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Embedding a long fault-free cycle in a crossed cube with more faulty nodes

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ABSTRACT

The crossed cube is an important variant of the most popular hypercube network for parallel computing. In this paper, we consider the problem of embedding a long fault-free cycle in a crossed cube with more faulty nodes. We prove that for $n \geqslant 5$ and $f \leqslant 2n-7$, a fault-free cycle of length at least $2^n - f - (n-5)$ can be embedded in an n-dimensional crossed cube with f faulty nodes. Our work extends some previously known results in the sense of the maximum number of faulty nodes tolerable in a crossed cube.

1. Introduction

The interconnection network provides an effective mechanism for exchanging data between processors in a parallel computing system. An interconnection network is usually represented as a graph, where nodes and edges correspond processors and communication links between processors, respectively.

In the design and analysis of an interconnection network, its graph embedding ability is a major concern. An ideal interconnection network (host graph) is expected to possess excellent graph embedding ability, which helps efficiently execute parallel algorithms with regular task graphs (guest graphs) on this network [13]. The cycle and path are recognized as important guest graphs because a great number of parallel algorithms have been developed on cycle/path-structured task graphs.

As the size of a parallel computing system increases, it becomes much likely that there exist faulty processors and faulty communication links in such a system. Consequently, the graph-embedding ability of an interconnection network should be studied in the situation where faulty elements arise.

Due to some appealing features such as shorter diameter than their hypercube counterpart, the crossed cube network has received considerable research attention [1,3,4,6,18]. In particular, the fault-tolerant cycle/path embedding ability of the crossed cube has been examined extensively in the literature [7–12,15–17].

However, almost all the previous results on fault-tolerant cycle/path embedding in the crossed cube could not tolerate faulty nodes more than the node degree n of the network. Yang et al. [17] proved that there exists a fault-free cycle of every length from 4 to $2^n - f_v$ in an n-dimensional crossed cube (CQ_n) with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leqslant n-2$. Ma et al. [12] showed that there exists a fault-free path of every length from $2^{n-1}-1$ to $2^n - f_v - 1$ between any two distinct fault-free nodes in CQ_n with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leqslant n-3$. Huang et al. [11] broke that limitation for link faults, they found that CQ_n is fault-tolerant Hamiltonian with up to $2n-5$ faulty edges. To our knowledge, no results has been reported on the fault-

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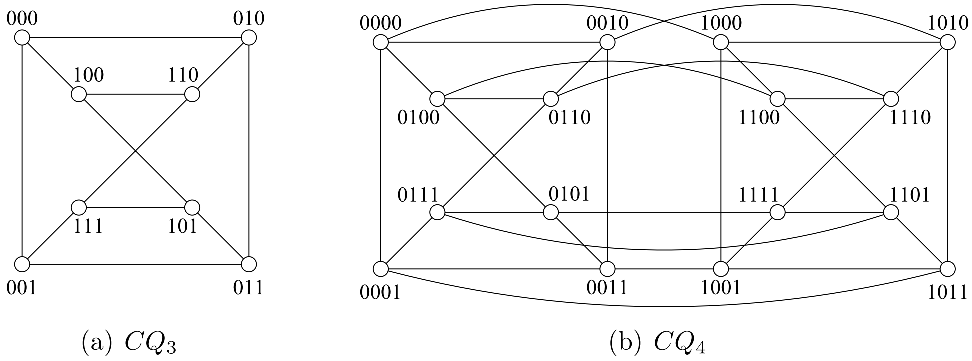

Fig. 1. Two examples of crossed cubes.

tolerant cycle embedding in CQ_n with more faulty nodes than the node degree n of the network.

In this paper, we consider the problem of embedding a long fault-free cycle in CQ_n with more faulty nodes. We prove that a fault-free cycle of length at least $2^n - f - (n-5)$ can be embedded in CQ_n with f faulty nodes, where $n \geqslant 5$ and $f \leqslant 2n-7$. Our work extends some previously known results in the sense of the maximum number of faulty nodes tolerable in a crossed cube.

The rest of this paper is organized as follows. Section 2 gives definitions and other preliminaries. Section 3 establishes the main result. Section 4 concludes the paper.

