = 0$ holds, further, $\mathcal{R}(J) \subseteq \mathcal{R}(D)$, which combining with $\mathcal{R}(D) \subset \mathcal{R}(J)$ yields (1.2). Thus, we complete the proof of the necessity.

Proof of Sufficiency. Assume $Py, Qy \in \mathcal{S}^*_{\mathrm{H}}$ and Q satisfies conditions (1) and (2) simultaneously. We will show in the following that Py is not MSE-superior over Qy:

- $QX = L'$: Noting that $\omega(P) = \beta'(PX - L')'(PX - L')\beta \ + \sigma^2 \mathrm{tr}\{PX(S^- - U)X'P' - 2PXS^- X'T^- NM\}$, $\omega(Q) = \sigma^2 \mathrm{tr}\{L'(S^- - U)L - 2L'S^- X'T^- NM\}$, we obtain $\omega(Q) = \omega(P)$ if $PX = L'$, and $\omega(Q) < \omega(P)$ for some (β_0, σ_0^2) if $PX \neq L'$, which follows from the fact (see Appendix) that for symmetric matrices A, B and scalars a, b, $x'Ax + a \leq x'Bx + b$ holds for any x if and only if $A \preccurlyeq B$ and $a \leq b$.
- $QX \neq L'$:

1. When $PX = L'$, then $[\omega(Q) - \omega(P)]_{\beta=0} \leq -\sigma^2 \mathrm{tr}(D)$ combining condition (1). It is observed that $D \succcurlyeq 0$ and $\mathcal{R}(D) = \mathcal{R}(J) \neq \{0\}$, which means $\mathrm{tr}(D) > 0$, and therefore we obtain $[\omega(Q) - \omega(P)]_{\beta=0} < 0$. Thus, Py is not MSE-superior over Qy.

2. When $\boldsymbol{PX} \neq \boldsymbol{L}'$ and $\boldsymbol{PX} = \boldsymbol{QX}$, then $\omega(\boldsymbol{Q}) = \omega(\boldsymbol{P})$.

3. Provided $\boldsymbol{PX} \neq \boldsymbol{L}'$ and $\boldsymbol{PX} \neq \boldsymbol{QX}$. Owing to the fact stated in the Appendix, we need only to consider when $(\boldsymbol{PX}-\boldsymbol{L}')'(\boldsymbol{PX}-\boldsymbol{L}') \preccurlyeq \boldsymbol{J}'\boldsymbol{J}$. Combining the fact that if $\boldsymbol{A} \succcurlyeq \boldsymbol{0}$ and $\boldsymbol{B} \succcurlyeq \boldsymbol{0}$, then implication $\boldsymbol{A} \preccurlyeq \boldsymbol{B} \Rightarrow \mathscr{R}(\boldsymbol{A}) \subset \mathscr{R}(\boldsymbol{B})$ holds inherently, we get $\mathscr{R}[(\boldsymbol{PX}-\boldsymbol{L}')'] \subset \mathscr{R}(\boldsymbol{J}')$, or, equivalently, $\boldsymbol{PX}-\boldsymbol{L}' = \boldsymbol{CJ}$ with $\boldsymbol{C} = (\boldsymbol{PX}-\boldsymbol{L}')\boldsymbol{J}^+$. Clearly, $\boldsymbol{C}'\boldsymbol{C} \preccurlyeq \boldsymbol{I}$. We shall show $[\omega(\boldsymbol{Q})-\omega(\boldsymbol{P})]_{\boldsymbol{\beta}=\boldsymbol{0}} < 0$ in the following. Actually, noting that $\boldsymbol{0} \preccurlyeq \boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}' \preccurlyeq \boldsymbol{F}$, $\boldsymbol{I}-\boldsymbol{C}'\boldsymbol{C} \succcurlyeq \boldsymbol{0}$ and $\boldsymbol{PX} = \boldsymbol{CQX} + (\boldsymbol{I}-\boldsymbol{C})\boldsymbol{L}'$, and writing $\varpi = [\omega(\boldsymbol{Q})-\omega(\boldsymbol{P})]_{\boldsymbol{\beta}=\boldsymbol{0}}$, we obtain

