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A fuzzy logic system based on Schweizer-Sklar t-norm

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Abstract Based on the Schweizer-Sklar t-norm, a fuzzy logic system UL^* is established, and its soundness theorem and completeness theorem are proved. The following facts are pointed out: the well-known formal system $SBL_{\sim}$ is a semantic extension of UL^*; the fuzzy logic system $IMTL_{\Delta}$ is a special case of UL^* when two negations in UL^* coincide. Moreover, the connections between the system UL^* and some fuzzy logic formal systems are investigated. Finally, starting from the concepts of "the strength of an 'AND' operator" by R.R. Yager and "the strength of fuzzy rule interaction" by T. Whalen, the essential meaning of a parameter p in UL^* is explained and the use of fuzzy logic system UL^* in approximate reasoning is presented.

Keywords: t-norm, fuzzy logic system UL^*, completeness, UL^*-algebras, approximate reasoning.

In recent years, the basic research of fuzzy logic and fuzzy reasoning is growing actively day by day, such as the basic logic system BL proposed by Hajek[1]; fuzzy logic system MTL proposed by Esteva and Godo[2]; fuzzy reasoning, implication operators and fuzzy quantifiers were deeply investigated by Ying[3,4]. Moreover, Wang[5−7] combined fuzzy logic with fuzzy reasoning and built up the formal deductive system L^*, created fuzzy reasoning triple I method which is superior to CRI method, and so on.

In fuzzy logic, how to choose implication operators has direct influence on the result of the fuzzy reasoning. Because the t-norm reflects the property of the logic connective "AND" to a large extent, most of the implication operators chosen in the fuzzy logic systems are related to certain t-norms. However, in order to describe the flexibility in human's logical reasoning, choosing some parametric t-norms may be more reasonable. In fact, Klement and Navara[8] had studied the fuzzy logic system based on the parametric t-norm—Frank t-norm. Wu[9] and Wang[10] studied the parametric Kleene system and H_α system respectively. Moreover, the residual implications induced by the Schweizer-Sklar t-norm were studied by Whalen[11], where the parameter p is connected with the strength of the interaction among fuzzy rules. These methods and results are the bases of investigating the fuzzy logic system based on the Schweizer-Sklar t-norm in the present paper.

This paper is organized as follows. The properties of the Schweizer-Sklar t-norm and its residual implication are briefly reviewed in section 1. The formal deductive system UL^* is proposed in section 2, where the soundness theorem of UL^* is proved. Then the completeness theorem of system UL^* is proved in section 3 by introducing the notion of UL^*-algebras and constructing the filter theory of UL^*-algebras. In section 4, the connections between the system UL^* and some other fuzzy logic formal systems are established. And the significance of the fuzzy logic system UL^* and its application in approximate reasoning are presented.

It should be pointed out that in the fuzzy logic system UL^*, the logic negation "$-$" is involutive, but "$-$" and "$\neg$" stand alone from each other, where "$\neg$" is defined via the implication $\to_p$ as $\neg A = A \to_p 0$. In addition, although there is an operator $\wedge$ in UL^*, it is not "logic and", another operator $\&_p$ is actually the abstract of "logic and". Lately the formalization theorem of the general fuzzy reasoning was given in ref. [12], and it reflects the academic idea of combining fuzzy logic with fuzzy reasoning. In ref. [12], an operator "$\otimes$", which corresponds to the operator $\&_p$ in this paper (reflects the features of t-norm), was introduced and investigated in details. By giving a reasonable explanation to the parameters in the parametric t-norms, the flexibility in fuzzy reasoning is reflected (in fact, many scholars are studying the parametric fuzzy logic controls, as seen in refs. [13—16]), this may provide a new way to combine fuzzy logic with fuzzy reasoning.

1 Schweizer-Sklar t-norm and its residual implication

In refs. [17, 18] (see also refs. [11, 19]), Schweizer and Sklar proposed the definitions of the t-norm as follows:

$$T_p^{SS}(x,y) = \begin{cases} (x^{-p}+y^{-p}-1)^{-\frac{1}{p}}, & p\in(0,\infty), \\ \begin{cases} (x^{-p}+y^{-p}-1)^{-\frac{1}{p}}, & x^{-p}+y^{-p}\geqslant 1, \\ 0, & \text{otherwise}. \end{cases} & p\in(-\infty,0), \end{cases}$$

Note that $(x^{-p}+y^{-p}-1)$ may be undefined when $p>0$, for example while $x=0$ or $y=0$. In this case, we denote $T_p^{SS}(x,y)=0$. When $p\to-\infty$, $p\to 0$, $p\to\infty$, Schweizer-Sklar t-norms are defined as follows respectively:

$$T_{-\infty}^{SS}(x,y) = \begin{cases} 0, & (x,y)\in[0,1)\times[0,1), \\ \min(x,y), & \text{otherwise}, \end{cases}$$

$$T_{\infty}^{SS}(x,y)=\min(x,y), \quad T_0^{SS}(x,y)=xy.$$

Definition 1.1[5]. Let P be a partial order set. An adjoint couple $(\otimes, \to)$ on P is defined as a couple $(\otimes, \to)$ of binary operations on P such that:

(M0) $\otimes$ is isotone on $P\times P$;

(R0) $\to$ is antitone in the first and isotone in the second variable;

(A) $a\otimes b\leqslant c$ if and only if $a\leqslant b\to c$ for all a, b, $c\in P$.