2. Definitions and preliminaries

For our purpose, an interconnection network is represented by a graph, where nodes and edges represent processors and communication links between processors, respectively. The node set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. A *Hamiltonian cycle* (respectively, a *Hamiltonian path*) in a graph is a cycle (respectively, a path) that passes every node of the graph exactly once. Two paths (respectively, two cycles) in a graph are *disjoint* if they do not share any common nodes. For other basic graph–theoretic notations and terminology, the reader is referred to Ref. [2].

Definition 2.1. (See [5].) An n-dimensional (n-D, for short) crossed cube, denoted CQ_n, is defined recursively as follows: CQ_1 is a complete graph on two nodes labeled with 0 and 1, respectively. For $n \geqslant 2$, let CQ^0_{n-1} (respectively, CQ^1_{n-1}) denote an $(n-1)$-D crossed cube with the label of each node being preceded by 0 (respectively, 1). Then, CQ_n is built from CQ^0_{n-1} and CQ^1_{n-1} by adding all those edges (u,v) such that

(1) $u = u_{n-1}u_{n-2}\cdots u_0 \in V(CQ^0_{n-1})$, $V = v_{n-1}v_{n-2}\cdots v_0 \in V(CQ^1_{n-1})$,
(2) $u_{n-2} = v_{n-2}$ if n is even, and
(3) for $0 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor - 1$, $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in \{(00,00), (10,10), (01,11), (11,01)\}$.

Fig. 1 plots two examples of crossed cubes.

An edge $(u,v) \in E(CQ_n)$ such that $u \in V(CQ^0_{n-1})$ and $v \in V(CQ^1_{n-1})$ is called a *bridge edge*, and u and v are called *counterparts* of each other. There exist 2^{n-1} disjoint bridge edge between CQ^0_{n-1} and CQ^1_{n-1}.

As mentioned before, we assume that only node faults occurs in this paper. An edge $(u,v) \in E(CQ_n)$ is regarded to be fault-free if and only if both u and v are fault-free. The following properties of the crossed cube will be useful in this paper.

Lemma 2.1. *(See [10].) For $n \geqslant 3$, there exists a fault-free Hamiltonian cycle in a CQ_n with $f \leqslant n-2$ faulty nodes, and there exists a fault-free Hamiltonian path between any pair of fault-free nodes in a CQ_n with $f \leqslant n-3$ faulty nodes.*

Lemma 2.2. *(See [14].) For any two pairs of nodes (x_1, x_2) and (y_1, y_2) in CQ_n, $n \geqslant 5$, there exist two disjoint paths P and Q satisfying that (1) P joins x_1 to y_1 and Q joins x_2 to y_2, and (2) $V(P \cup Q) = V(CQ_n)$.*

3. Main result

This section deals with the fault-free cycle embedding in a crossed cube with faulty nodes. The main result of this paper is established as follows.

Theorem 3.1. *For $n \geqslant 5$ and $f \leqslant 2n-7$, a fault-free cycle of length at least $2^n - f - (n-5)$ can be embedded in CQ_n with f faulty nodes.*

Proof. We argue by induction on n. By Lemma 2.1, the assertion holds for $n = 5$. Suppose the assertion is true for $n = k$ $(k \geqslant 5)$. Now, we consider CQ_{k+1} with $f \leqslant 2k-5$ faulty nodes. Let f_0 and f_1 denote the numbers of faulty nodes in CQ^0_{k+1} and CQ^1_{k+1}, respectively. Without loss of generality, we may assume that $f_0 \geqslant f_1$. Next we will construct a fault-free cycle of length at least $2^{k+1} - f - (k-4)$ in CQ_{k+1}.

Case 1. $f_0 \leqslant k-3$ (see Fig. 2).

Since there exist $2^k > 2k-3 \geqslant f+2$ bridge edges between CQ^0_{k+1} and CQ^1_{k+1}, we can find two fault-free bridge edges (x_0, x_1) and (y_0, y_1). According to Lemma 2.1, there exists a fault-free Hamiltonian path P_0 between x_0 and y_0 in CQ^0_{k+1}, and there exists a fault-free Hamiltonian path P_1 between x_1 and y_1 in CQ^1_{k+1}. Hence,

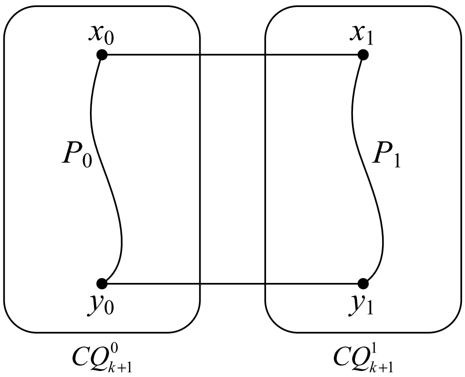

Fig. 2. The fault-free cycle in Case 1.

