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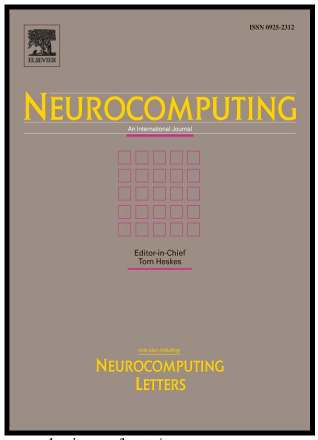

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Stability and Hopf bifurcation of a three-layer neural network model with delays [1]

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Abstract

A kind of three-layer neural network with time delays is introduced. By analyzing its associated characteristic equation, local stability and the existence of Hopf bifurcation of the system are investigated. By using the normal form method and center manifold theorem, formulas to determine the direction of the Hopf bifurcation and the stability of bifurcating periodic solution are obtained. Numerical simulation results are also given to support our theoretical predictions.

Keywords:

Neural network, Hopf bifurcation, Stability, Delays.

1. Introduction

Neural network is the abstraction and modeling of the human brain or biological neural network. It has ability to learn from the environment, and adapt to the environment in a interactive mode as creatures. In 1984, Hopfield proposed a simplified neural network model [2]. For time delays often occur during the signal transmission, Marcus and Westervelt put forward an Artificial neural network (ANN) model with delay [3]. Since then, many scientists pay attention to the dynamical characteristics of neural network(see[4][6][19][20][22][23][27]).

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Many scholars researched extensively in stability and multi-stablility of ANN (see[7]-[11][13]). Meanwhile some scientist discussed the dynamical behaviors such as bifurcation (see[12][14]-[16] [21]), periodic phenomenon and almost periodic solutions (see[5][24]-[26][31]), and chaos (see[18][27]-[30]).

Due to the existence of time delay, the delayed neural network is different from the non-delayed neural network, meanwhile, its solution space to be infinite dimensional as time delay. Therefore, many results of the non-delayed neural network cant be directly applied to the delayed neural network. Hopf bifurcation as a kind of common bifurcation phenomenon, is one of the important characteristics of delayed neural network, and also is one of the important research directions. We research the Hopf bifurcation of a delayed neural network, which generally takes the time delay as the bifurcation parameter, and obtain the bifurcation point. A limit cycle is generated after a tiny disturbance of the bifurcation point, and we can obtain the condition for losing stability of delayed neural network. By using frequency domain approach, Yu and Cao [34] studied a BAM $(2-2)$ neural network model with delays. Xu and He [35] studied a six-neuron BAM neural network model with discrete delays $(3-3)$, by using the normal form method and center manifold theorem, they get the condition of Hopf bifurcation occurs, and the direction, stability and the period of the bifurcating periodic solution. In [17], Xiao and Cao studied Hopf bifurcation of a $1-n$ neural network with the same method,they also give the Hopf bifurcation occurs condition and other results.

We can take the three layered neural network as a back propagation (BP) neural network. BP neural network is a multilayer feedforward network trained by the error back propagation algorithm [32]. As one of the most widely used neural networks, it includes input layer, hidden layer and output layer. The BP neural network has many outstanding advantages. Firstly, the three layer neural network is able to approximate any nonlinear continuous function with any accuracy, that is, the BP neural network has a strong nonlinear mapping ability. Secondly, BP neural network has a high degree of self-learning and adaptive ability. Thirdly, the BP neural network has the ability to apply the learning results and the new knowledge. Finally, the BP neural network has a fault tolerance ability, and the global training results will not be affected by the local neurons. In BP neural network, the hidden layer is connected with the input layer and the output layer, which has a very important position. According to the existing research results, there is no theoretical guidance on the choice of the number of neurons in the network. In this paper, we study the situation of two neurons in the hidden layer, and lay the foundation for the next step.

According to the existing literature, this paper is the first time to discuss the Hopf bifurcation of the three layered neural network. At present, the results of the Hopf bifurcation of the neural network are mainly concentrated in the two-layer of neural network and other simple two neurons , three neurons or four neurons network model. The research results of high dimensional delayed neural networks are limited, and the research of the three-layer network has great significance.

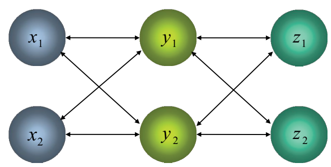

Fig.1. three-layer neural network model

In this paper, we consider the situation of a three-layer neuron network with six neurons. The architecture of this special case of system (1) is illustrated in Fig.1. This simplified three layer-neural network model can be described by the following system:

$$\begin{cases} \dot{x}_1(t) = -kx_1(t) + c_{13}f_{13}(y_1(t-\tau_3)) + c_{14}f_{14}(y_2(t-\tau_3)), \\ \dot{x}_2(t) = -kx_2(t) + c_{23}f_{23}(y_1(t-\tau_3)) + c_{24}f_{24}(y_2(t-\tau_3)), \\ \dot{y}_1(t) = -ky_1(t) + c_{31}f_{31}(x_1(t-\tau_1)) + c_{32}f_{32}(x_2(t-\tau_1)) \\ \qquad\qquad + c_{35}f_{35}(z_1(t-\tau_4)) + c_{36}f_{36}(z_2(t-\tau_4)), \\ \dot{y}_2(t) = -ky_2(t) + c_{41}f_{41}(x_1(t-\tau_1)) + c_{42}f_{42}(x_2(t-\tau_1)) \\ \qquad\qquad + c_{45}f_{45}(z_1(t-\tau_4)) + c_{46}f_{46}(z_2(t-\tau_4)), \\ \dot{z}_1(t) = -kz_1(t) + c_{53}f_{53}(y_1(t-\tau_2)) + c_{54}f_{54}(y_2(t-\tau_2)), \\ \dot{z}_2(t) = -kz_2(t) + c_{63}f_{63}(y_1(t-\tau_2)) + c_{64}f_{64}(y_2(t-\tau_2)) \end{cases} \tag{1}$$

where x_i is the input layer, y_i is the hidden layer and z_i is the output layer. k is the attenuation coefficient of all neurons, c_{ij} $(i, j = 1, 2, ..., 6)$ is the connection weights, τ_1 and τ_2 are the transmission delays between different layers, τ_3 and τ_4 are the feedback delays between different layers. The triggering nonlinear function of the neurons takes the hyperbolic tangent function, i.e. $f(\cdot) = \tanh(\cdot)$. Our aim in this paper is to study the stability and Hopf bifurcations of the system (1). Taking the sum of the delays, $\tau = \tau_1 + \tau_3 = \tau_2 + \tau_4$, as a parameter, we shall show that when the delay τ passes through a critical value, the zero solution loses its stability and a Hopf bifurcation occurs.

The remainder of this paper is organized as follows. In the next section, we shall consider the stability and the local Hopf bifurcation. In Section 3, based on the normal form method and the center manifold reduction introduced by Hassard et al. [1], we derive the formulae determining the direction, stability and the period of the bifurcating periodic solution at the critical value of τ. To verify the theoretic analysis, numerical simulations are given in Section 3. Finally, a conclusion is drawn in Section 4.

2. Local stability and Hopf bifurcation

For simplicity, let $u_1(t) = x_1(t + \tau_3)$, $u_2(t) = x_2(t + \tau_3)$, $u_3(t) = y_1(t)$, $u_4(t) = y_2(t)$, $u_5(t) = z_1(t + \tau_2)$, $u_6(t) = z_2(t + \tau_2)$, then system (1) can be transformed into the following equations with delay:

$$\begin{cases} \dot{u}_1(t) = -ku_1(t) + c_{13}f_{13}(u_3(t)) + c_{14}f_{14}(u_4(t)), \\ \dot{u}_2(t) = -ku_2(t) + c_{23}f_{23}(u_3(t)) + c_{24}f_{24}(u_4(t)), \\ \dot{u}_3(t) = -ku_3(t) + c_{31}f_{31}(u_1(t-\tau)) + c_{32}f_{32}(u_2(t-\tau)) \\ \qquad\qquad +c_{35}f_{35}(u_5(t-\tau)) + c_{36}f_{36}(u_6(t-\tau)), \\ \dot{u}_4(t) = -ku_4(t) + c_{41}f_{41}(u_1(t-\tau)) + c_{42}f_{42}(u_2(t-\tau)) \\ \qquad\qquad +c_{45}f_{45}(u_5(t-\tau)) + c_{46}f_{46}(u_6(t-\tau)), \\ \dot{u}_5(t) = -ku_5(t) + c_{53}f_{53}(u_3(t)) + c_{54}f_{54}(u_4(t)), \\ \dot{u}_6(t) = -ku_6(t) + c_{63}f_{63}(u_3(t)) + c_{64}f_{64}(u_4(t)) \end{cases} \tag{2}$$

where $\tau = \tau_1 + \tau_3 = \tau_2 + \tau_4$. We make the following assumption on function f_{ij}:

(H1): $f_{ij} \in C^2,\ f_{ij}(0) = 0,\ i, j = 1, 2, 3, 4, 5, 6.$

Under the hypothesis (H_1), the linearization of system (2) at equilibrium $O(0,0,0,0,0,0)$ is

$$\begin{cases} \dot{u}_1(t) = -ku_1(t) + \alpha_{13}(u_3(t)) + \alpha_{14}(u_4(t)), \\ \dot{u}_2(t) = -ku_2(t) + \alpha_{23}(u_3(t)) + \alpha_{24}(u_4(t)), \\ \dot{u}_3(t) = -ku_3(t) + \alpha_{31}(u_1(t-\tau)) + \alpha_{32}(u_2(t-\tau)) \\ \qquad + \alpha_{35}(u_5(t-\tau)) + \alpha_{36}(u_6(t-\tau)), \\ \dot{u}_4(t) = -ku_4(t) + \alpha_{41}(u_1(t-\tau)) + \alpha_{42}(u_2(t-\tau)) \\ \qquad + \alpha_{45}(u_5(t-\tau)) + \alpha_{46}(u_6(t-\tau)), \\ \dot{u}_5(t) = -ku_5(t) + \alpha_{53}(u_3(t)) + \alpha_{54}(u_4(t)), \\ \dot{u}_6(t) = -ku_6(t) + \alpha_{63}(u_3(t)) + \alpha_{64}(u_4(t)) \end{cases} \tag{3}$$

where $\alpha_{ij} = c_{ij} f'_{ij}(0)$. The associated characteristic equation of system (3) is

$$(\lambda + k)^4 e^{\lambda\tau} + B(\lambda + k)^2 + Ae^{-\lambda\tau} = 0 \tag{4}$$

where

$$A = -(a_{13}a_{31} + a_{32}a_{23} + a_{14}a_{41} + a_{42}a_{24} + a_{63}a_{36} + a_{64}a_{46} + a_{35}a_{53} + a_{45}a_{54}),$$

$$\begin{aligned} B = {} & (a_{54}a_{23} - a_{53}a_{24})(a_{32}a_{45} - a_{42}a_{35}) + (a_{54}a_{63} - a_{64}a_{53})(a_{36}a_{45} - a_{46}a_{35}) \\ & + (a_{53}a_{14} - a_{54}a_{13})(a_{41}a_{35} - a_{31}a_{45}) + (a_{14}a_{23} - a_{24}a_{13})(a_{41}a_{32} - a_{31}a_{42}) \\ & + (a_{14}a_{63} - a_{64}a_{13})(a_{41}a_{36} - a_{31}a_{46}) + (a_{64}a_{23} - a_{24}a_{63})(a_{46}a_{32} - a_{42}a_{36}). \end{aligned}$$

There are several types of bifurcation.

Case 1: $\tau = 0$.

Equation (4) becomes

$$\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0, \tag{5}$$

where $p_1 = 4k, p_2 = 6k^2 + B, p_3 = 4k^3 + 2Bk, p_4 = k^4 + Bk^2 + A$. Suppose

(H2): $p_i > 0, \ (i = 1, 2, 3, 4,), \quad (p_1p_2 - p_3)p_3 - p_1^2p_4 > 0.$

The Routh-Hurwitz criterion implies that the equilibrium $O(0,0,0,0,0,0)$ is locally asymptotically stable if (H2) holds.

Case 2: $\tau \neq 0$.

Lemma 2.1 [33] *Consider the exponential polynomial*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m}) &= \lambda^n + p_1^0\lambda^{n-1} + \cdots + p_{n-1}^0\lambda + p_n^0 + [p_1^1\lambda^{n-1} + \cdots + p_{n-1}^1\lambda + p_n^1]e^{-\lambda\tau_1} \\ &\quad + \cdots + [p_1^m\lambda^{n-1} + \cdots + p_{n-1}^m\lambda + p_n^m]e^{-\lambda\tau_m} \end{aligned} \tag{6}$$

where $\tau_i \geq 0$ $(i = 1, 2, ..., m)$, p_j^i $(i = 1, 2, ..., m; j = 1, 2, ..., n)$ *are constants. As* $(\tau_1, \tau_2, ..., \tau_m)$ *vary, the sum of the order of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m})$ *on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

The associated characteristic equation of system (3) is

$$(\lambda^4 + 4k\lambda^3 + 6k^2\lambda^2 + 4k^3\lambda + k^4)e^{\lambda\tau} + B(\lambda^2 + 2k\lambda + k^2) + Ae^{-\lambda\tau} = 0. \tag{7}$$

Let $\lambda = \omega\mathrm{i}$ $(\omega > 0)$ be the root of (7), separating the real and imaginary parts, then we have

$$\sin\omega\tau = \frac{2Bk\omega}{\omega^4 + 2k^2\omega^2 + k^4 + A}, \tag{8}$$

$$\cos\omega\tau = \frac{B(\omega^2 - k^2)}{\omega^4 + 2k^2\omega^2 + k^4 + A}. \tag{9}$$

Then we can obtain

$$\omega^8 + h_1\omega^6 + h_2\omega^4 + h_3\omega^2 + h_4 = 0, \tag{10}$$

where

$$h_1 = 4k^2, \quad h_2 = 6k^4 + 2A - B^2, \quad h_3 = 4k^6 + (4A - 2B^2)k^2, \quad h_4 = (k^4 + A)^2 - B^2k^4.$$

If $h_i(i = 1, ..., 8)$ of equation (10) are given, it is easy to calculate the roots of (10). From (9), we derive

$$\tau_k^{(j)} = \frac{1}{\omega_k}[\arccos\frac{B(\omega^2 - k^2)}{\omega^4 + 2k^2\omega^2 + k^4 + A} + 2j\pi], k = 1, 2, ..., 8, j = 0, 1, 2, ... \tag{11}$$

then $\pm\omega_k\mathrm{i}$ are a pair of purely imaginary root of (7) with $\tau = \tau_k^{(j)}$. Define

$$\tau_0 = \tau_k^{(0)} = \min_{\{k \in 1,2\}}\{\tau_k^{(0)}\}, \ \omega_0 = \omega_{k0}.$$

Let

$$\lambda(\tau) = \xi(\tau) + \mathrm{i}\omega(\tau) \tag{12}$$

be the root of (7) near $\tau = \tau_0$ satisfying

$$\xi(\tau_0) = 0, \ \omega(\tau_0) = \omega_0. \tag{13}$$

Theorem 2.1. *Suppose **(H1) (H2)** holds. Then as τ increases from zero, there is a value τ_0 such that the positive equilibrium E^* is locally asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Further, for system (3) the the positive equilibrium E^* undergoes a Hopf bifurcation when $\tau = \tau_0$.*

Proof We know that when **(H2)** holds, all the roots of (7) have negative real parts at $\tau = 0$. Hence, when $\tau = 0$ the system (3) is asymptotically stable. Taking the derivative of λ with respect to τ in (7), we have

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{P}{Q} - \frac{\tau}{\lambda}. \tag{14}$$

Where

$$\begin{aligned} P &= (4\lambda^3 + 12k\lambda^2 + 12k^2\lambda + 4k^3)e^{\lambda\tau} + B(2\lambda + 2k), \\ Q &= -(\lambda^4 + 4k\lambda^3 + 6k^2\lambda^2 + 4k^3\lambda + k^4)\lambda e^{\lambda\tau} + A\lambda e^{-\lambda\tau}. \end{aligned} \tag{15}$$

Let $\lambda = \pm\mathrm{i}\omega_0$ at the roots of equation (7) at $\tau = \tau_0$, we should compute $\frac{dRe(\lambda(\tau_0))}{d\tau}$. By calculated, we can get

$$(\frac{d\lambda}{d\tau})^{-1}|_{\tau=\tau_0} = \frac{M_1 + \mathrm{i}M_2}{N_1 + \mathrm{i}N_2} - \frac{\tau}{\lambda}, \tag{16}$$

Where

$$\begin{aligned} M_1 &= \cos\omega\tau(-12k\omega^2 + 4k^3) + \sin\omega\tau(4\omega^3 - 12k^2\omega) + 2Bk, \\ M_2 &= \cos\omega\tau(-4\omega^3 + 12k^2\omega) + \sin\omega\tau(-12k\omega^2 + 4k^3) + 2B\omega, \\ N_1 &= \cos\omega\tau(-4k\omega^4 + 4k^3\omega^2) + \sin\omega\tau(\omega^5 - 6k^2\omega^3 + (k^4 + A)\omega), \\ N_2 &= \cos\omega\tau(-\omega^5 + 6k^2\omega^3 - (k^4 - A)\omega) + \sin\omega\tau(-4k\omega^4 + 4k^3\omega^2). \end{aligned} \tag{17}$$

