A Simple Model for Measuring Refractive Index of a Liquid Based upon Fresnel Equations *

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Due to many experimental data required and a lot of calculations involved, it is very complex and cumbersome to model prism-based liquid-refractive-index-measuring methods. We develop a new method of mathematical modelling for measuring refractive index of a liquid based upon the Fresnel formula and prism internal reflection at an incident angle less than the critical angle. With this method, only two different concentrations measurements for a kind of solution can lead to the determination of computational model. Measurements are performed to examine the validity of the theoretical model. Experimental results indicate the feasibility of the theoretical model with an error of 1%. The method is also capable of measuring even smaller changes in the optical refractive index of the material on a metal surface by the surface plasma resonance sensing techniques.

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The measurement methods based on the optical reflection and refraction have attracted a great deal of attention in the past few years. The main applications include detection limits of an internal-reflection sensor for the optical beam deflection method,[1,2] the optical sensor techniques based on surface plasma resonance (SPR),[3−8] thin-film enhanced Goos–Hänchen shift in total internal reflection,[9,10] and salinity measurement based on optical refraction method.[11,12] The measurement is based on Fresnel formula while the angle of incidence is smaller than the critical angle. A significant change in the intensity of reflected wave with the angle of incidence when incident angle close to critical angle ensures a high sensitivity, which leads to a fast, simple and efficient on-line measurement. For an incidence of an angle greater than the critical angle, measurement can be carried out by use of SPR sensing techniques. The main approaches include analysis of wavelength modulation,[13] angular modulation,[14] and intensity modulation.[15] Generally speaking, the above-mentioned two methods have all the developing potential. However, the approaches have the following two major drawbacks. Firstly, these intensity-based methods suffer from the fluctuation of intensity in light sources, the sensitivity and accuracy are degraded.[10] Secondly, it is very complex and cumbersome to model for measuring refractive index of a liquid. In this Letter, we present a new method of mathematical modelling that can overcome the above-mentioned two shortcomings. The model is compared with the experimental data and the experimental results indicate the feasibility of the theoretical model with an error of 1%.

The configuration considered here is shown in Fig. 1, where the refractive index of the prism is n_g, the included angle at the apex is $180^\circ - 2\alpha$, and let the medium at the two refracting surfaces of the prism be air and the refractive index of the medium at the bottom of the prism be n_x. Assume that a plane electromagnetic wave is incident onto the first surface of the prism at the incidence angle θ, the vector $\boldsymbol{k}$ determining the direction of propagation of the incident wave is in the drawing plane (Fig. 1). The direction of a normal to the first surface of the prism will be characterized by the vector $\boldsymbol{n}$. The plane in which vector $\boldsymbol{k}$ and $\boldsymbol{n}$ are is called the plane of incidence of the wave. It is obvious that vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3$ and $\boldsymbol{k}_4$ can only be in the plane of incidence. Assume that vector $\boldsymbol{E}$ is the field in the incident wave propagating in the direction of vector $\boldsymbol{k}$, vector $\boldsymbol{E}_1$ is the field in the refracted wave propagating in the direction of the vector $\boldsymbol{k}_1$ at the first surface, vectors $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are the fields in the reflected and refracted waves propagating in the directions of vectors $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ at the interface between an optically denser medium (refractive index, n_g) and an optically less dense one (refractive index, n_x, $n_g > n_x$), vector $\boldsymbol{E}_4$ is the field in the refracted wave propagating in the direction of vector $\boldsymbol{k}_4$ at the second surface of the prism. Assuming that the direction of vector $\boldsymbol{E}$ makes an arbitrary angle with respect to the plane of incidence, and it can be divided into $\boldsymbol{E}_s$ and $\boldsymbol{E}_p$,[16,17] in which vector $\boldsymbol{E}_s$ is a TM-polarized wave and vector $\boldsymbol{E}_p$ is a TE-polarized wave, i.e.

$$\boldsymbol{E} = \boldsymbol{E}_p + \boldsymbol{E}_s,$$
$$\boldsymbol{E}_i = \boldsymbol{E}_{ip} + \boldsymbol{E}_{is},$$