The operator $\to$ which corresponds to a t-norm $\otimes$ is called a residual implication. It is easy to prove that the residual implication corresponding to the Schweizer-Sklar t-norm is defined as follows (see ref. [11]):

$$I_p^{SS}(x,y)=\min(1,(1-x^{-p}+y^{-p})^{-\frac{1}{p}}),\quad p\in(-\infty,0)\cup(0,\infty),$$

$$I_0^{SS}(x,y)=\begin{cases}1, & x\leqslant y,\\ y/x, & \text{otherwise},\end{cases}\qquad I_\infty^{SS}(x,y)=\begin{cases}1, & x\leqslant y,\\ y, & \text{otherwise},\end{cases}$$

$$I_{-\infty}^{SS}(x,y)=\begin{cases}y, & x=1,\\ 1, & \text{otherwise}.\end{cases}$$

Remark. For several special points while $p\in(-\infty, 0)\cup(0, \infty)$ was not discussed in ref. [11], which caused undefined in the last formula of the residual implication. Thus, we make the following analysis:

(1) For $p>0$, we have $I_p(x, 0)=\begin{cases}1 & x=0,\\ 0 & x\neq 0.\end{cases}$

In fact, by the definition of the residual implication we have $0\to 0=\sup\{z \mid T_p(0, z)\leqslant 0\}=\sup\{z|0\leqslant 0\}=1$. If $x\neq 0$, then $x\to 0=\sup\{z \mid T_p(x, z)\leqslant 0\}=\sup\{z| T_p(x, z)=0\}=0$, because only $z=0$ satisfies $T_p(x, z)=0$ when $p>0$, $x\neq 0$.

(2) For $(1-x^{-p}+y^{-p})$, it is possible to be non-positive (for example, $1-0.1^{-2}+0.5^{-2} = -95$, $1-0.2^{-1}+0.25^{-1} = 0$). Then, according to the definition of the residual implication we have $I_p(x, y)=1$ (in fact, in this case we have $x<y$).

Of course, we can still adopt one expression to denote a residual implication adjoint with the Schweizer-Sklar t-norm, i.e.

$$I_p(x,y)=\min(1,\ (1-x^{-p}+y^{-p})^{-\frac{1}{p}}),\ p\in(-\infty,0)\cup(0,\infty).$$

But the following stipulations need to be made (when $p>0$): $I_p(0, 0)=1$; and $I_p(x, 0)=0$ when $x\neq 0$; $I_p(x, y)=1$ when $(1-x^{-p}+y^{-p})$ is non-positive.

In the following, the Schweizer-Sklar t-norms are denoted by $\otimes_p$, and their corresponding residual implications are denoted by $\to_p$. Since $(\otimes_p, \to_p)$ is an adjoint couple, using Proposition 7.2.3, Theorem 7.2.8 and 7.2.10 in ref. [5] we get

Theorem 1.2. For any real number p, the following properties hold ($\forall x,y,z\in[0,1]$):

(SS1) $x\otimes_p y=y\otimes_p x$,

(SS2) $(x\otimes_p y)\otimes_p z=x\otimes_p(y\otimes_p z)$,

(SS3) $1\to_p x=x, 1\otimes_p x=x$,

(SS4) $x\leqslant y$ iff $x\to_p y=1$,

(SS5) $x \to_p y \leqslant (z \to_p x) \to_p (z \to_p y)$,

(SS6) $(x \otimes_p y) \to_p z = x \to_p (y \to_p z)$,

(SS7) $a \otimes_p (\vee_i x_i) = \vee_i (a \otimes_p x_i)$, $a \to_p \wedge_i y_i = \wedge_i (a \to_p y_i)$, $(\vee_i y_i) \to_p a = \wedge_i (y_i \to_p a)$,

(SS8) $x \to_p y \leqslant (x \otimes_p z) \to_p (y \otimes_p z)$,

(SS9) $(x \to_p y) \vee (y \to_p x) = 1$, $(x \otimes_p (x \to_p y)) \leqslant x \wedge y$, $(x \to_p y) \to_p z \leqslant ((y \to_p x) \to_p z) \to_p z$,

where $\vee$, $\wedge$ represent the supremum and the infimum, respectively.

Now, we introduce a 0–1 projective operator "Δ" and a logic negative operator "–" as follows:

$$\Delta x = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad -x = 1 - x,\ \forall x \in [0,1].$$

It is easy to prove the following results:

Theorem 1.3. For any real number p, we have ($\forall x, y, z \in [0,1]$):

(SS10) $\Delta x \vee (\Delta x \to_p 0) = 1$,

(SS11) $\Delta(x \vee y) \leqslant \Delta(x) \vee \Delta(y)$,

(SS12) $\Delta x \leqslant x$, $\Delta 1 = 1$,

(SS13) $\Delta x \leqslant \Delta(\Delta x)$,

(SS14) $\Delta(x \to_p y) \leqslant \Delta x \to_p \Delta y$,

(SS15) $-(-x) = x$,

(SS16) $x \leqslant y \Rightarrow (-y) \leqslant (-x)$,

(SS17) $\Delta(x \to_p y) = \Delta((-y) \to_p (-x))$.