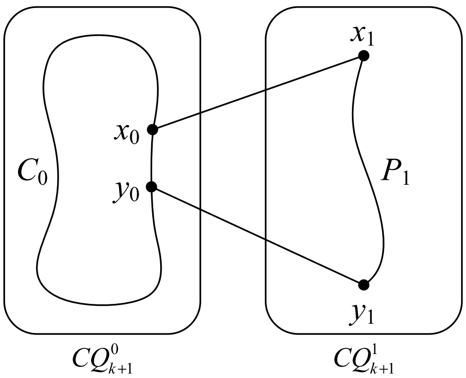

Fig. 3. The fault-free cycle in Case 2 and Case 3.1.

$\langle x_0, P_0, y_0, y_1, P_1, x_1, x_0\rangle$ forms a fault-free Hamiltonian cycle in CQ_{k+1}.

Case 2. $k-2 \leqslant f_0 \leqslant 2k-7$ (see Fig. 3).

According to the inductive hypothesis, CQ^0_{k+1} admits a fault-free cycle C_0 of length at least $2^k - f_0 - (k-5)$. We claim that we can find an edge (x_0, y_0) on C_0, such that (x_0, x_1) and (y_0, y_1) are both fault-free bridge edges. The existence of such an edge is due to the fact that there are at least $2^k - f_0 - (k-5)$ candidate edges on C_0, and there are $f_1 \leqslant k-3 < \lceil (2^k - f_0 - (k-5))/2\rceil$ faulty nodes in CQ^1_{k+1}, each of which can "block" at most two candidates. We may write C_0 as $\langle x_0, P_0, y_0, x_0\rangle$, then the length of path P_0 is $l_0 \geqslant 2^k - f_0 - (k-5) - 1$.

Note that $f_1 = f - f_0 \leqslant k-3$. By Lemma 2.1, there exists a fault-free path P_1 of length $l_1 = 2^k - f_1 - 1$ between x_1 and y_1 in CQ^1_{k+1}. Thus, $\langle x_0, P_0, y_0, y_1, P_1, x_1, x_0\rangle$ forms a fault-free cycle of length $l_0 + l_1 + 2 \geqslant 2^k - f_0 - (k-4)$ in CQ_{k+1}.

Case 3. $f_0 = 2k-6$.

Imagine one arbitrary faulty node z in CQ^0_{k+1} to be fault-free. Then according to the inductive hypothesis, there exists a fault-free cycle C_0 of length at least $2^k - (f_0 - 1) - (k-5)$ in CQ^0_{k+1}.

Case 3.1. z is not incident to C_0.

The proof of this case is similar to that of Case 2 (see Fig. 3).

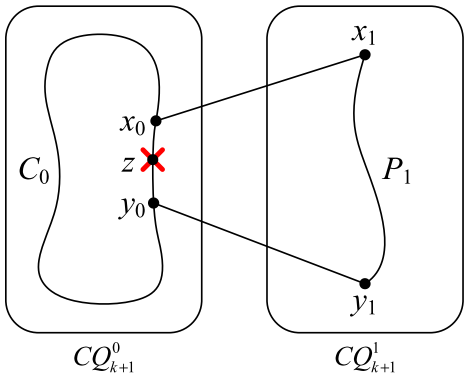

Fig. 4. The fault-free cycle in Case 3.2.1.

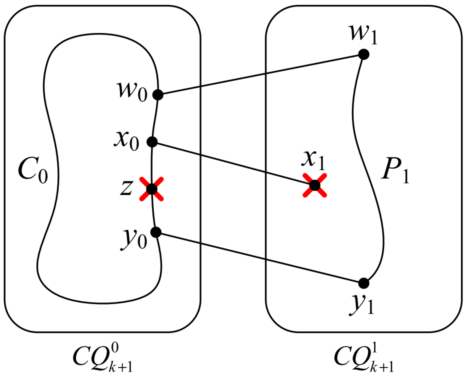

Fig. 5. The fault-free cycle in Case 3.2.2.