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where $i = 1$–4, vector $\boldsymbol{E}_{is}$ are the TM-polarized waves, vector $\boldsymbol{E}_{ip}$ are the TE-polarized waves. According to Snell's law[18] and Fresnel reflection formula,[19] vectors $\boldsymbol{E}_{4p}$, $\boldsymbol{E}_{4s}$, $\boldsymbol{E}_p$ and $\boldsymbol{E}_s$ satisfy the relations

$$E_{4p} = AE_p,$$
$$E_{4s} = BE_s,$$

where

$$A = \left[4\cos\theta\cos\theta_4\sin\theta_1\sin\theta_5\tan(\theta_2-\theta_3)\right] \cdot \Big[\tan(\theta_2+\theta_3)\sin(\theta+\theta_1)\cos(\theta-\theta_1) \cdot \sin(\theta_4+\theta_5)\cos(\theta_4-\theta_5)\Big]^{-1}$$
$$B = -\frac{4\cos\theta\cos\theta_4\sin\theta_1\sin\theta_5\sin(\theta_2-\theta_3)}{\sin(\theta_2+\theta_3)\sin(\theta+\theta_1)\sin(\theta_4+\theta_5)}.$$

According to the definition of the intensity of light, we have

$$I \propto |\boldsymbol{E}|^2 = |\boldsymbol{E}_p|^2 + |\boldsymbol{E}_s|^2 = \boldsymbol{E}_p^2 + \boldsymbol{E}_s^2, \tag{1}$$
$$I_4 \propto |\boldsymbol{E}_4|^2 = |\boldsymbol{E}_{4p}|^2 + |\boldsymbol{E}_{4s}|^2 = \boldsymbol{E}_{4p}^2 + \boldsymbol{E}_{4s}^2, \tag{2}$$

where I is the intensity of the incident wave, and I_4 is the intensity of the refracted wave when the ray passes through the prism symmetrically.

It is known from Eqs. (1) and (2) that when the light intensity I of the incident wave and the incident angle θ are constants and if θ_2 is constant, then θ_1, θ_4, θ_5 would also be constant, thereafter the light intensity I_4 is only the function of the refractive index n_x. At the bottom of the prism, according to Snell's law,[18]

$$n\sin\theta_2 = n_x\sin\theta_3$$

we arrive at the formula

$$\theta_3 = \sin^{-1}\Big(\frac{n_g}{n_x}\sin\theta_2\Big).$$

The light intensity I_4 will therefore vary with the refractive index n_x of the measured medium.

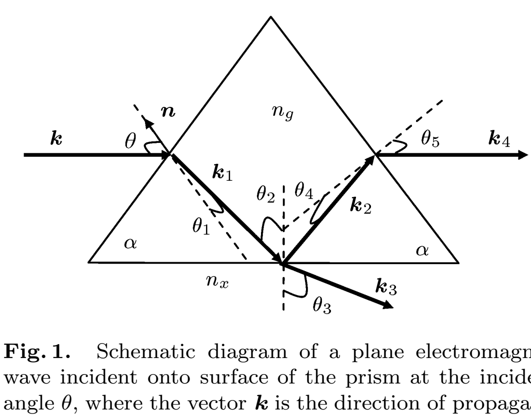


Fig. 1. Schematic diagram of a plane electromagnetic wave incident onto surface of the prism at the incidence angle θ, where the vector $\boldsymbol{k}$ is the direction of propagation of the incident wave, α are the two basic angles of the prism, n_g is the refractive index of the prism, n_x is the refractive index of the measured medium.

As shown in Fig. 1, the angle α depends on the range of the refractive index n_x of the measured medium on the basis of considering the high sensitivity of the sensor system. Assume that the range of the measured refractive index n_x satisfy the relation

$$n_{\min} \le n_x \le n_{\max}.$$

The prism in Fig. 1 has a refractive index of 1.5163. According to the law of refraction of light, both the basic angles α of the prism can be written as

$$\alpha < \theta_c = \sin^{-1}(n_{\min}/1.5163).$$

In the case of θ_2 close to θ_c, the intensity of reflected wave will vary more sensitively to the medium refractive index n_x. At $\theta = 0$, we have $\alpha = \theta_2$, and $\theta_1 = \theta_4 = \theta_5 = 0$. It is related to the angle θ_3, subsequently refractive index n_x of measured medium is given by