Remark. Generally, $\Delta x \neq (-x) \to_p 0$. For example, let $p = -2$, $x = 0.4$. Then $\Delta x = 0$, $(-x) \to_p 0 = 0.8$. This means that the projective operator "Δ" is not defined by $\Delta x = \neg\sim x$, i.e., $\Delta x = (-x) \to 0$, which is the definition of a projective operator in ref. [20].

2 Fuzzy logic system UL* and its soundness theorem

Based on the Schweizer-Sklar t-norm and its residual implication, we introduce the fuzzy logic system UL* as follows. In fact, fuzzy logic system UL* can be regarded as an extension of the formal deductive system MTL by adding a new involutive negation "–" and a projective operator "Δ".

Definition 2.1. Let S be the set of all propositional variables, and $F(S)$ be the ($-$, Δ, $\&_p$, $\to_p$, $\wedge$)-type free algebra generated by S, where $-$ and Δ are unary connectives, and $\&_p$, $\to_p$, $\wedge$ are binary connectives, 0 is a truth constant. The fuzzy logic system UL* consists of the following axioms and inference rules:

Axioms:

(UL1) $(A \to_p B) \to_p ((B \to_p C) \to_p (A \to_p C))$,

(UL2) $(A\&_pB)\to_pA$,
(UL3) $(A\&_pB)\to_p(B\&_p A)$,
(UL4) $(A\wedge B)\to_pA$,
(UL5) $(A\wedge B)\to_p(B\wedge A)$,
(UL6) $(A\&_p(A\to_pB))\to_p(A\wedge B)$,
(UL7) $((A\&_pB)\to_pC)\to_p(A\to_p(B\to_pC))$,
(UL8) $(A\to_p(B\to_pC))\to_p((A\&_p B)\to_pC)$,
(UL9) $((A\to_p B)\to_pC)\to_p(((B\to_p A)\to_pC)\to_pC)$,
(UL10) $0\to_pA$,
(UL11) $(A\to_p (--A))\ \&_p((--A)\to_pA)$,
(UL12) $\Delta A\vee(\neg\Delta A)$,
(UL13) $\Delta(A\vee B)\to_p(\Delta A\vee\Delta B)$,
(UL14) $\Delta A\to_pA$,
(UL15) $\Delta A\to_p\Delta\Delta A$,
(UL16) $\Delta(A\to_pB)\to_p(\Delta A\to_p\Delta B)$,
(UL17) $\Delta(A\to_pB)\to_p\Delta((-B)\to_p(-A))$,

where, $\neg A$ is an abbreviation of $A\to_p0$, $A\vee B$ is an abbreviation of $((A\to_pB)\to_pB)\wedge((B\to_pA)\to_pA)$.

Inference rules:

MP rule—from A and $A\to_pB$ infer B.

Necessity inference rule—from A infer ΔA.

A system constructed by $F(S)$, axioms (UL1)—(UL17) and the above inference rules is called a fuzzy logic system UL^*.

In a natural manner[5], we can introduce the concepts in the system UL^* such as proof, theorem, deduction from a formula set Γ, Γ-consequence. Similar to the way in ref. [5], we use the notations $\vdash A$, $\Gamma\vdash A$ to represent that A is a theorem of system UL^* and a Γ-consequence respectively.

We may easily prove the following results.

Theorem 2.2. The formulas with the following forms are the theorems of the system UL^*:

(T1) $A\to_p(B\to_pA)$,
(T2) $(A\to_p (B\to_p C))\to_p (B\to_p (A\to_p C))$,
(T3) $A\to_p A$,
(T4) $(B\to_pC)\to_p(A\&_pB\to_p A\&_p C)$,
(T5) $A\&_p(B\&_pC)\to_p(A\&_pB)\&_pC$, $(A\&_pB)\&_pC\to_pA\&_p(B\&_pC)$,
(T6) $(B\to_p C)\to_p ((A\to_p B)\to_p (A\to_p C))$,
(T7) $(--A)\to_p A$, $A\to_p (--A)$,
(T8) $A\to_p(A\vee B)$, $B\to_p(A\vee B)$,
(T9) $(A\&_pB)\to_p(A\wedge B)$, $A\to_p (A\wedge A)$,
(T10) $(A\to_p B)\vee(B\to_p A)$,
(T11) $(A\wedge B)\to_p(B\wedge A)$, $(A\vee B)\to_p(B\vee A)$,
(T12) $((A\to_pC)\wedge(B\to_pC))\to_p((A\vee B)\to_pC)$,

(T13) $((A\to_p B)\wedge(A\to_p C))\to_p(A\to_p(B\wedge C))$.

Theorem 2.3. In the system UL*, the following properties hold:

(T14) If $\vdash B\to_p C$, then $\vdash (A\to_p B)\to_p (A\to_p C)$.