$$\begin{aligned}
\varpi &= \sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C}'\boldsymbol{C})\boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}'\} - 2\sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{QXS}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{NM}\} \\
&\quad - \sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C})' \times (\boldsymbol{I}-\boldsymbol{C})\boldsymbol{L}'(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L} - 2(\boldsymbol{I}-\boldsymbol{C})'\boldsymbol{CQX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L}\} \\
&\quad + 2\sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{L}'\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{NM}\}, \\
&\leq \sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C}'\boldsymbol{C})\boldsymbol{F}\} - 2\sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{QXS}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{NM}\} \\
&\quad - \sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C})'(\boldsymbol{I}-\boldsymbol{C})\,\boldsymbol{L}'(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L} \\
&\quad - 2(\boldsymbol{I}-\boldsymbol{C})'\boldsymbol{CQX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L}\} + 2\sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{L}'\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{NM}\} \\
&= -\sigma^2\mathrm{tr}\{(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{C})'\}
\end{aligned}$$

by direct operations. Further, one can readily justify that $\varpi \leq 0$ since $\boldsymbol{D} \succcurlyeq \boldsymbol{0}$. On the other hand, $(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{J} = \boldsymbol{J} - (\boldsymbol{PX}-\boldsymbol{L}') = \boldsymbol{QX} - \boldsymbol{PX} \neq \boldsymbol{0}$, which combined with $\mathscr{R}(\boldsymbol{D}) = \mathscr{R}(\boldsymbol{J})$ yields $\varpi < 0$.

The proof of the sufficiency is also finished. □

Corollary 2.4. *Assume ϱ is estimable under model* (1.1). *Then $\varrho^* \in \mathscr{A}_{H\text{-}MSE}$.*

The following theorem gives a concise result of Theorem 2.3.

Theorem 2.3′. *Assume ϱ is estimable under* (1.1), *and $\boldsymbol{Qy} \in \mathscr{S}_H$. Then $\boldsymbol{Qy} \in \mathscr{A}_{H\text{-}MSE}$ if and only if $\boldsymbol{Qy} \in \mathscr{S}^*_H$,* (1) (*stated in Theorem* 2.3) *and $rk(\boldsymbol{J}) = rk(\boldsymbol{D})$* (*say* (3)) *hold.*

Noting $\boldsymbol{D} = \boldsymbol{D}' = \boldsymbol{J}[\boldsymbol{S}^-\boldsymbol{X}'\boldsymbol{T}^-\boldsymbol{NM} - (\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L}']$, Theorem 2.3′ holds clearly. As a consequence, we state the following corollary. For a trivial modified version, one may see Wu (1986).

Corollary 2.5. *Assume $\boldsymbol{L}'\boldsymbol{\beta}$ is linearly estimable under* (1.1). *Then $\boldsymbol{Qy}$ is MSE-admissible for $\boldsymbol{L}'\boldsymbol{\beta}$ if and only if the following conditions hold simultaneously*:

(4) $\boldsymbol{QT} = \boldsymbol{QXS}^-\boldsymbol{X}'$;

(5) $\boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}' \preccurlyeq \boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L}$;

(6) $rk(\boldsymbol{J}) = rk(\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}')$.

Proof. Employing Theorem 2.3′, it follows that the necessary and sufficient conditions for $\mathbf{Q}\boldsymbol{y}$ to be MSE-admissible for $\boldsymbol{L}'\boldsymbol{\beta}$ are (4) (5) and $\mathrm{rk}(\boldsymbol{J}) = \mathrm{rk}(\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L})$, say (7). To establish the corollary, it suffices to show the equivalence (4) (5) (6) $\Leftrightarrow$ (4) (5) (7). Actually, the implication (4) (5) (7) $\Rightarrow$ (4) (5) (6) holds clearly since $\mathscr{R}(\boldsymbol{L}) \subset \mathscr{R}(\boldsymbol{X}')$. Conversely, one can justify that $\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{J}' \preccurlyeq$

$-\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L}$ by (5). This fact combining $\boldsymbol{S}^+ - \boldsymbol{U} \succcurlyeq \boldsymbol{0}$ and $\boldsymbol{L} = P_{X'}\boldsymbol{L}$ would imply that $\mathrm{rk}[\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'] \le \mathrm{rk}[\boldsymbol{J}(\boldsymbol{S}^+ - \boldsymbol{U})] = \mathrm{rk}[\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{J}'] \le \mathrm{rk}[-\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L}] \le \mathrm{rk}[\boldsymbol{J}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}']$, which means (7) holds and thereby (4) (5) (6) $\Rightarrow$ (4) (5) (7). □