$$\theta_3 = \sin^{-1}(n_g\sin\alpha/n_x).$$

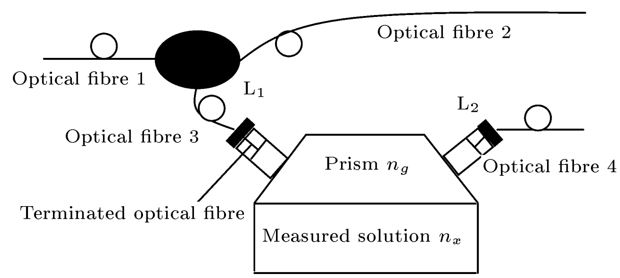


Fig. 2. Schematic diagram of light path of the sensor system, where the light path consists of two GRIN lenses, a high refractive index prism (refractive index $n_g = 1.5163$), a Y-shaped coupler and the sample with lower refractive index (refractive index n_x, $n_x < n_g$).

In Fig. 2, both the gradient index (GRIN) rod lenses L_1 and L_2 are placed symmetrically on the two surfaces of the prism with Canada balsam. Assume that a beam of light enters into optical fibre 1, later it is broken down into two beams in the coupler. The beam passing through optical fibre 2 acts as the reference optical path, where intensity of light is I_0. The other beam passing through the optical fibre 3 is known as the sensing optical path, and it enters into the prism.

The intensity of light entering the prism then passing through the GRIN lens L_1 is expressed by

$$I_1 = \frac{n_g}{2c\mu_0}(|\boldsymbol{E}_p|^2 + |\boldsymbol{E}_s|^2), \tag{3}$$

where vector $\boldsymbol{E}_s$ is the TM polarized wave, vector $\boldsymbol{E}_p$ is the TE polarized wave, n_g is the refractive index of the prism, c is the velocity of light in vacuum.

The intensity of light entering the optical fibre 4 then passing through the GRIN lens L_2 is

$$I_2 = \frac{n_g}{2c\mu_0}\cdot(r_p^2|\boldsymbol{E}_p|^2 + r_s^2|\boldsymbol{E}_s|^2), \tag{4}$$

where the r_p is the amplitude reflection coefficient of the TE polarized wave, the r_s is the amplitude reflection coefficient of the TM polarized wave.

According to Fresnel reflection formula,[19] they can be written as follows:

$$r_p^2 \propto \frac{\tan^2\{\alpha - \sin^{-1}(n_g \sin\alpha/n_x)\}}{\tan^2\{\alpha + \sin^{-1}(n_g \sin\alpha/n_x)\}},$$

$$r_s^2 \propto \frac{\sin^2\{\sin^{-1}(n_g \sin\alpha/n_x) - \alpha\}}{\sin^2\{\sin^{-1}(n_g \sin\alpha/n_x) + \alpha\}}.$$

Assume that the light entering optical fibre 1 is modulated, V_r and V_k are the voltages of the reference optical path with light on and off, respectively, in a modulating period. V_m and V_n are the counterpart voltages of the sensing optical path. K_1 is the product of the light-to-electric conversion coefficient of photodiode 1 and the gain of the afterwards circuitry, K_2 is the product of the light-to-electric conversion coefficient of photodiode 2 and the gain of the afterwards circuitry. The output signal R can be described as

$$R = \frac{(V_n - V_m)}{(V_r - V_k)} = \frac{K_2 \cdot I_2}{K_1 \cdot I_0}. \quad (5)$$

where R is displayed by the system of computer processing and controlling.

Substituting Eq. (4) into Eq. (5) gives

$$R = \frac{n_g}{2c\mu_0} \cdot \frac{KK_2}{K_1 I_0} \cdot (r_p^2|\boldsymbol{E}_p|^2 + r_s^2|\boldsymbol{E}_s|^2), \quad (6)$$

where K is the constant relating to the refractivity index n_g of the prism. The form of Eq. (6) shows that the output signal R of the system is dependent only on the refractive index n_x of the measured medium, while it is independent of the dark-current of photodiode, intensity in light sources and temperature drifts of processing circuit.