(T15) If $\vdash A\to_p B$, then $\vdash (B\to_p C)\to_p (A\to_p C)$.

(T16) If $\Gamma=\{A\to_p B, B\to_p C\}$, then $\Gamma\vdash(A\to_p C)$.

(T17) If $\vdash A\to_p B$, then $\vdash (-B)\to_p (-A)$.

(T18) $\vdash A\to_p (B\to_p (A\wedge B))$.

(T19) If $\Gamma=\{A\to_p C, B\to_p C\}$, then $\Gamma\vdash (A\vee B)\to_p C$.

Definition 2.4. Let A, $B\in F(S)$. If $\vdash A\to_p B$ and $\vdash B\to_p A$, we call that A and B is provable equivalent, denoted by $A\approx B$.

Theorem 2.5. The relation $\approx$ is an equivalent relation on $F(S)$.

Theorem 2.6. The provable equivalent relation $\approx$ is a $(-, \Delta, \&_p, \to_p, \wedge)$-type congruence relation on $F(S)$, that is,

i) If $A\approx B$, then $(-A)\approx(-B)$.

ii) If $A\approx B$, then $\Delta A\approx\Delta B$.

iii) If $A\approx B$ and $C\approx D$, then $A\&_p C\approx B\&_p D$.

iv) If $A\approx B$ and $C\approx D$, then $A\to_p C\approx B\to_p D$.

v) If $A\approx B$ and $C\approx D$, then $A\wedge C\approx B\wedge D$.

Theorem 2.7. In the system UL*, the following De Morgan laws hold:

(T20) $\neg(A\wedge B)\approx(\neg A)\vee(\neg B)$.

(T21) $\neg(A\vee B)\approx(\neg A)\wedge(\neg B)$.

(T22) $-(A\wedge B)\approx(-A)\vee(-B)$.

(T23) $-(A\vee B)\approx(-A)\wedge(-B)$.

Theorem 2.8. For a provable equivalent relation $\approx$ on $F(S)$, the following properties hold:

(T24) $A\&_p(B\&_p C)\approx(A\&_p B)\&_p C$, $(A\&_p B)\to_p C\approx A\to_p(B\to_p C)$.

(T25) $A\vee(B\vee C)\approx(A\vee B)\vee C$, $A\wedge(B\wedge C)\approx(A\wedge B)\wedge C$.

(T26) $\Delta A\approx\Delta(A\&_p A)$.

(T27) $\Delta A\approx\Delta A\&_p\Delta A$.

(T28) $\Delta(A\&_p B)\approx\Delta A\&_p\Delta B$.

Definiton 2.9. A function v: $F(S)\to[0, 1]$ is called a SS-valuation of $F(S)$, if it satisfies (where p is any real number and follows the stipulations made in section 1) the following conditions:

(i) $v(-A)=1-v(A)$;

(ii) $v(\Delta A)=\Delta(v(A))$, where $\Delta x=\begin{cases}1, & \text{if } x=1\\ 0, & \text{otherwise}\end{cases}$, $x\in[0,1]$;

(iii) $v(A\&_p B)=\max(v(A)^{-p}+v(B)^{-p}-1)^{-\frac{1}{p}},0)$;

(iv) $v(A\to_p B)=\min((1-v(A)^{-p}+v(B)^{-p})^{-\frac{1}{p}},1)$;

(v) $v(A\wedge B)=\min\{v(A), v(B)\}$.

We can define the operators $-$, Δ, $\&_p$, $\rightarrow_p$, $\wedge$ on [0,1], such that a SS-valuation of $F(S)$ is a ($-$, Δ, $\&_p$, $\rightarrow_p$, $\wedge$)-type homomorphism from $F(S)$ to [0,1].

The set of all SS-valuations of $F(S)$ is denoted by Ω.

Definition 2.10. Let A be a formulae in $F(S)$. A is called a tautology, denoted by $\models A$, if $v(A)=1$ for every $v\in\Omega$. If $A\rightarrow_p B$ and $B\rightarrow_p A$ are tautologies, then A and B is called logically equivalent. It is easy to verify that A and B is logically equivalent if and only if $v(A)=v(B)$ for every $v\in\Omega$.

The following theorem shows that the syntax of the system UL^* is sound with respect to the semantic Ω.

Theorem 2.11 (Soundness Theorem of UL^*). Every theorem of the system UL^* is a tautology, i.e. for any $A\in F(S)$, if $\vdash A$, then $\models A$.

Proof. According to Theorem 1.2 and Theorem 1.3, every axiom of the system UL^* is a tautology. Moreover, MP rule and Necessity rule preserve the tautologies. In fact, similar to Theorem 4.1.1 in ref. [5], we can prove that MP rule preserves the tautologies. Now, let A be a tautology. Then $v(A)=1$ for every $v\in\Omega$. Thus, $v(\Delta A)=\Delta(v(A))=\Delta 1=1$, this means that ΔA is a tautology, i.e. Necessity rule preserves the tautologies. Hence, for any $A\in F(S)$, $\vdash A$ implies $\models A$.