So, we have

$$\frac{dRe(\lambda(\tau_0))}{d\tau} = Re[(\frac{d\lambda}{d\tau})^{-1}]|_{\tau=\tau_0} = \frac{M_1N_1 + M_2N_2}{N_1^2 + N_2^2}. \tag{18}$$

Obviously, we know that $\frac{dRe(\lambda(\tau_0))}{d\tau} \neq 0$. The conditions of Hopf bifurcation theorem contain $\frac{dRe(\lambda(\tau_0))}{d\tau}$ and $\lambda = \pm\mathrm{i}\omega_0$. So Hopf bifurcation occurs at $\tau = \tau_0$ in system (3), and when τ

passes through τ_0, a family of periodic solutions appear in system (3). This completes the proof. □

3. Direction and stability of Hopf bifurcation

In this section, we shall study the direction of the Hopf bifurcation and the stability of bifurcation periodic solution of system (3). The using approach here is the normal-form method and center manifold theorem introduced by Hassard et al. [1]. More precisely, we will calculate the reduced system on the center manifold with the pair of conjugate complex, purely imaginary solutions of the characteristic equation (7). We can determine the Hopf bifurcation direction, i.e., to answer the question of supercritical bifurcation or subcritical bifurcation whether the bifurcation branch of periodic solution exists locally.

We assume that $f_i \in C^3(\mathbb{R})$, $i = 1, 2, 3, 4, 5, 6$. For convenience, let $x_i = u_i(\tau t)$ and $\tau = \tau^{(j)} + \mu$, where $\tau^{(j)}$ is defined by (11) and $\mu \in \mathbb{R}$, the the system (3) can be written as an FDE in $C = C([-1, 0], \mathbb{R}^6)$ as

$$\dot{x}(t) = L_\mu x_t + F(\mu, x_t), \tag{19}$$

where $x_t(\theta) \in C$, and $L_\mu : C \to \mathbb{R}$, $F : R \times C \to \mathbb{R}$ are given, respectively, by

$$L_\mu(\theta) = (\tau^{(j)}+\mu) \begin{pmatrix} -k & 0 & \alpha_{13} & \alpha_{14} & 0 & 0 \\ 0 & -k & \alpha_{23} & \alpha_{24} & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & \alpha_{53} & \alpha_{54} & -k & 0 \\ 0 & 0 & \alpha_{63} & \alpha_{64} & 0 & -k \end{pmatrix} \phi(0) + (\tau^{(j)}+\mu) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 0 & 0 & \alpha_{35} & \alpha_{36} \\ \alpha_{41} & \alpha_{42} & 0 & 0 & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \phi(-1)$$

and

$$F(\mu, \phi) = (\tau^{(j)} + \mu)\Lambda \tag{20}$$

where Λ are defined by Appendix A.

From the discussions in Section 2, we know that if $\mu = 0$, then system (19) undergoes a Hopf bifurcation at the zero equilibrium and the associated characteristic equation of system (19) has a pair of simple imaginary roots $\pm i\tau^{(j)}\omega_0$.

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, 0)\phi(\theta) \qquad for\ \phi \in C. \tag{21}$$

In fact, we can choose

$$\eta(\theta,\mu)=(\tau^{(j)}+\mu)\begin{pmatrix} -k & 0 & \alpha_{13} & \alpha_{14} & 0 & 0 \\ 0 & -k & \alpha_{23} & \alpha_{24} & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & \alpha_{53} & \alpha_{54} & -k & 0 \\ 0 & 0 & \alpha_{63} & \alpha_{64} & 0 & -k \end{pmatrix}\delta(\theta)-(\tau^{(j)}+\mu)\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 0 & 0 & \alpha_{35} & \alpha_{36} \\ \alpha_{41} & \alpha_{42} & 0 & 0 & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\delta(\theta+1). \tag{22}$$

where δ is defined by

$$\delta(\theta)=\begin{cases} 0, & \theta\neq 0, \\ 1, & \theta=0. \end{cases} \tag{23}$$

Then (19) is satisfied.

Next, for $\phi\in C([-1,0],\mathbb{R}^3)$, we define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta)=\begin{cases} \dfrac{d\phi}{d\theta}, & \theta\in[-1,0), \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta=0. \end{cases} \tag{24}$$

and

$$R(\mu)\phi(\theta)=\begin{cases} 0, & \theta\in[-1,0), \\ F(\mu,\phi), & \theta=0. \end{cases} \tag{25}$$

Since $\frac{du_t}{d\theta}=\frac{du_t}{dt}$, (19) can be rewritten as

$$\dot{x}(t)=A(\mu)x_t+R(\mu)x_t \tag{26}$$

where $x_t=x(t+\theta)$, for$\theta\in[-1,0]$, which is an equation of the form we desired. For $\psi\in C'([-1,0],(R^3)^*)$, we further define the adjoint A^* of A as

$$A^*(\mu)\psi(s)=\begin{cases} -\dfrac{d\psi}{ds}, & s\in(0,1], \\ \displaystyle\int_{-1}^{0} d\psi(-s)d\eta(s,\mu), & s=0. \end{cases} \tag{27}$$

we define a bilinear form

$$\langle\psi(s),\phi(\theta)\rangle=\overline{\psi}^T(0)\phi(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}^T(0)(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi. \tag{28}$$

where $\eta(\theta)=\eta(\theta,0)$. We have the following result on the relation between the operators $A=A(0)$ and A^*.

Lemma 3.1 *$A=A(0)$ and A^* are adjoint operators.*

Proof. Let $\phi \in C^l([-1,0], R^6)$ and $\psi \in C^l([0,1], (\mathbb{R}^6)^*)$. It follows from (28) and the definitions of $A = A(0)$ and A^* that

$$\begin{aligned}
&\langle \psi(s), A(0)\phi(\theta)\rangle \\
&= \overline{\psi}(0)A(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \overline{\psi}\int_{-1}^{0} d\eta(\theta)\phi(\theta) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \overline{\psi}\int_{-1}^{0} d\eta(\theta)\phi(\theta) - \int_{-1}^{0} [\overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)]_{\xi=0}^{\theta} + \int_{-1}^{0}\int_{\xi=0}^{\theta} \frac{d\overline{\psi}(\xi-\theta)}{d\xi}d\eta(\theta)\phi(\xi)d\xi \qquad (29)\\
&= \int_{-1}^{0} \overline{\psi}(-\theta)d\eta(\theta)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} [-\frac{d\overline{\psi}(\xi-\theta)}{d\xi}]d\eta(\theta)\phi(\xi)d\xi \\
&= A^*\overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} A^*\overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \\
&= \langle A^*\psi(s), \phi(\theta)\rangle.
\end{aligned}$$

This shows that $A = A(0)$ and A^* are adjoint operators and the proof is complete. □

From the above analysis, we obtain that $\pm \mathrm{i}\omega_0$ are the eigenvalues of $A(0)$. Let $q(\theta)$ be eigenvector of $A(0)$ corresponding to $\mathrm{i}\omega_0$, then we have

$$A(0)q(\theta) = \mathrm{i}\omega_0 q(\theta). \qquad (30)$$

Since $\pm \mathrm{i}\omega_0$ are the eigenvalues of $A(0)$, and other eigenvalues have strictly negative real parts, $\mp \mathrm{i}\omega_0$ are the eigenvalues of $A^*(0)$. Then we have the following lemma.