From the proving of Corollary 2.5, the condition (7) can be replaced with $\mathrm{rk}(\boldsymbol{J}) = \mathrm{rk}[\boldsymbol{J}(\boldsymbol{S}^+ - \boldsymbol{U})]$, say (8). Further, (4) (5) (6) $\Leftrightarrow$ (4) (5) (7) $\Leftrightarrow$ (4) (5) (8). Specially, if $\boldsymbol{M} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} \succ \boldsymbol{0}$, taking $\boldsymbol{U} = \boldsymbol{0}$, we have the following result obtained by Rao (1976). See also Wang (1987).

Corollary 2.6. *Assume that $\boldsymbol{L}'\boldsymbol{\beta}$ is linearly estimable under the model* (1.1) *with $\boldsymbol{\Sigma} \succ \boldsymbol{0}$. Then $\boldsymbol{Q}\boldsymbol{y}$ is MSE-admissible for $\boldsymbol{L}'\boldsymbol{\beta}$ if and only if the following conditions hold simultaneously*:

(9) $\boldsymbol{Q}\boldsymbol{\Sigma} = \boldsymbol{Q}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^-\boldsymbol{X}'$;

(10) $\boldsymbol{Q}\boldsymbol{\Sigma}\boldsymbol{Q}' \preccurlyeq \boldsymbol{Q}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^-\boldsymbol{L}$.

Remark 2.1. It can be concluded that $\boldsymbol{Q}\boldsymbol{y}$ is linearly sufficient in some sense iff $\mathscr{R}(\boldsymbol{X}, \boldsymbol{N}) \subseteq \mathscr{R}(\boldsymbol{T}\boldsymbol{Q}')$. On the other hand, if $\boldsymbol{Q}\boldsymbol{y}$ is admissible for ϱ in the sense of any version proposed in the present article, then $\mathscr{R}(\boldsymbol{T}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M}) \subseteq \mathscr{R}(\boldsymbol{X})$ is inherently satisfied. We write $\boldsymbol{T}\boldsymbol{Q}' - \boldsymbol{N}\boldsymbol{M} = \boldsymbol{X}\boldsymbol{A}$ for some matrix $\boldsymbol{A}$. Further, $\boldsymbol{T}\boldsymbol{Q}' = (\boldsymbol{X}, \boldsymbol{N})\,(\boldsymbol{A}', \boldsymbol{M}')'$. Together with the above statements, we obtain a necessary condition for $\boldsymbol{Q}\boldsymbol{y}$ to be admissible and linearly sufficient as $\mathscr{R}(\boldsymbol{X}, \boldsymbol{N}) = \mathscr{R}(\boldsymbol{T}\boldsymbol{Q}')$.

2.2. *Admissibility of Inhomogeneous Linear Estimation*

Denote by $\mathscr{A}_{\text{INH-MSEM}}$ and $\mathscr{A}_{\text{INH-MSE}}$ the sets of all MSEM-admissible and MSE-admissible inhomogeneous linear estimators of ϱ, respectively. It follows that $\mathscr{A}_{\text{INH-MSE}} \subset \mathscr{A}_{\text{INH-MSEM}} \subset \mathscr{S}_{\text{INH}}$. In a similar fashion, we obtain the following three theorems concerning $\mathscr{A}_{\text{INH-MSEM}}$ and $\mathscr{A}_{\text{INH-MSE}}$. The proofs are similar to Theorems 2.1, 2.2, 2.3 and are thus omitted here.