Remark. (1) In ref. [20], the authors introduced a logic system $SBL_{\sim}$ with an involutive negation "$\sim$" which is similar to "$-$" in this paper. For $SBL_{\sim}$, it is based on the strict continuous t-norm, and there the 0−1 projective operator "Δ" is defined by two negation operators, i.e. $\Delta\varphi=\neg(\sim\varphi)$. But, in system UL^*, $-$, $\neg$ and Δ are independent.

(2) The logic system IMTL was introduced in ref. [2], and it has only one negation that is involutive. We can establish the extended system $IMTL_{\Delta}$ by adding a projective operator "Δ" to IMTL using the similar way extended from the system BL to the system BL_{Δ} in ref. [1]. It is worthy to point out that the system UL^* just reduces to the system $IMTL_{\Delta}$ when the two negations in UL^* coincide.

3 UL^*-algebra and the completeness theorem of fuzzy logic system UL^*

It is easy to show that $[F]=F(S)/\approx=\{[A]\mid A\in F(S)\}$ is a ($-$, Δ, $\&_p$, $\rightarrow_p$, $\wedge$)-type quotient algebra, where $[A]=\{B\in F(S)\mid A\approx B\}$. Based on the quotient algebra [F], we introduce a UL^*-algebra and establish its filter theory. By the prime filter theorem of UL^*-algebra, we prove the completeness theorem of UL^*.

Definition 3.1. Let M be a ($-$, Δ, $\&_p$, $\rightarrow_p$, $\wedge$)-type algebra. M is called a UL^*-algebra, if the following conditions hold:

i) (M, $\&_p$, $\rightarrow_p$, $\wedge$, $\vee$, 0, 1) is a residuated lattice with an order $\leqslant$, where 1 is the largest element and 0 is the least element, $\vee$ is defined by

(P1) $a\vee b=((a\rightarrow_p b)\rightarrow_p b)\wedge((b\rightarrow_p a)\rightarrow_p a)$, $\forall a, b, c\in M$.

And, $\vee$ satisfies

(P2) $(a\rightarrow_p b)\vee(b\rightarrow_p a)=1$, $\forall a, b, c\in M$.

ii) "$-$" is an order-reversing involution on M with respect to $\leqslant$, that is, $a\leqslant b\Rightarrow(-b)$

$\leqslant(-a)$, and $(--a)=a$ for any $a\in M$.

iii) "Δ" is a unary operation on M, and it satisfies ($\forall a, b, c\in M$):

(P3) $\Delta a\vee(\neg(\Delta a))=1$,

(P4) $\Delta(a\vee b)\leqslant(\Delta a)\vee(\Delta b)$,

(P5) $\Delta a\leqslant a$,

(P6) $\Delta a\leqslant\Delta\Delta a$,

(P7) $\Delta(a\rightarrow_p b)\leqslant(\Delta a)\rightarrow_p(\Delta b)$,

(P8) $\Delta(a\rightarrow_p b)=\Delta((-b)\rightarrow_p(-a))$,

(P9) $\Delta 1=1$.

where $\neg a= a\rightarrow_p 0$, $\forall a\in M$.

It is easy to verify that $[F]=F(S)/\approx$ is a UL*-algebra.

Theorem 3.2. Let M be a UL*-algebra. Then the following properties hold ($\forall a, b, c\in M$):

(P10) If $a\leqslant b$, then $(-b)\leqslant(-a)$.

(P11) $(-1)=0$, $(-0)=1$.

(P12) If $a\leqslant b$, then $\Delta a\leqslant\Delta b$.

(P13) $\Delta a=\Delta\Delta a$.

(P14) $\Delta a\&_p\Delta(\neg a)=0$.

(P15) $\Delta a = 1$ if and only if $a=1$.

(P16) $\Delta a\&_p\Delta b=\Delta(a\&_p b)$.

(P17) $\Delta(-a) = \Delta(\neg a)$.

Remark. (1) The proof of Theorem 3.2 is similar to Lemma 4 and Lemma 6 in ref. [20], but we have to revise some place. For example, the proof of (P11) must be revised as follows (because A~2 in ref. [20] does not hold for a UL*-algebra):

Since 1 is the largest element, then $(-0)\leqslant 1$. Using (P10), $(-1)\leqslant-(-0)=0$. Thus $(-1)=0$, and $(-0)=-(-1)=1$. Moreover, (P16) can be proved similar to Theorem 2.8 (T28) in this paper.

(2) (P17) is not the same as Lemma 6(10) in ref. [20], i.e. generally, $\Delta(-a)\neq\neg a$. For example, let $M=[0,1]$, "–" is defined by $(-x)=1-x$, Δ is a 0–1 projective operator, $\&_p$ is a Schweizer-Sklar t-norm and $\rightarrow_p$ is its residual implication, where $p=-2$, $\wedge$=min. Then $(M, -, \Delta, \&_p, \rightarrow_p, \wedge)$ is a UL*-algebra, but $\Delta(-0.6)=0\neq 0.8=\neg 0.6$. From this example, we also know that $\Delta x=\neg(-x)$ does not hold in a UL*-algebra, because $\Delta(0.4)=0\neq 0.8=\neg(-0.4)$.