Lemma 3.2 *The vector*

$$q(\theta) = (1, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^T e^{\mathrm{i}\omega_0\tau_0\theta}, \quad \theta \in [-1, 0].$$

where

$$\rho_1 = \frac{E_1}{E}, \rho_2 = \frac{E_2}{E}, \rho_3 = \frac{E_3}{E}, \rho_4 = \frac{E_4}{E}, \rho_5 = \frac{E_5}{E}.$$

is the eigenvector of $A(0)$ corresponding to the eigenvalue $\mathrm{i}\omega_0$, and

$$q^*(\theta) = (1, \rho_1^*, \rho_2^*, \rho_3^*, \rho_4^*, \rho_5^*)^T e^{\mathrm{i}\omega_0\tau_0\theta}, \quad \theta \in [0, 1].$$

where

$$\rho_1^* = \frac{E_1^*}{E^*}, \rho_2^* = \frac{E_2^*}{E^*}, \rho_3^* = \frac{E_3^*}{E^*}, \rho_4^* = \frac{E_4^*}{E^*}, \rho_5^* = \frac{E_5^*}{E^*}.$$

is the eigenvector of A^* corresponding to the eigenvalue $-\mathrm{i}\omega_0$, moreover, $\langle q^*(s), q(\theta)\rangle = 1$, where

$$\begin{aligned}\overline{D} = \{1 + \sum_{i=1}^{5} \rho_i \overline{\rho_i^*} + (\alpha_{31} + \rho_1\alpha_{32} + \rho_4\alpha_{35} + \rho_5\alpha_{36})\overline{\rho}_2^* e^{-\mathrm{i}\omega_0\tau_0} \\ + (\alpha_{41} + \rho_1\alpha_{42} + \rho_4\alpha_{45} + \rho_5\alpha_{46})\overline{\rho}_3^* e^{-\mathrm{i}\omega_0\tau_0})\}^{-1}.\end{aligned} \tag{31}$$

Proof Since $q(\theta)$ is eigenvector of $A(0)$ corresponding to $\mathrm{i}\omega_0$, then we have

$$A(0)q(\theta) = \mathrm{i}\omega_0 q(\theta). \tag{32}$$

So (28) can be rewritten as

$$\begin{cases} \dfrac{dq(\theta)}{d\theta}, & \theta \in [-\tau, 0), \\ L(0)q(0) = \mathrm{i}\omega_0 q(0), & \theta = 0. \end{cases} \tag{33}$$

Therefore, we have

$$q(\theta) = q(0)e^{\mathrm{i}\omega_0\theta}, \qquad \theta \in [-\tau, 0]. \tag{34}$$

in addition

$$\begin{aligned}\int_{-1}^{0} d\eta(\theta)q(\theta) = \tau_0 \begin{pmatrix} -k & 0 & \alpha_{13} & \alpha_{14} & 0 & 0 \\ 0 & -k & \alpha_{23} & \alpha_{24} & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & \alpha_{53} & \alpha_{54} & -k & 0 \\ 0 & 0 & \alpha_{63} & \alpha_{64} & 0 & -k \end{pmatrix} q(0) + \tau_0 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 0 & 0 & \alpha_{35} & \alpha_{36} \\ \alpha_{41} & \alpha_{42} & 0 & 0 & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} q(-1) \\ = A(0)q(0) = \mathrm{i}\omega_0\tau_0 q(0).\end{aligned} \tag{35}$$

that is

$$\begin{pmatrix} -k + \alpha_{13}\rho_2 + \alpha_{14}\rho_3 \\ -k\rho_1 + \alpha_{23}\rho_2 + \alpha_{24}\rho_3 \\ -k\rho_2 + (\alpha_{31} + \rho_1\alpha_{32} + \rho_4\alpha_{35} + \rho_5\alpha_{36})e^{-\mathrm{i}\omega_0\tau_0} \\ -k\rho_3 + (\alpha_{41} + \rho_1\alpha_{42} + \rho_4\alpha_{45} + \rho_5\alpha_{46})e^{-\mathrm{i}\omega_0\tau_0} \\ -k\rho_4 + \alpha_{53}\rho_2 + \alpha_{54}\rho_3 \\ -k\rho_5 + \alpha_{63}\rho_2 + \alpha_{64}\rho_3 \end{pmatrix} = \begin{pmatrix} \mathrm{i}\omega_0 \\ \mathrm{i}a_1\omega_0 \\ \mathrm{i}a_2\omega_0 \\ \mathrm{i}a_3\omega_0 \\ \mathrm{i}a_4\omega_0 \\ \mathrm{i}a_5\omega_0 \end{pmatrix}.$$

Therefore, we can easily obtain

$$\rho_1 = \frac{E_1}{E}, \rho_2 = \frac{E_2}{E}, \rho_3 = \frac{E_3}{E}, \rho_4 = \frac{E_4}{E}, \rho_5 = \frac{E_5}{E}.$$

where $E, E_1, E_2, E_3, E_4, E_5$ are defined by Appendix B.

On the other hand

$$\begin{aligned}\int_{-1}^{0} q^*(-t)d\eta(t) = \tau_0 \begin{pmatrix} -k & 0 & \alpha_{13} & \alpha_{14} & 0 & 0 \\ 0 & -k & \alpha_{23} & \alpha_{24} & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & \alpha_{53} & \alpha_{54} & -k & 0 \\ 0 & 0 & \alpha_{63} & \alpha_{64} & 0 & -k \end{pmatrix}^T q^*(0) + \tau_0 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 0 & 0 & \alpha_{35} & \alpha_{36} \\ \alpha_{41} & \alpha_{42} & 0 & 0 & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T q^*(-1) \\ = A^*(0)q^*(0)\tau_0 = -\mathrm{i}\omega_0\tau_0 q^*(0).\end{aligned} \tag{36}$$

Namely

$$\begin{pmatrix} -k+\alpha_{13}\rho_2^*+\alpha_{14}\rho_3^* \\ -k+\alpha_{23}\rho_2^*+\alpha_{24}\rho_3^* \\ -k\rho_2^*+(\alpha_{31}+\alpha_{32}\rho_1^*+\alpha_{35}\rho_4^*+\alpha_{36}\rho_5^*)e^{-\mathrm{i}\omega_0\tau_0} \\ -k\rho_3^*+(\alpha_{41}+\alpha_{42}\rho_1^*+\alpha_{45}\rho_4^*+\alpha_{46}\rho_5^*)e^{-\mathrm{i}\omega_0\tau_0} \\ -k\rho_4^*+\alpha_{53}\rho_2^*+\alpha_{54}\rho_3^* \\ -k\rho_5^*+\alpha_{63}\rho_2^*+\alpha_{64}\rho_3^* \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\omega_0 \\ -\mathrm{i}\omega_0\rho_1^* \\ -\mathrm{i}\omega_0\rho_2^* \\ -\mathrm{i}\omega_0\rho_3^* \\ -\mathrm{i}\omega_0\rho_4^* \\ -\mathrm{i}\omega_0\rho_5^* \end{pmatrix}.$$

Therefore, we can easily obtain

$$\rho_1^* = \frac{E_1^*}{E^*}, \rho_2^* = \frac{E_2^*}{E^*}, \rho_3^* = \frac{E_3^*}{E^*}, \rho_4^* = \frac{E_4^*}{E^*}, \rho_5^* = \frac{E_5^*}{E^*}.$$

where $E^*, E_1^*, E_2^*, E_3^*, E_4^*, E_5^*$ are defined by Appendix C.

In the sequel, we shall verify that $\langle q^*(\theta), q(\theta)\rangle = 1$. In fact, from (28), we have

$$\begin{aligned} &\langle q^*(\theta), q(\theta)\rangle = (1, \overline{\rho}_1^*, \overline{\rho}_2^*, \overline{\rho}_3^*, \overline{\rho}_4^*, \overline{\rho}_5^*)(1, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{0}(1, \overline{\rho}_1^*, \overline{\rho}_2^*, \overline{\rho}_3^*, \overline{\rho}_4^*, \overline{\rho}_5^*)e^{-\mathrm{i}\omega_0\tau(\xi-\theta)}d\eta(\theta)\times(1, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^T e^{\mathrm{i}\omega_0\tau\xi}d\xi \\ &= \overline{D}[1+\sum_{i=1}^{5}\rho_i\overline{\rho}_i^* - \int_{-1}^{0}(1, \overline{\rho}_1^*, \overline{\rho}_2^*, \overline{\rho}_3^*, \overline{\rho}_4^*, \overline{\rho}_5^*)\theta e^{\mathrm{i}\omega_0\tau\theta}d\eta(\theta)\times(1, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)^T] \\ &= \overline{D}\{1+\sum_{i=1}^{5}\rho_i\overline{\rho}_i^* + (\alpha_{31}+\rho_1\alpha_{32}+\rho_4\alpha_{35}+\rho_5\alpha_{36})\overline{\rho}_2^* e^{-\mathrm{i}\omega_0\tau_0} + (\alpha_{41}+\rho_1\alpha_{42}+\rho_4\alpha_{45}+\rho_5\alpha_{46})\overline{\rho}_3^* e^{-\mathrm{i}\omega_0\tau_0}\}. \end{aligned} \tag{37}$$

where

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 0 & 0 & \alpha_{35} & \alpha_{36} \\ \alpha_{41} & \alpha_{42} & 0 & 0 & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, when

$$\overline{D} = \{1+\sum_{i=1}^{5}\rho_i\overline{\rho}_i^* + (\alpha_{31}+\rho_1\alpha_{32}+\rho_4\alpha_{35}+\rho_5\alpha_{36})\overline{\rho}_2^* e^{-\mathrm{i}\omega_0\tau_0} + (\alpha_{41}+\rho_1\alpha_{42}+\rho_4\alpha_{45}+\rho_5\alpha_{46})\overline{\rho}_3^* e^{-\mathrm{i}\omega_0\tau_0}\}^{-1}. \tag{38}$$

we can get $\langle q^*, q\rangle = 1$. On the other hand, since $\langle \psi, A\phi\rangle = \langle A^*\psi, \phi\rangle$, we can obtain

$$\mathrm{i}\omega_0\langle q^*, \overline{q}\rangle = \langle q^*, A\overline{q}\rangle = \langle A^*q^*, \overline{q}\rangle = \langle -\mathrm{i}\omega_0 q^*, \overline{q}\rangle = -\mathrm{i}\omega_0\langle q^*, \overline{q}\rangle.$$