Theorem 2.5. *Assume that ϱ is not estimable under the model* (1.1), *and $\boldsymbol{Q}\boldsymbol{y} + \boldsymbol{q} \in \mathscr{S}_{INH}$. Then $\boldsymbol{Q}\boldsymbol{y} + \boldsymbol{q} \in \mathscr{A}_{INH\text{-}MSEM}$ if and only if $\boldsymbol{Q}\boldsymbol{y} \in \mathscr{S}_H^*$.*

Theorem 2.6. *Assume that ϱ is estimable under the model* (1.1), *and $\boldsymbol{Q}\boldsymbol{y} + \boldsymbol{q} \in \mathscr{S}_{INH}$. Then $\boldsymbol{Q}\boldsymbol{y} + \boldsymbol{q} \in \mathscr{A}_{INH\text{-}MSEM}$ if and only if $\boldsymbol{Q}\boldsymbol{y} \in \mathscr{S}_H^*$ and either $\boldsymbol{q} = \boldsymbol{0}$ when $\boldsymbol{Q}\boldsymbol{X} = \boldsymbol{L}'$, or $\tau(\alpha, \boldsymbol{Q}) \not\succcurlyeq \boldsymbol{0}$ for any $\alpha \in (0, 1)$ when $\boldsymbol{Q}\boldsymbol{X} \neq \boldsymbol{L}'$.*

Theorem 2.7. *Assume that ϱ is estimable under the model* (1.1), *and $\boldsymbol{Q}\boldsymbol{y} + \boldsymbol{q} \in \mathscr{S}_{INH}$. Then $\boldsymbol{Q}\boldsymbol{y} + \boldsymbol{q} \in \mathscr{A}_{INH\text{-}MSE}$ if and only if $\boldsymbol{Q}\boldsymbol{y} \in \mathscr{A}_{H\text{-}MSE}$ and $\boldsymbol{q} \in \mathscr{R}(\boldsymbol{J})$.*

3. Concluding Remarks

By virtue of the method utilized in part by Baksalary and Markiewicz (1990), Groß and Markiewicz (1999), and Wu (1988), one can consider the admissibility problem with respect to generalized MSE (GMSE) criterion in the sense that $\mathrm{GMSE}(\tilde{\varrho}, \varrho; \boldsymbol{\Xi}) = \mathscr{E}(\tilde{\varrho} - \varrho)'\boldsymbol{\Xi}(\tilde{\varrho} - \varrho)$ with $\boldsymbol{\Xi}$ being a given $k \times k$ nonzero s.n.n.d. matrix. Accordingly, we have the symbols $\mathscr{A}_{\text{H-GMSE}}$ and $\mathscr{A}_{\text{INH-GMSE}}$. Let us now write

$\boldsymbol{\Xi} = \boldsymbol{\Xi}_0'\boldsymbol{\Xi}_0$ for some $\boldsymbol{\Xi}_0$ satisfying the above representation. Clearly, such $\boldsymbol{\Xi}_0$ may be not unique. Denoting $\widetilde{\boldsymbol{Q}} = \boldsymbol{\Xi}_0\boldsymbol{Q}$, $\widetilde{\boldsymbol{L}} = \boldsymbol{L}\boldsymbol{\Xi}_0'$, $\widetilde{\boldsymbol{M}} = \boldsymbol{M}\boldsymbol{\Xi}_0'$, we find

$$\mathrm{GMSE}(\boldsymbol{Qy}, \varrho(\boldsymbol{L}, \boldsymbol{M}); \boldsymbol{\Xi}) = \mathrm{MSE}\left(\widetilde{\boldsymbol{Q}}\boldsymbol{y}, \varrho\left(\widetilde{\boldsymbol{L}}, \widetilde{\boldsymbol{M}}\right)\right).$$

Thus, the admissibility problem under GMSE may be operated as an admissibility problem under MSE. Being similar to Theorem 2.3, it follows that

Theorem 3.1. *Assume that ϱ is estimable under the model* (1.1). *Then $\boldsymbol{Qy} \in \mathscr{A}_{H\text{-}GMSE}$ if and only if $\mathscr{R}(\boldsymbol{\Sigma Q}'\boldsymbol{\Xi}_0' - \boldsymbol{NM\Xi}_0') \subseteq \mathscr{R}(\boldsymbol{X})$ and the following conditions hold simultaneously*:

- $\boldsymbol{\Xi}_0\boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}'\boldsymbol{\Xi}_0' \preccurlyeq \boldsymbol{\Xi}_0\boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L\Xi}_0' + \boldsymbol{\Xi}_0\boldsymbol{M}'\boldsymbol{N}'\boldsymbol{T}^-\boldsymbol{XS}^-\boldsymbol{J}'\boldsymbol{\Xi}_0'$ $(= \widetilde{\boldsymbol{F}}_0$, *say*);
- *Either* $\boldsymbol{\Xi}_0\boldsymbol{QX} = \boldsymbol{\Xi}_0\boldsymbol{L}'$, *or* $\boldsymbol{\Xi}_0\boldsymbol{QX} \neq \boldsymbol{\Xi}_0\boldsymbol{L}'$ *but* $\mathscr{R}(\widetilde{\boldsymbol{J}}_0) = \mathscr{R}(\widetilde{\boldsymbol{D}}_0)$ *with* $\widetilde{\boldsymbol{J}}_0 = \boldsymbol{\Xi}_0\boldsymbol{J}$, *where the matrix* $\widetilde{\boldsymbol{D}}_0$ *refers to* $\widetilde{\boldsymbol{D}}_0 = \widetilde{\boldsymbol{J}}_0(\boldsymbol{S}^- - \boldsymbol{U})\widetilde{\boldsymbol{J}}_0' + \widetilde{\boldsymbol{F}}_0 - \boldsymbol{\Xi}_0\boldsymbol{QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}'\boldsymbol{\Xi}_0'$.

Obviously, the above conclusion is invariant regarding the choice of $\boldsymbol{\Xi}_0$. Actually, we have

Theorem 3.2. *Assume that ϱ is estimable under the model* (1.1). *Then $\boldsymbol{Qy} \in \mathscr{A}_{H\text{-}GMSE}$ if and only if $\mathscr{R}(\boldsymbol{\Sigma Q}'\boldsymbol{\Xi} - \boldsymbol{NM\Xi}) \subset \mathscr{R}(\boldsymbol{X})$ and the following conditions hold simultaneously*:

- $\boldsymbol{\Xi QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}'\boldsymbol{\Xi} \preccurlyeq \boldsymbol{\Xi QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{L\Xi} + \boldsymbol{\Xi M}'\boldsymbol{N}'\boldsymbol{T}^-\boldsymbol{XS}^-\boldsymbol{J}'\boldsymbol{\Xi}$ $(= \widetilde{\boldsymbol{F}}$, *say*);
- *Either* $\boldsymbol{\Xi QX} = \boldsymbol{\Xi L}'$, *or* $\boldsymbol{\Xi QX} \neq \boldsymbol{\Xi L}'$ *but* $\mathscr{R}(\widetilde{\boldsymbol{J}}) = \mathscr{R}(\widetilde{\boldsymbol{D}})$ *with* $\widetilde{\boldsymbol{J}} = \boldsymbol{\Xi J}$, *where the matrix* $\widetilde{D}$ *refers to* $\widetilde{\boldsymbol{D}} = \widetilde{\boldsymbol{J}}(\boldsymbol{S}^- - \boldsymbol{U})\widetilde{\boldsymbol{J}}' + \widetilde{\boldsymbol{F}} - \boldsymbol{\Xi QX}(\boldsymbol{S}^- - \boldsymbol{U})\boldsymbol{X}'\boldsymbol{Q}'\boldsymbol{\Xi}$.

With similar fashion, we can deal with $\mathscr{A}_{\mathrm{INH\text{-}GMSE}}$ utilizing Theorem 2.7 and omit it here.

Appendix

Here, we offer an alternative proof of the following two algebraic facts: For symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ of suitable orders, the statements below hold: (1) $\boldsymbol{A} + \boldsymbol{Bxx}'\boldsymbol{B}' \preccurlyeq \boldsymbol{C} + \boldsymbol{Dxx}'\boldsymbol{D}'$ holds for any $\boldsymbol{x}$ if and only if $\boldsymbol{A} \preccurlyeq \boldsymbol{C}$ and $\boldsymbol{B} = \alpha\boldsymbol{D}$ for some $\alpha \in [-1, 1]$; (2) $\boldsymbol{x}'\boldsymbol{Ax} + \boldsymbol{a} \leq \boldsymbol{x}'\boldsymbol{Cx} + \boldsymbol{b}$ holds for any $\boldsymbol{x}$ if and only if $\boldsymbol{A} \preccurlyeq \boldsymbol{C}$ and $\boldsymbol{a} \leq \boldsymbol{b}$.