When the refractive index n_x is chosen to be $n_x = n_0$ and n_1, respectively. Equation (6) changes into

$$R_i = \frac{n_g}{2c\mu_0} \cdot \frac{KK_2}{K_1 I_0} \cdot (r_{\mathrm{Pi}}^2|\boldsymbol{E}_p|^2 + r_{\mathrm{Si}}^2|\boldsymbol{E}_S|^2), \quad (7)$$

where $i = 0, 1$,

$$r_{\mathrm{Si}}^2 = \frac{\sin^2\{\sin^{-1}(n_g \sin\alpha/n_i) - \alpha\}}{\sin^2\{\sin^{-1}(n_g \sin\alpha/n_i) + \alpha\}},$$

$$r_{\mathrm{Pi}}^2 = \frac{\tan^2\{\alpha - \sin^{-1}(n_g \sin\alpha/n_i)\}}{\tan^2\{\alpha + \sin^{-1}(n_g \sin\alpha/n_i)\}}.$$

Equation (7) can be rewritten as

$$\frac{n_g}{2c\mu_0} \cdot \frac{KK_2}{K_1 I_0} \cdot |\boldsymbol{E}_S|^2 = \frac{(R_1 r_{p0}^2 - R_0 r_{p1}^2)}{(r_{s1}^2 r_{p0}^2 - r_{s0}^2 r_{p1}^2)}, \quad (8)$$

$$\frac{n_g}{2c\mu_0} \cdot \frac{KK_2}{K_1 I_0} \cdot |\boldsymbol{E}_P|^2 = \frac{(R_0 r_{s1}^2 - R_1 r_{s0}^2)}{(r_{s1}^2 r_{p0}^2 - r_{s0}^2 r_{p1}^2)}. \quad (9)$$

Substituting Eqs. (8) and (9) into Eq. (6) gives

$$R = \frac{(R_0 r_{s1}^2 - R_1 r_{s0}^2) \cdot r_P^2}{(r_{s1}^2 r_{p0}^2 - r_{s0}^2 r_{p1}^2)} + \frac{(R_1 r_{p0}^2 - R_0 r_{p1}^2) \cdot r_S^2}{(r_{s1}^2 r_{p0}^2 - r_{s0}^2 r_{p1}^2)}, \quad (10)$$

where R is the output signal, R_0 and R_1 obtained by experiment are the output signal R at $n_x = n_0$ and n_1 respectively, r_{s0} and r_{s1} are the amplitude reflection coefficients of the TM polarized wave at $n_x = n_0$ and n_1, respectively, r_{p0} and r_{p1} are the amplitude reflection coefficients of the TE polarized wave at $n_x = n_0$ and n_1, respectively. Equation (10) shows that the dependency of the output signal R on the refractive index n_x can be determined by R_0 and R_1.

To examine the validity of the theoretical model, an experimental setup was carried out, as shown in Fig. 3. A semiconductor laser with an output wavelength of 1650 nm was selected as the light source, the laser beam was modulated by chopper enters a Y-shaped spectroscope where it is bifurcated into two light beams, i.e. reference light and sensing light. The photodiodes 1 and 2 have the same peak wavelength. According to the range of concentration c_x of the measured solution

$$5\% \le c_x \le 25\%,$$

two basic angles α of the prism can be determined to be

$$\alpha = 62°8'10'', \quad \text{at } c_x = 4\%.$$

As an example, we measured the salt solution by the experimental setup shown in Fig. 3. In the experiment, the measured salt water made up using chemically pure sodium chloride (NaCl) and distilled water, the samples with various concentrations were provided. Thus, a sodium chloride solution of distilled water was provided to give a salinity from 0 to 25% at intervals of 5%, while $n_x = 1.3331 + 0.185c_x$ was also obtained by the refractometer. For the solutions of the different salinity, output of the sensor system was examined and the monitoring time was about 10 min, respectively at 25°C. The experimental results were recorded as shown in Table 1.

Table 1. Experimental results for examining the validity of the theoretical model by the experimental setup shown in Fig. 3.