Case 3.2. z is incident to C_0.

Let the two neighbor nodes of z on C_0 be x_0 and y_0, and let the counterparts of x_0 and y_0 in CQ^1_{k+1} be x_1 and y_1.

Case 3.2.1. x_1 and y_1 are both fault-free (see Fig. 4).

We may write C_0 as $\langle x_0, z, y_0, P_0, x_0\rangle$, then the length of path P_0 is $l_0 \geqslant 2^k - (f_0 - 1) - (k-5) - 2$. Note that $f_1 \leqslant f - f_0 \leqslant 1 \leqslant k-3$. By Lemma 2.1, there exists a fault-free path P_1 of length $l_1 = 2^k - f_1 - 1$ between x_1 and y_1 in CQ^1_{k+1}. Hence, $\langle x_0, P_0, y_0, y_1, P_1, x_1, x_0\rangle$ forms a fault-free cycle of length $l_0 + l_1 + 2 \geqslant 2^k - f_0 - (k-4)$ in CQ_{k+1}.

Case 3.2.2. Either x_1 or y_1 is faulty (see Fig. 5).

Without loss of generality, we may assume that x_1 is a faulty node. Let the other neighbor of x_0 on C_0 (excluding the faulty node z) be w_0. The counterpart of w_0 in CQ^1_{k+1} must be fault-free. We may write C_0 as $\langle w_0, x_0, z, y_0, P_0, w_0\rangle$. Then the length of path P_0 is $l_0 \geqslant 2^k - (f_0 - 1) - (k-5) - 3$.

By Lemma 2.1, there exists a fault-free path P_1 of length $l_1 = 2^k - f_1 - 1$ between w_1 and y_1 in CQ^1_{k+1}. Therefore, $\langle w_0, P_0, y_0, y_1, P_1, w_1, w_0\rangle$ forms a fault-free cycle of length $l = l_0 + l_1 + 2 \geqslant 2^{k+1} - f - (k-4)$ in CQ_{k+1}.

Case 4. $f_0 = 2k-5$.

Imagine two arbitrary faulty nodes z and w in CQ^0_{k+1} to be fault-free. Then according to the inductive hypothesis, there exists a fault-free cycle C_0 of length at least

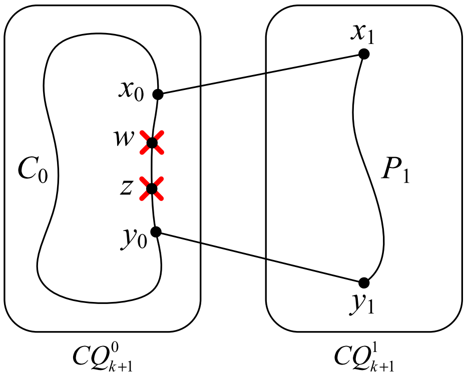

Fig. 6. The fault-free cycle in Case 4.1.

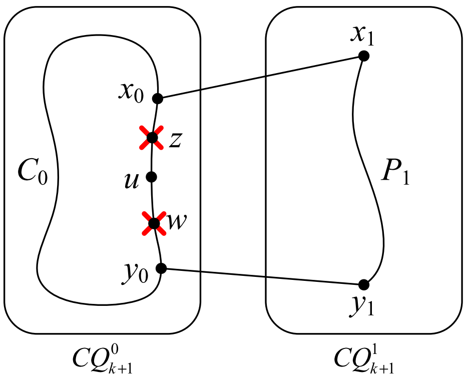

Fig. 7. The fault-free cycle in Case 4.2.

$2^k-(f_0-2)-(k-5)$ in CQ^0_{k+1}. If either z or w is not incident to C_0, the proof is similar to that of Cases 1–3. Next we consider the situation that both z and w are incident to C_0. We use $d_{C_0}(z,w)$ to denote the shorter distance traversing from z to w along C_0.

Case 4.1. $d_{C_0}(z,w)=1$ (see Fig. 6).

We may write C_0 as $\langle x_0,z,w,y_0,P_0,x_0\rangle$. Then the length of path P_0 is $l_0 \geqslant 2^k-(f_0-2)-(k-5)-3$. Let the counterparts of x_0 and y_0 in CQ^1_{k+1} be x_1 and y_1. Since $f_1=0$, x_1 and y_1 are obviously fault-free nodes.