Proof of (1). The sufficiency holds inherently. We need only to show the necessity. Actually, provided that $\boldsymbol{A} + \boldsymbol{Bxx}'\boldsymbol{B}' \preccurlyeq \boldsymbol{C} + \boldsymbol{Dxx}'\boldsymbol{D}'$ holds for any $\boldsymbol{x}$, we can get $\boldsymbol{A} \preccurlyeq \boldsymbol{C}$ by taking $\boldsymbol{x} = \boldsymbol{0}$, and $\boldsymbol{Bxx}'\boldsymbol{B}' \preccurlyeq \boldsymbol{Dxx}'\boldsymbol{D}'$ for any $\boldsymbol{x}$. Otherwise, there is some $\boldsymbol{t}_0$ such that $\boldsymbol{t}_0'\boldsymbol{Bxx}'\boldsymbol{B}'\boldsymbol{t}_0 > \boldsymbol{t}_0'\boldsymbol{Dxx}'\boldsymbol{D}'\boldsymbol{t}_0$, taking k_0 large enough and putting $\boldsymbol{y}_0 = k_0\boldsymbol{x}$, we get $\boldsymbol{t}_0'(\boldsymbol{A} + \boldsymbol{By}_0\boldsymbol{y}_0'\boldsymbol{B}')\boldsymbol{t}_0 > \boldsymbol{t}_0'(\boldsymbol{C} + \boldsymbol{Dy}_0\boldsymbol{y}_0'\boldsymbol{D}')\boldsymbol{t}_0$, which contradicts $\boldsymbol{A} + \boldsymbol{Bxx}'\boldsymbol{B}' \preccurlyeq \boldsymbol{C} + \boldsymbol{Dxx}'\boldsymbol{D}'$. Because $\boldsymbol{Bxx}'\boldsymbol{B}' \preccurlyeq \boldsymbol{Dxx}'\boldsymbol{D}'$ holds for any $\boldsymbol{x}$, it follows that $\boldsymbol{B}'\boldsymbol{B} \preccurlyeq \boldsymbol{D}'\boldsymbol{D}$ and therefore $\mathscr{R}(\boldsymbol{B}') \subset \mathscr{R}(\boldsymbol{D}')$. Further, $\boldsymbol{B} = \boldsymbol{FD}$ for some matrix $\boldsymbol{F}$. On the other hand, $\boldsymbol{Bxx}'\boldsymbol{B}' \preccurlyeq \boldsymbol{Dxx}'\boldsymbol{D}'$ implies $\mathscr{R}(\boldsymbol{Bx}) = \mathscr{R}(\boldsymbol{Bxx}'\boldsymbol{B}') \subset \mathscr{R}(\boldsymbol{Dxx}'\boldsymbol{D}') = \mathscr{R}(\boldsymbol{Dx})$, i.e., there exists α_x such that $\boldsymbol{Bx} = \alpha_x\boldsymbol{Dx}$, which combined with $\boldsymbol{B} = \boldsymbol{FD}$ yields $\boldsymbol{FDx} = \alpha_x\boldsymbol{Dx}$. Further, $\boldsymbol{Dx}$ is the eigenvector of $\boldsymbol{F}$ with respect to the eigenvalue, α_x, of $\boldsymbol{F}$ if $\boldsymbol{Dx} \neq \boldsymbol{0}$.

If there exist x and y such that $Dx \neq 0$, $Dy \neq 0$ and $\alpha_x \neq \alpha_y$, then Dx and Dy are linearly independent. Noting that $\alpha_x Dx + \alpha_y Dy = FDx + FDy = FD(x + y) = \alpha_{x+y} Dx + \alpha_{x+y} Dy$, or equivalently, $(\alpha_x - \alpha_{x+y})Dx + (\alpha_y - \alpha_{x+y})Dy = 0$, we get $\alpha_x = \alpha_{x+y} = \alpha_y$, which contradicts $\alpha_x \neq \alpha_y$. Thus, for any x and y, if $Dx \neq 0$ and $Dy \neq 0$, then $\alpha_x = \alpha_y = \alpha$, say. Clearly, $Bx = \alpha Dx$ holds for any x. Again by $Bxx'B' \preccurlyeq Dxx'D'$ for any x, we have $\alpha \in [-1, 1]$. □

Proof of (2). It is with similar fashion.

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