It is easy to prove that the class of all UL*-algebras is a variety of algebras. Based on the theory of the universal algebra, we have

Theorem 3.3. The class of all UL*-algebra is closed under subalgebras, homomorphism images and products.

Definition 3.4. Let M be a UL*-algebra. A nonempty subset F is called a filter of M, if it satisfies

(F1) If $a, b\in F$, then $a\&_p b\in F$.

(F2) If $a\in F$, $a\leqslant b$, then $b\in F$.

(F3) If $a\in F$, then $\Delta a\in F$.

(F4) If $a\to_p b\in F$, then $(-b)\to_p(-a)\in F$.

A filter is said to be prime if it satisfies

(F5) for any $a, b\in M$, $a\to_p b\in F$ or $b\to_p a\in F$.

Remark. Since the equation $\Delta x=\neg(-x)$ does not hold in a UL*-algebra, then the condition (F3) above is independent, i.e. it cannot be inferred from other conditions.

Theorem 3.5. Let M be a UL*-algebra and F a nonempty subset of M. Then F is a filter of M if and only if it satisfies

(F1′) $1\in F$.

(F2′) If $a\in F$, $a\to_p b\in F$, then $b\in F$.

(F3) If $a\in F$, then $\Delta a\in F$.

(F4) If $a\to_p b\in F$, then $(-b)\to_p(-a)\in F$.

Similar to Lemma 7, Lemma 8 in ref. [20], we can prove the following theorem.

Theorem 3.6. Let M be a UL*-algebra. Then

1) If M is linearly ordered, then $\Delta a=0$ for all $a\in M-\{1\}$.

2) The relation $\equiv_F$ is a congruence relation over M, if $\equiv_F$ is defined as follows:

$$a\equiv_F b \text{ if and only if } (a\to_p b\in F,\ b\to_p a\in F), \text{ where } F \text{ is a filter of } M.$$

3) The quotient algebra $M/\equiv_F$ of M with respect to $\equiv_F$ is a UL*-algebra.

4) The quotient algebra $M/\equiv_F$ is linearly ordered if and only if F is a prime filter of M.

5) If M is linearly ordered, then M is simple, that is, the only filters of M are $\{1\}$ and itself.

Theorem 3.7. Let M be a UL*-algebra, $a\in M-\{1\}$. Then there exists a prime filter F such that $a\notin F$.

Proof. Firstly, we prove the following result:

(P18) The least filter containing another filter F and an element x is $F'=\{u\mid\exists v\in F, u\geqslant v\&_p(\Delta x)\}$.

In fact, for a subset F' of M, (F1) and (F2) hold. We check that the conditions (F3) and (F4) also hold. If $u\in F'$, then there exists $v\in F$ such that $u\geqslant v\&_p(\Delta x)$, by Theorem 3.2 (P12), (P16) and (P13) we have

$$\Delta u\geqslant\Delta(v\&_p(\Delta x))=(\Delta v)\&_p(\Delta\Delta x)=(\Delta v)\&_p(\Delta x).$$

Since $\Delta v\in F$ (using (F3)), thus $\Delta u\in F'$, this means that (F3) holds for F'. If $a\to_p b\in F'$, then there exists $v\in F$ such that $a\to_p b\geqslant v\&_p(\Delta x)$, by (P5), (P8), (P12) and (P13) we get

$$(-b)\to_p(-a)\geq\Delta((-b)\to_p(-a))=\Delta(a\to_p b)\geqslant\Delta(v\&_p(\Delta x))=(\Delta v)\&_p(\Delta x).$$

Therefore, $(-b)\to_p(-a)\in F'$, F' satisfies (F4). Hence F' is a filter of M. Obviously, F' is a least filter containing F and x.

Now, we prove that there exists a prime filter F such that $a\notin F$, where $a\in M-\{1\}$. Let F be a filter not containing a (there exists at least one, $F=\{1\}$). Let $x, y\in M$ such that $x\to_p y\notin F$ and $y\to_p x\notin F$. Using the above result (P18), we can build F_1 and F_2 as the least filters containing F, $x\to_p y$ and $y\to_p x$ respectively. We can prove that at least one of these two filters does not contain a. In fact, if $a \in F_1$ and $a \in F_2$, then there exists $v_1, v_2 \in F$ such that $a \geqslant v_1 \&_p (\Delta(x \to_p y)$, $a \geqslant v_2 \&_p (\Delta(y \to_p x)$. Thus

$$\begin{aligned}
a &\geqslant (v_1 \&_p (\Delta(x \to_p y))) \vee (v_2 \&_p (\Delta(y \to_p x))) \\
&\geqslant ((v_1 \&_p v_2) \&_p (\Delta(x \to_p y))) \vee ((v_1 \&_p v_2) \&_p (\Delta(y \to_p x))) && \text{(since } \&_p \text{ is isotone)} \\
&= (v_1 \&_p v_2) \&_p (\Delta(x \to_p y) \vee \Delta(y \to_p x)) && \text{(by Proposition 1(30) in ref. [2])} \\
&\geqslant (v_1 \&_p v_2) \&_p (\Delta((x \to_p y) \vee (y \to_p x))) && \text{(by P4)} \\
&= (v_1 \&_p v_2) \&_p (\Delta 1) = (v_1 \&_p v_2) \&_p 1 && \text{(by P2, P9)} \\
&= v_1 \&_p v_2 \in F && \text{(by Definition 3.4 (F1)).}
\end{aligned}$$

Therefore, $a \in F$, this is a contradiction. Hence, a sequence of nested filters not containing a can be built. Finally, the prime filter which we are looking for is the union of that sequence of filters.