Therefore, $\langle q^*, \overline{q}\rangle = 0$. This completes the proof. $\square$

In the remainder of this section, by using the same notations as in Hassard et al. [1], we first compute the coordinates to describe the center manifold center C_0 at $\mu = 0$, which is locally invariant, attracting three-dimensional manifold in C_0. Let u_t be solution of (26) when $\mu = 0$. Define

$$\begin{cases} z(t) = \langle q^*, x_t\rangle, \\ W(t,\theta) = u_t - zq - \overline{zq} = x_t - 2Rez(t)q(\theta). \end{cases} \tag{39}$$

On the center manifold C_0, we have

$$W(z,\overline{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots \tag{40}$$

z and $\overline{z}$ are local coordinates of center manifold C_0 in direction of q^* and $\overline{q}^*$. Note the W is real if u_t is real. We only consider real solutions.

$$\begin{aligned}\dot{z}(t) &= \langle q^*, u_t\rangle = \langle q^*, A(0)u_t + R(0)u_t\rangle \\ &= \langle A^*(0)q^*, u_t\rangle + \overline{q}^*(0)F(u_t,0) \\ &\doteq \mathrm{i}\omega_0 z(t) + \overline{q}^{*^T}(0)f_0(z,\overline{z})\end{aligned} \tag{41}$$

We rewrite in abbreviated form as

$$\dot{z}(t) = \mathrm{i}\omega_0 z(t) + g(z,\overline{z}), \tag{42}$$

where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots, \tag{43}$$

Hence we have

$$\begin{aligned}g(z,\overline{z}) &= \overline{q}^{*T}(0)f_0(z,\overline{z}) = \overline{q}^{*T}(0)f(0,x_t) \\ &= \overline{D}(1,\overline{\rho_1^*},\overline{\rho_2^*},\overline{\rho_3^*},\overline{\rho_4^*},\overline{\rho_5^*}) \times (f_1(0,x_t), f_2(0,x_t),\cdots,f_6(0,x_t))^T,\end{aligned} \tag{44}$$

where $f_1(0,x_t)$, $f_2(0,x_t)$, $f_3(0,x_t)$, $f_4(0,x_t)$, $f_5(0,x_t)$, $f_6(0,x_t)$ are defined by Appendix D. By (40), we have

$$x_t(\theta) = (x_{1t},x_{2t},x_{3t},x_{4t},x_{5t},x_{6t}) = W(t,\theta) + zq(\theta) + \overline{z}\overline{q}(\theta), \tag{45}$$

and

$$q(\theta) = (1,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5)^T e^{\mathrm{i}\theta\omega_0\tau^{(j)}}, \tag{46}$$

we can obtain

$$\begin{aligned}
x_{3t}(0) &= z\rho_2 + \overline{z}\overline{\rho}_2 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\overline{z} + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + O(|(z,\overline{z})|^3),\\
x_{4t}(0) &= z\rho_2 + \overline{z}\overline{\rho}_3 + W_{20}^{(4)}(0)\frac{z^2}{2} + W_{11}^{(4)}(0)z\overline{z} + W_{02}^{(4)}(0)\frac{\overline{z}^2}{2} + O(|(z,\overline{z})|^3),\\
x_{1t}(-1) &= ze^{-\mathrm{i}\omega_0\tau^{(j)}} + \overline{z}e^{\mathrm{i}\omega_0\tau^{(j)}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + O(|(z,\overline{z})|^3),\\
x_{2t}(-1) &= z\rho_1 e^{-\mathrm{i}\omega_0\tau^{(j)}} + \overline{z}\overline{\rho_1}e^{\mathrm{i}\omega_0\tau^{(j)}} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + O(|(z,\overline{z})|^3),
\end{aligned}$$

$$x_{5t}(-1)=z\rho_4 e^{-\mathrm{i}\omega_0\tau^{(j)}}+\overline{z\rho_4}e^{\mathrm{i}\omega_0\tau^{(j)}}+W_{20}^{(5)}(-1)\frac{z^2}{2}+W_{11}^{(5)}(-1)z\overline{z}+W_{02}^{(5)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3),$$

$$x_{6t}(-1)=z\rho_5 e^{-\mathrm{i}\omega_0\tau^{(j)}}+\overline{z\rho_5}e^{\mathrm{i}\omega_0\tau^{(j)}}+W_{20}^{(6)}(-1)\frac{z^2}{2}+W_{11}^{(6)}(-1)z\overline{z}+W_{02}^{(6)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3).$$

From (42) and (43), we get the result of $g(z,\overline{z})$ (see Appendix E) and then we obtain

$$\begin{aligned}
g_{20} = {} & \overline{D}\{[a_1\rho_2^2+a_3\rho_3^2]+\overline{\rho}_1^*[b_1\rho_2^2+b_3\rho_3^2]+\overline{\rho}_2^*[(c_1+c_3\rho_1^2+c_5\rho_4^2+c_7\rho_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}] \\
& +\overline{\rho}_3^*[(d_1+d_3\rho_1^2+d_5\rho_4^2+d_7\rho_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}]+\overline{\rho}_4^*[e_1\rho_2^2+e_3\rho_2^2]+\overline{\rho}_5^*[f_1\rho_2^2+f_3\rho_2^2]\}, \\
g_{11} = {} & \overline{D}\{[a_1|\rho_2|^2+a_3|\rho_3|^2]+\overline{\rho}_1^*[b_1|\rho_2|^2+b_3|\rho_3|^2] \\
& +\overline{\rho}_2^*[c_1+c_3|\rho_1|^2+c_5|\rho_4|^2+c_7|\rho_5|^2]+\overline{\rho}_3^*[d_1+d_3|\rho_1|^2+d_5|\rho_4|^2+d_7|\rho_5|^2] \\
& +\overline{\rho}_4^*[e_1|\rho_2|^2+e_3|\rho_3|^2]+\overline{\rho}_5^*[f_1|\rho_2|^2+f_3|\rho_3|^2]\}, \\
g_{02} = {} & \overline{D}\{[a_1\overline{\rho}_2^2+a_3\overline{\rho}_3^2]+\overline{\rho}_1^*[b_1\overline{\rho}_2^2+b_3\overline{\rho}_2^2]+\overline{\rho}_2^*[(c_1+c_3\overline{\rho}_1^2+c_5\overline{\rho}_4^2+c_7\overline{\rho}_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}] \\
& +\overline{\rho}_3^*[(d_1+d_3\overline{\rho}_1^2+d_5\overline{\rho}_4^2+d_7\overline{\rho}_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}]+\overline{\rho}_4^*[e_1\overline{\rho}_2^2+e_3\overline{\rho}_2^2]+\overline{\rho}_5^*[f_1\overline{\rho}_2^2+f_3\overline{\rho}_2^2]\}, \\
g_{21} = {} & \overline{D}\{[2a_1\rho_2W_{11}^{(3)}(0)+2a_3\rho_3W_{11}^{(4)}(0)+a_1\overline{\rho}_2W_{20}^{(3)}(0)+a_3\overline{\rho}_2W_{20}^{(3)}(0) \\
& +3a_2|\rho_2|^2\rho_2+3a_4|\rho_3|^2\rho_3]+\overline{\rho}_1^*[2b_1\rho_2W_{11}^{(3)}(0)+2b_3\rho_3W_{11}^{(4)}(0) \\
& +b_1\overline{\rho}_2W_{20}^{(3)}(0)+b_3\overline{\rho}_2W_{20}^{(3)}(0)+3b_2|\rho_2|^2\rho_2+3b_4|\rho_3|^2\rho_3] \\
& +\overline{\rho}_2^*[2e^{-\mathrm{i}\omega_0\tau^{(j)}}(c_1W_{11}^{(1)}(-1)+c_3W_{11}^{(2)}(-1)\rho_1+c_5W_{11}^{(5)}(-1)\rho_4+c_7W_{11}^{(6)}(-1)\rho_5) \\
& +e^{\mathrm{i}\omega_0\tau^{(j)}}(c_1W_{20}^{(1)}(-1)+c_3W_{20}^{(2)}(-1)\overline{\rho}_1+c_5W_{20}^{(5)}(-1)\overline{\rho}_4+c_7W_{20}^{(6)}(-1)\overline{\rho}_5) \\
& +3e^{-\mathrm{i}\omega_0\tau^{(j)}}(c_2+c_4|\rho_1|^2\rho_1+c_6|\rho_4|^2\rho_4+c_8|\rho_5|^2\rho_5)] \\
& +\overline{\rho}_3^*[2e^{-\mathrm{i}\omega_0\tau^{(j)}}(d_1W_{11}^{(1)}(-1)+d_3W_{11}^{(2)}(-1)\rho_1+d_5W_{11}^{(5)}(-1)\rho_4+d_7W_{11}^{(6)}(-1)\rho_5) \\
& +e^{\mathrm{i}\omega_0\tau^{(j)}}(d_1W_{20}^{(1)}(-1)+d_3W_{20}^{(2)}(-1)\overline{\rho}_1+d_5W_{20}^{(5)}(-1)\overline{\rho}_4+d_7W_{20}^{(6)}(-1)\overline{\rho}_5) \\
& +3e^{-\mathrm{i}\omega_0\tau^{(j)}}(d_2+d_4|\rho_1|^2\rho_1+d_6|\rho_4|^2\rho_4+d_8|\rho_5|^2\rho_5)] \\
& +\overline{\rho}_4^*[2e_1\rho_2W_{11}^{(3)}(0)+2e_3\rho_3W_{11}^{(4)}(0)+e_1\overline{\rho}_2W_{20}^{(3)}(0)+e_3\overline{\rho}_2W_{20}^{(3)}(0) \\
& +3e_2|\rho_2|^2\rho_2+3e_4|\rho_3|^2\rho_3]+\overline{\rho}_5^*[2f_1\rho_2W_{11}^{(3)}(0)+2f_3\rho_3W_{11}^{(4)}(0) \\
& +f_1\overline{\rho}_2W_{20}^{(3)}(0)+f_3\overline{\rho}_2W_{20}^{(3)}(0)+3f_2|\rho_2|^2\rho_2+3f_4|\rho_3|^2\rho_3]\}.
\end{aligned}$$