c_x (%)	5	10	15	20	25
R	0.953	0.695	0.562	0.436	0.341

From Table 1, it is known that R_0 is 0.953 at $c_x = 5\%$ and R_1 is 0.695 at $c_x = 10\%$, r_{p0}^2 and r_{s0}^2 calculated by Eq. (7) are 0.596 and 0.667, respectively; at $c_x = 5\%$, r_{p1}^2 and r_{s1}^2 calculated by Eq. (7) are 0.271 and 0.359, respectively, at $c_x = 10\%$. We thus arrive at the expression

$$r_{s1}^2 r_{p0}^2 - r_{s0}^2 r_{p1}^2 = 0.033.$$

Equation (10) can therefore be written in the form

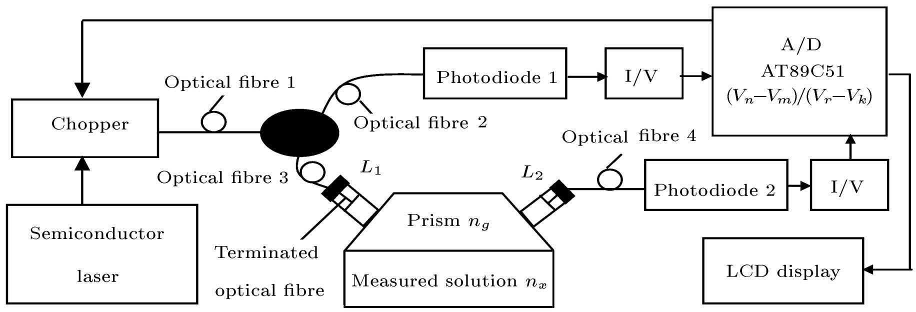
$$R = 4.73 r_S^2 - 3.68 r_P^2, \quad (11)$$

where

$$r_S^2 = \frac{\sin^2\{\sin^{-1}(1.3405/n_x) - 62°8'10''\}}{\sin^2\{\sin^{-1}(1.3405/n_x) + 62°8'10''\}},$$

$$r_P^2 = \frac{\tan^2\{62°8'10'' - \sin^{-1}(1.3405/n_x)\}}{\tan^2\{62°8'10'' + \sin^{-1}(1.3405/n_x)\}},$$

$$n_x = 1.3331 + 0.185c_x.$$



mination of validity of the theoretical model.

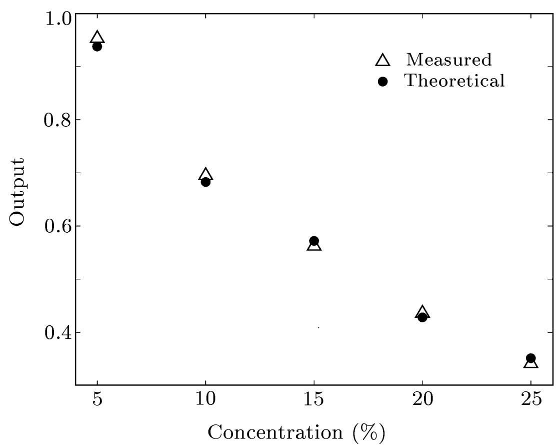


Fig. 4. Dependence of the output signal on concentration of the salt solution, where the temperature T of salt solution is 25°C. Circles: calculated by Eq. (11). Triangles: measured by the optic fibre sensor based on the method of the prism internal reflection (see Fig. 3).

Figure 4 shows the dependence of the output signal R measured at 25°C on the salinity by the circles, where the salinity from 0 to 25% at intervals of 5%. It is clearly seen from Fig. 4 that the output signal R changes significantly with the salinity when the salinity close to 5% and the output signal R changes slowly when the salinity is greater than 15%. For comparison, the experimental results are also shown in Fig. 4 by the triangles. From Table 1, R is 0.562, 0.436 and 0.341 at $c_x = 15\%$, 20% and 25%, respectively, R calculated by Eq. (11) is 0.572, 0.428 and 0.351 at $c_x = 15\%$, 20% and 25%, respectively. From the results, the theoretical model has error of 1%.

The calculated results are in agreement with the measured ones. In fact, the refractive index of the measured salt solution and the one calculated by equation $n_x = 1.3331 + 0.185c_x$ are also not exactly the same, so the output signal R has error, too. In order to determine R_0 and R_1, it is required that the solutions of the refractive indices $n_x = n_0$ and n_1 are at the same temperature, so that the influence of temperature difference between the solution of refractive index n_0 and the one of refractive index n_1 on the refractive index n_g of the prism is small enough to be ignored, the measurement error of the system can be decreased to the minimum, and then the theoretical model is reasonable in an error within 1%.

The new method of mathematical modelling for measuring refractive index of a liquid is also meaningful to the measuring even smaller changes in the optical refractive index of a material on the metal surface by SPR sensing techniques.

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