Definition 3.8. Let M be a UL^*-algebra. A mapping $v: F(S) \to M$ is called an M-evaluation, if it is a $(-, \Delta, \&_p, \to_p, \wedge)$-type homomorphism.

The set of all M-valuations is denoted by $\Omega(M)$, where M is a UL^*-algebra. If $v(A)=1$ for any $v \in \Omega(M)$, then A is called an M-tautology, denoted by $\models_M A$. If M is a linearly ordered UL^*-algebra, then $\models_M A$ is simply denoted by $\models A$. Similar to Theorem 2 in ref. [2] and Theorem 6 in ref. [20], we can prove the following completeness theorem.

Theorem 3.9 (Completeness Theorem of UL^*). The logic system UL^* is complete with respect to a UL^*-algebra, i.e., for each formula A, the following statements are equivalent:

1) A is a theorem in the system UL^*, i.e. $\vdash A$.

2) A is an M-tautology for each UL^*-algebra M, i.e. $\models_M A$.

3) A is an M-tautology for each linearly ordered UL^*-algebra M, i.e. $\models A$.

4 Significance of fuzzy logic system UL^* and its application in approximate reasoning

This section mainly interprets the relation of fuzzy logic system UL^* and some other fuzzy logic formal systems so as to indicate the theoretical significance of UL^*. Moreover, we use the concepts of "the strength of an 'and' operator" by Yager[21] and "the strength of fuzzy rule interaction" by Whalen[11], to explain the essential meaning of the parameter p in the system UL^* and then to elaborate the application value of UL^*.

As for fuzzy logic formal systems, the most famous ones are BL[1], MTL[2], L^*[5], $SBL_{\sim}$[20], etc. Among them MTL is a rather weak system based on a general left-continual t-norm. The system MTL consists of MP rule and the following axioms (see ref. [2]):

(MTL1) $(A \to B) \to ((B \to C) \to (A \to C))$,

(MTL2) $(A \& B) \to A$,

(MTL3) $(A \& B) \to (B \& A)$,

(MTL4) $(A \wedge B) \to A$,

(MTL5) $(A \wedge B) \to (B \wedge A)$,

(MTL6) $(A \& (A \to B)) \to (A \wedge B)$,

(MTL7) $((A \& B) \to C) \to (A \to (B \to C))$,

(MTL8) $(A \to (B \to C)) \to ((A \& B) \to C)$, (MTL9) $((A\to B)\to C)\to(((B\to A)\to C)\to C)$,

(MTL10) $0\to A$.

It is easily seen that (MTL1)—(MTL10) just corresponds to (UL1)—(UL10) in UL^*. Why extending MTL to set up UL^*? The reason lies in the following aspects:

Firstly, MTL system has no involutive negation and this is far away from the human's intuitive meaning that $\neg\neg A\approx A$, and it also forms a great contrast to the well-defined structure of the system L^*. In order to retain the generality of MTL (based on a general left-continual t-norm), and to possess an involutive negation of the system L^* as well, we are urged to set up the UL^* system. As stated early in this paper, the UL^* system has really attained its expected goal.

Secondly, as pointed out in ref. [20], the traditional fuzzy logic systems (such as BL, MTL) take the negation as $\neg A=A\to 0$, i.e. the negation is defined by the implication, therefore this negation behaves quite differently based on the different implications. To improve this unreasonable situation is just the reason for ref. [20] to set up the new logic system $\mathrm{SBL}_\sim$. To guarantee the completeness of the system, however, $\mathrm{SBL}_\sim$ is restrained under a strict continuous t-norm ("S" in "$\mathrm{SBL}_\sim$" is just an abbreviation of the English word "strict"). UL^* system set up in this paper, however, takes off the restriction from "strict", and it still possesses the completeness.

In addition the method setting up a UL^* system with the completeness has a direct generalized value to the formulization of a non-communicative fuzzy logic. In recent years influenced by the non-communicative linear logics (which have an important applications in the fields of logic programming, computation linguistics, fuzzy relation database and so on), several non-communicative fuzzy logic systems based on pseudo t-norm, such as *ps*BL, *ps*MTL (cf. refs. [22, 23]) have been put forward one after another. Because of the limit of non-communicative "logic AND", new axioms have to be added to make these logic systems complete, such as the complete $ps\mathrm{BL}^r$, $ps\mathrm{MTL}^r$ systems respectively. In fact, using the method presented in this paper, the author has preliminarily proved that by introducing certain appropriate projective operators we can obtain a complete non-commutative fuzzy logic formal system without adding any new axiom.