In order to determine g_{21}, in the sequel, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From

(26) and (40), we have

$$\begin{aligned}\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = A(0)u_t + R(0)u_t - (i\omega_0 z + g)q - (-\mathrm{i}\omega_0\bar{z} + \bar{g})\bar{q} \\ &= A(0)u_t + R(0)u_t - 2\mathrm{Re}(gq) \\ &= \begin{cases} A(0)W - 2\mathrm{Re}(\bar{q}^*(0)f_0q(\theta)), \theta \in [-\tau, 0), \\ A(0)W - 2\mathrm{Re}(\bar{q}^*(0)f_0q(0)) + f_0, \theta = 0, \end{cases} \\ &= A(0)W + H(z, \bar{z}, \theta).\end{aligned}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z_2}{2} + H_{11}(\theta)z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots \tag{47}$$

On the other hand, note that on the center manifold C_0 near to the origin,

$$\dot{W} = W_z\dot{z} + W_{\dot{z}} + W_{\overline{z}}\dot{\overline{z}}. \tag{48}$$

Substituting (28) and (42) into (48), and comparing the coefficients of the above equation with those of (41), we get

$$\begin{cases} (A(0) - 2\mathrm{i}\omega_0)W_{20}(\theta) = -H_{20}(\theta), \\ A(0)W_{11} = -H_{11}(\theta), \\ (A(0) + 2\mathrm{i}\omega_0)W_{11}(\theta) = -H_{02}(\theta). \end{cases} \tag{49}$$

By (47), we know that for $\theta \in [-1, 0)$,

$$\begin{aligned}&H(z,\overline{z},\theta) = -2Re(\overline{q}^*(0)f_0q(\theta)) = -2Re(g(z,\overline{z})q(\theta)) = -g(z,\overline{z})q(\theta) - \overline{g}(z,\overline{z})\overline{q}(\theta) \\ &= -(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2})q(\theta) - (\overline{g}_{20}\frac{\overline{z}^2}{2} + g_{11}z\overline{z} + \overline{g}_{02}\frac{z^2}{2} + \overline{g}_{21}\frac{\overline{z}^2z}{2})\overline{q}(\theta)\cdots\end{aligned} \tag{50}$$

Comparing the coefficients of the above equation with those in (47), it is obvious that

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \end{aligned} \tag{51}$$

It follows from (27) and (49) that

$$\begin{aligned}\dot{W}_{20}(\theta) &= AW_{20}(\theta) = 2\mathrm{i}\omega_0W_{20}(\theta) - H_{20}(\theta) = 2\mathrm{i}\omega_0W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta) \\ &= 2\mathrm{i}\omega_0W_{20}(\theta) + g_{20}q(0)e^{\mathrm{i}\omega_0\theta} + \overline{g}_{02}\overline{q}(0)e^{-\mathrm{i}\omega_0\theta}.\end{aligned}$$

Solving for $W_{20}(\theta)$, we can obtain

$$W_{20}(\theta) = \frac{\mathrm{i}g_{20}q(0)}{\omega_0}e^{\mathrm{i}\omega_0\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\omega_0}e^{-\mathrm{i}\omega_0\theta} + G_1 e^{2\mathrm{i}\omega_0\theta}. \tag{52}$$

By similar way, we get

$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}q(0)}{\omega_0}e^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\overline{g}_{11}\overline{q}(0)}{\omega_0}e^{-\mathrm{i}\omega_0\theta} + G_2 \tag{53}$$

Where

$$G_1 = (G_1^{(1)}, G_1^{(2)}, G_1^{(3)}, G_1^{(4)}, G_1^{(5)}, G_1^{(6)})^T$$

and

$$G_2 = (G_2^{(1)}, G_2^{(2)}, G_2^{(3)}, G_2^{(4)}, G_2^{(5)}, G_2^{(6)})^T$$

are both constant vectors, and can be determined by setting $\theta = 0$ in $H(z, \overline{z}, \theta)$. In what follows, we shall seek appropriate G_1 and G_2. From the definition of A and (49), we obtain

$$\begin{aligned} \textstyle\int_{-1}^{0} d\eta(0,\theta)W_{20}(\theta) &= 2\mathrm{i}\omega_0 W_{20}(0) - H_{20}(0), \\ \textstyle\int_{-1}^{0} d\eta(0,\theta)W_{11}(\theta) &= -H_{11}(0). \end{aligned} \tag{54}$$

where $\eta(\theta) = \eta(0, \theta)$. From (47), we have

$$H_{20}(\theta) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + (H_1, H_2, \cdots, H_6)^T. \tag{55}$$

where

$$\begin{aligned} &H_1 = a_1\rho_2^2 + a_3\rho_3^2, H_2 = b_1\rho_2^2 + b_3\rho_3^2, H_3 = (c_1 + c_3\rho_1^2 + c_5\rho_4^2 + c_7\rho_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}, \\ &H_4 = (d_1 + d_3\rho_1^2 + d_5\rho_4^2 + d_7\rho_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}, H_5 = e_1\rho_2^2 + e_3\rho_3^2, H_6 = f_1\rho_2^2 + f_3\rho_3^2, \\ &H^* = (H_1, H_2, H_3, H_4, H_5, H_6)^T. \end{aligned} \tag{56}$$

and

$$H_{11}(\theta) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + (P_1, P_2, \cdots, P_6)^T. \tag{57}$$

where

$$\begin{aligned} &P_1 = a_1|\rho_2|^2 + a_3|\rho_3|^2, P_2 = b_1|\rho_2|^2 + b_3|\rho_3|^2, P_3 = c_1 + c_3|\rho_1|^2 + c_5|\rho_4|^2 + c_7|\rho_5|^2, \\ &P_4 = d_1 + d_3|\rho_1|^2 + d_5|\rho_4|^2 + d_7|\rho_5|^2, P_5 = e_1|\rho_2|^2 + e_3|\rho_3|^2, P_6 = f_1|\rho_2|^2 + f_3|\rho_3|^2, \\ &P^* = (-P_1, -P_2, -P_3, -P_4, -P_5, -P_6)^T. \end{aligned} \tag{58}$$

Substituting (52) and (55) into (54) and noticing that

$$
\begin{aligned}
(\mathrm{i}\omega_0 I - \int_{-1}^{0} d\eta(0,\theta)e^{\mathrm{i}\omega_0\theta})q(0) = 0,\\
(-\mathrm{i}\omega_0 I - \int_{-1}^{0} d\eta(0,\theta)e^{-\mathrm{i}\omega_0\theta})\overline{q}(0) = 0.
\end{aligned}
\tag{59}
$$

we can obtain

$$
(2\mathrm{i}\omega_0 I - \int_{-1}^{0} d\eta(0,\theta)e^{2\mathrm{i}\omega_0\theta})G_1 = (H_1, H_2, \cdots, H_6)^T. \tag{60}
$$

which leads to

$$
(L_1^{(1)}, L_1^{(2)}, L_1^{(3)}, L_1^{(4)}, L_1^{(5)}, L_1^{(6)})G_1 = H^*. \tag{61}
$$

where $L_1^{(1)}, L_1^{(2)}, L_1^{(3)}, L_1^{(4)}, L_1^{(5)}, L_1^{(6)} and G_1$ are defined by Appendix F.