By Lemma 2.1, there exists a fault-free path P_1 of length $l_1=2^k-1$ between x_1 and y_1 in CQ^1_{k+1}. Hence, $\langle x_0,P_0,y_0,y_1,P_1,x_1,x_0\rangle$ forms a fault-free cycle of length $l=l_0+l_1+2 \geqslant 2^{k+1}-f-(k-4)$ in CQ_{k+1}.

Case 4.2. $d_{C_0}(z,w)=2$ (see Fig. 7).

We may write C_0 as $\langle x_0,z,u,w,y_0,P_0,x_0\rangle$. Then the length of path P_0 is $l_0 \geqslant 2^k-(f_0-2)-(k-5)-4$. Let the counterparts of x_0 and y_0 in CQ^1_{k+1} be x_1 and y_1. x_1 and y_1 are obviously fault-free nodes.

By Lemma 2.1, there exists a fault-free path P_1 of length $l_1=2^k-1$ between x_1 and y_1 in CQ^1_{k+1}. Hence, $\langle x_0,P_0,y_0,y_1,P_1,x_1,x_0\rangle$ forms a fault-free cycle of length $l=l_0+l_1+2 \geqslant 2^{k+1}-f-(k-4)$ in CQ_{k+1}.

Case 4.3 $d_{C_0}(z,w)\geqslant 3$ (see Fig. 8).

We may write C_0 as $\langle x_0,z,y_0,P_{01},u_0,w,v_0,P_{02},x_0\rangle$. Let the lengths of paths P_{01} and P_{02} be l_{01} and l_{02}, then $l_{01}+l_{02} \geqslant 2^k-(f_0-2)-(k-5)-4$.

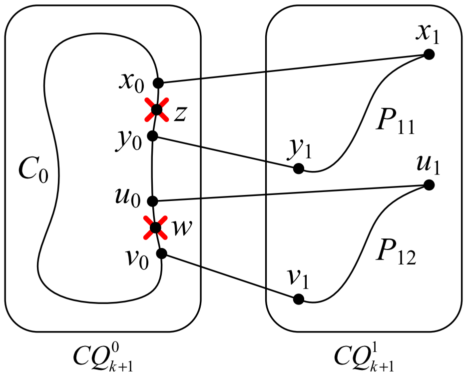

Fig. 8. The fault-free cycle in Case 4.3.

Let the counterparts of x_0, y_0, u_0 and v_0 in CQ^1_{k+1} be x_1, y_1, u_1 and v_1. Since $f_1=0$, x_1, y_1, u_1 and v_1 are obviously fault-free nodes. By Lemma 2.2, there exist two disjoint paths P_{11} and P_{12} satisfying that (1) P_{11} joins x_1 to y_1 and P_{12} joins u_1 to v_1, and (2) $V(P_{11}\cup P_{12})=V(CQ_{k+1})$. Let the lengths of paths P_{11} and Q_{12} be l_{11} and l_{12}, then $l_{11}+l_{12}=2^k-2$.

Therefore, $\langle x_0,x_1,P_{11},y_1,y_0,P_{01},u_0,u_1,P_{12},v_1,v_0,P_{02},x_0\rangle$ forms a fault-free cycle of length $l=l_{01}+l_{02}+l_{11}+l_{12}+4 \geqslant 2^{k+1}-f-(k-4)$ in CQ_{k+1}.

The inductive proof of this theorem is complete. □

4. Conclusions

In this paper, we proved that for $n \geqslant 5$ and $f \leqslant 2n-7$, a fault-free cycle of length at least $2^n-f-(n-5)$ can be embedded in an n-dimensional crossed cube with f faulty nodes. As compared to [10], this paper significantly improves the maximum number of faulty nodes tolerable. However, the two bounds, the cycle length $2^n-f-(n-5)$ and the number of faulty nodes $f \leqslant 2n-7$ are not optimal. According to our experience, we strongly conjecture the optimal result as follows.

Conjecture 4.1. *For $n \geqslant 3$ and $f \leqslant 2n-5$, a fault-free cycle of length at least 2^n-f-1 can be embedded in CQ_n with f faulty nodes.*

We tried to prove this conjecture but failed. In our opinion, the method developed in this paper is still useful for the proof of the conjecture, but some more lemmas should be established to support it.

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