In the following we will justify the meaning of the parameter p in UL^* and further to explain the application of UL^* starting from the two concepts of "the strength of an 'and' operator" and "the strength of fuzzy rule interaction".

Yager proposed for the first time in ref. [21] the concept of "the strength of an 'and' operator (shortly SA)", that is to say people have different requirements to logical "AND" or "OR", and this is an embodiment of human intelligence. In ref. [24], SA is defined as follows:

$$1-3\int_0^1\int_0^1 f(x,y)dxdy,$$

where $f(x, y)$ is fuzzy "AND" operator (usually a t-norm is chosen, and for a non-continuous t-norm Lebesgue integral can be taken as suggested in ref. [25]). From the logic point of view, SA can be understood as an association degree among the propositions. According to the definition stated just now, the SA of the largest "AND" operator

$\wedge$ is the least (just being 0), and this is just opposite to our intuitive sense, so we use its complement to 1 as the definition of more intuitive SA h (understood as an association degree among the propositions), i.e.

$$h = 3\int_0^1\int_0^1 f(x,y)dxdy.$$

Similar to Theorem 4.1 in ref. [24] we can prove that $0 \leqslant h \leqslant 1$. Since the h of the largest "AND" operator $\wedge$ is just 1/3, the geometric meaning of h is the volume ratio of fuzzy "AND" operator $f(x, y)$ to the largest "AND" operator $\wedge$. It is interesting that recently Whalen[11] published a long paper in which he thought that the interaction among fuzzy rules should be considered, and proposed a concept of the "strength" of the rule interaction, then associated this concept with the parameter p in the Schweizer-Sklar t-norm. This is just the similar idea as the above "the association degree among propositions". This is just the starting point for which Schweizer-Sklar t-norm based fuzzy logic systems are studied in this paper.

As stated in ref. [11], in fuzzy logic application, people focus on selecting proper fuzzy implication operators according to practical context, almost regardless of the interaction among If—Then rules, but in fact fuzzy rules do not stand alone from one another. In a practical application system there is the weak or strong difference among their interactions, and this happens to be described by the parameter p in the Schweizer-Sklar t-norm. The purpose of UL^* system put forward in this paper is to show the association degree among the propositions in the formal systems, and to offer possible flexibility logic foundation to the application fields of fuzzy control and so on. In the following, the role of system UL^* in approximate reasoning will be analyzed with the aid of "the association degree among the propositions" and the Triple I method in ref. [6].

Considering the following fuzzy reasoning question:

(GMP) From formula $A \to B$ and A^* infer B^*,

where the formula A, B, A^*, B^* stand for fuzzy propositions, that is, their truth values are real numbers in the unit interval [0,1], and generally, A and A^*, B and B^* may be different.

Based on the system UL^* and the Triple I method, a solution can be given to (GMP): as $(A \to_p B) \to_p (A^* \to_p B^*) = (A^* \&_p (A \to_p B)) \to_p B^*$, to make $(A \to_p B)$ closer to $(A^* \to_p B^*)$, it is natural to take $B^* = A^* \&_p (A \to_p B)$. This, actually, is consistent with the semantic "conjunctive" meaning of the connective $\&_p$.

Similarly we use the examples in ref. [26] to explain this question. Let a formula A stand for "the tomato is red", a formula B "the tomato is ripe", and A^* "the tomato is almost red". For the true value of $A \to B$, there may be many interpretations, as 0.7, 0.8, 0.9, etc. For the true value of A^*, there are also many interpretations, as 0.7, 0.9, 0.95, etc. There is certain association degree between A and B generally, and there is a relatively large association degree between A and B in this question, therefore $h \geqslant 0.75$ should be taken (accordingly $p>0$).

Through the calculation we have the following results (take $p=2$, accordingly $h \approx 0.92$ can be obtained by using the mathematic software MAPLE):

It can be seen from Table 1 that the system UL^* comparatively better reflects the

Table 1

A^*	$A \to B$	B^*	
		SS t-norm	R_0 t-norm
0.7	0.7	0.59	0.7
0.7	0.8	0.64	0.7
0.9	0.7	0.67	0.7
0.8	0.9	0.75	0.8
0.5	0.5	0.38	0
0.3	0.7	0.29	0

common sense reasoning. It is especially important that people can choose different parameters p according to their different understanding of the association degree between A and B to make their reasoning possessing flexibility (but the general tendencies are almost the same). The system L^* always takes the least true value in A^* and $A \to B$, and "too rigid". For example, for L^*, (0.9, 0.7) has no difference with (0.7, 0.7), and the results of (0.5, 0.5) and (0.3, 0.7) are all 0. Although this can be explained to certain extent by the principle of "more-than-half credibility" in ref. [7], it seems a little crudity to give up all "not to and more than half" situations when the association degree is relatively large between A and B.

To sum up, the UL^* system extends the applicability of some logic systems theoretically, and meanwhile retains its completeness; in application, the UL^* system reflects the association degree among the propositions through parameters, and holds certain reference value to the application such as fuzzy control, approximate reasoning and so on.

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