Similarly, substituting (53) and (57) into (54), we can get

$$
(L_2^{(1)}, L_2^{(2)}, L_2^{(3)}, L_2^{(4)}, L_2^{(5)}, L_2^{(6)})G_2 P^*. \tag{62}
$$

where $L_2^{(1)}, L_2^{(2)}, L_2^{(3)}, L_2^{(4)}, L_2^{(5)}, L_2^{(6)} and G_2$ are defined by Appendix G.

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from *lemma* 3.1 and (68). Furthermore, we can see that each g_{ij} in (50) is determined by parameters and delays in system (3). Thus, we can compute the following quantities.

$$
\begin{cases}
c_1(0) = \dfrac{\mathrm{i}}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \dfrac{|g_{02}|^2}{3}) + \dfrac{g_{21}}{2},\\
\mu_2 = -\dfrac{Re\{c_1(0)\}}{Re\{\lambda'(\tau_0)\}},\\
\beta_2 = 2Re\{c_1(0)\},\\
T_2 = -\dfrac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0}.
\end{cases}
\tag{63}
$$

Theorem 3.1. *In (51), the following results hold:*
(i) The sign of μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcation periodic solutions exist for $\tau > \tau^0$ ($\tau < \tau^0$);
(ii) The sign of β_2 determines the stability of the bifurcating periodic solution: the bifurcation periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$);
(iii) The sign of T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0$ ($T_2 < 0$).

4. Numerical examples

In this section, we give some numerical results of system (3) to support the analytic results obtained above. We consider the following system as:

$$
\begin{cases}
\dot{x}_1(t) = -0.7x_1(t) - 0.5\tanh y_1(t) + 0.6\tanh y_2(t), \\
\dot{x}_2(t) = 0.7x_2(t) - 0.8\tanh y_1(t) + 1.4\tanh y_2(t), \\
\dot{y}_1(t) = -0.7y_1(t) + 0.6\tanh(x_1(t-\tau)) - 0.8\tanh(x_2(t-\tau)) \\
\qquad\qquad + 0.6\tanh(z_1(t-\tau)) - 1.8\tanh(z_2(t-\tau)), \\
\dot{y}_2(t) = -0.7y_2(t) + 0.7\tanh(x_1(t-\tau)) - 1.5\tanh(x_2(t-\tau)) \\
\qquad\qquad + 0.9\tanh(z_1(t-\tau)) - 1.4\tanh(z_2(t-\tau)), \\
\dot{z}_1(t) = -0.7z_1(t) + 0.6\tanh y_1(t) - 1.5\tanh y_2(t), \\
\dot{z}_2(t) = -0.7z_2(t) + 1.2\tanh y_1(t) - 2\tanh y_2(t)
\end{cases}
\tag{64}
$$

Obviously, the system have the equilibrium $(0,0,0,0,0,0)$, from section 2. we compute that $\omega_0 = 0.9009$, from (11) we can obtain $\tau = 1.7546$, where $\tau = \tau_1 + \tau_3 = \tau_2 + \tau_4$. From Theorem 3.1, we know that the zero steady state of system (64) is asymptotically stable $\tau \in [0, 1.7546)$. This is illustrated by the numerical simulation shown in Figs.2 and 3 in which $\tau_1 = 1.3$. Further, when τ is increased to the critical value 1.7546, the origin losses its stability and Hopf bifurcation occurs (see Figs.4 and 5). By Theorem 3.1, the bifurcation is supercritical and the bifurcating periodic solution is asymptotically stable (see Figs.6 and 7).

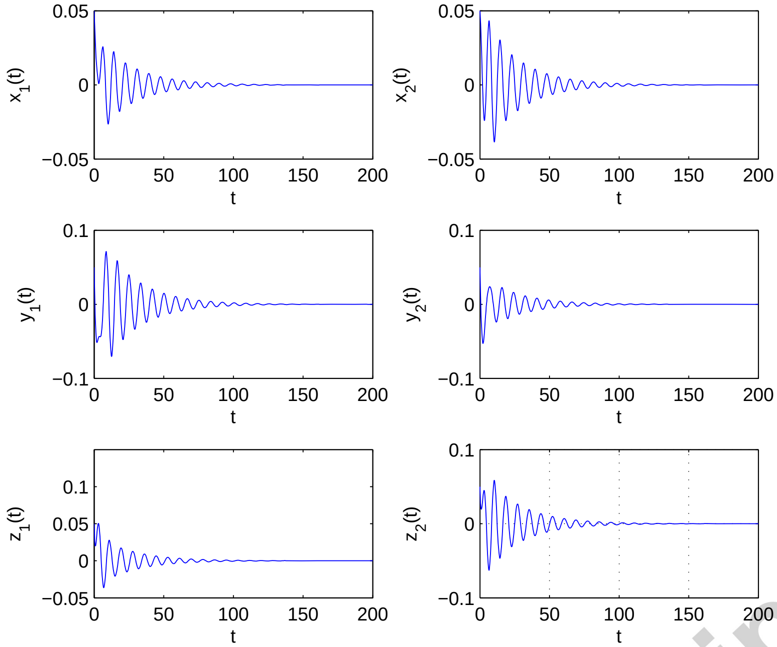

Fig.2. The trajectories graphs when $\tau = 1.3 < 1.7546$.

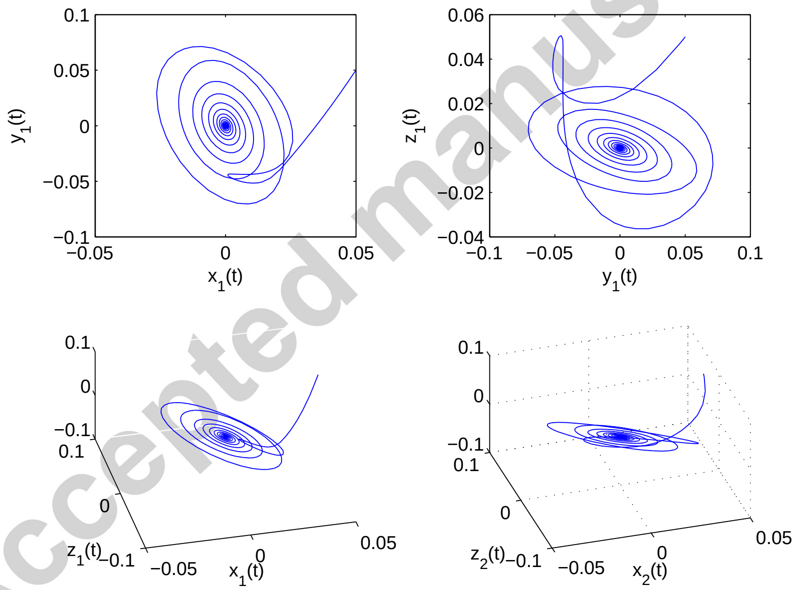

Fig.3. The phase graphs when $\tau = 1.3 < 1.7546$.

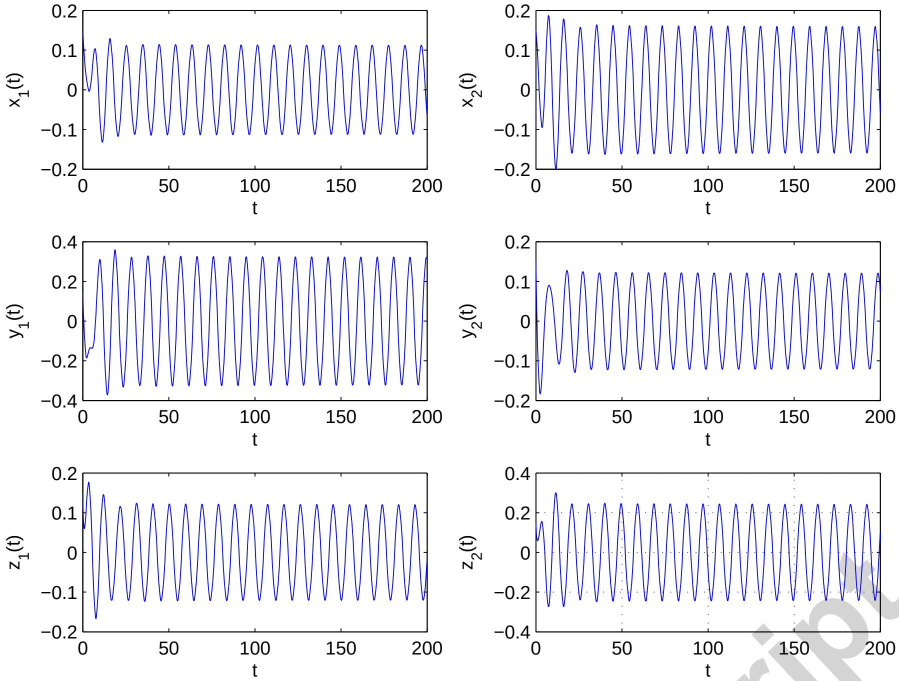

Fig.4. The trajectories graphs when $\tau = 1.75$.

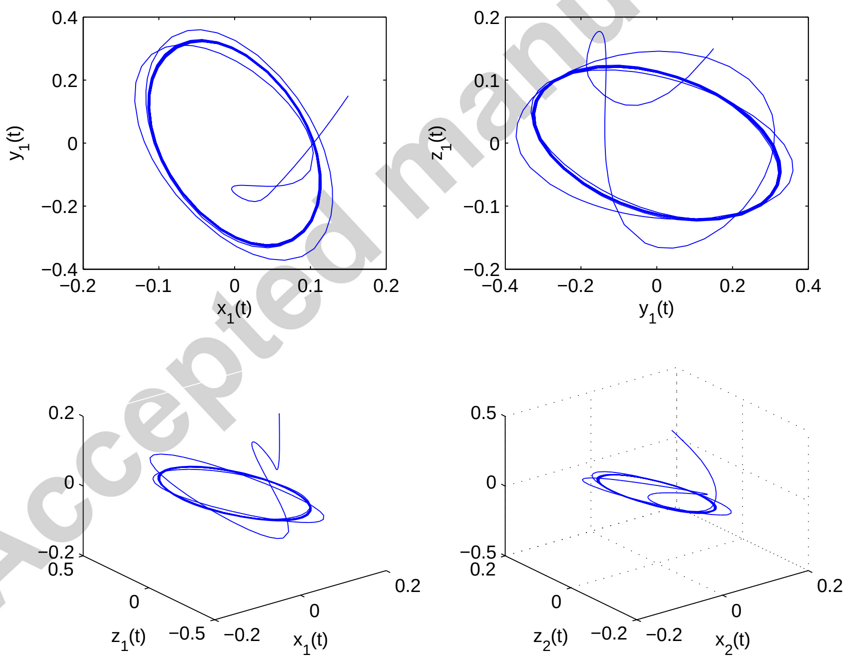

Fig.5. The phase graphs when $\tau = 1.75$.

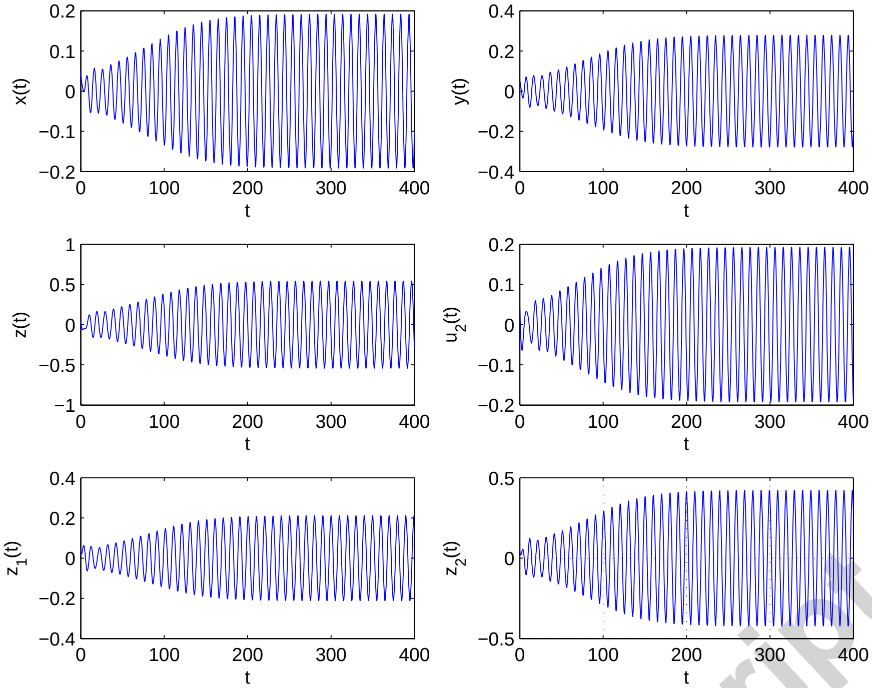

Fig.6. The trajectories graphs when $\tau = 1.9 > 1.7546$.

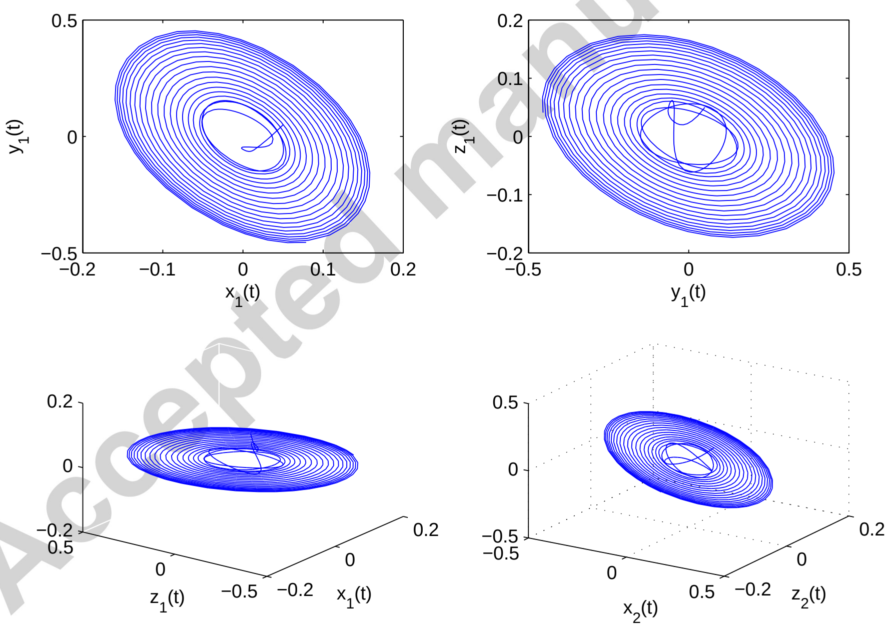

Fig.7. The phase graphs when $\tau = 1.9 > 1.7546$.

Finally, from section 4, we compute the direction of Hopf bifurcation. Noting that $\tanh''(0) = 0$, we can easily obtain g_{20}, g_{02}, g_{11}. Thus we can calculate the following values: $C_1(0) = 9.1486 + 2.8498i$. $\mu_2 = -3.3078$, $\beta_2 = 18.2972$, $T_2 = -12.7765$.

5. Conclusions

This paper studies a three-layer neural network model with six neurons. So far, the research in this area has been less fruitful. The main reason is that the three-layer neural network is more complex than other neural network models, and the it is difficult to describe. In general, the three-layer neural network has many neurons, and the more neurons in neural network, the more close to the actual neural network. As the BP neural network has many outstanding advantages, so it is necessary to study on it.

In this paper, we have investigated local stability of the equilibrium and local Hopf bifurcation in a three-layer neural network model with delays. We have showed that if the condition $(H1)$ $(H2)$ hold, the equilibrium of system (3) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. We also have showed that, if the condition $(H1)$ $(H2)$ hold, as the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur at (0,0,0,0,0,0), i.e., a family of periodic orbits bifurcates from the positive equilibrium. Then, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. At last, a numerical example illustrated the obtained results.

Appendix A

$$
\begin{aligned}
&\Lambda = (L_1, L_2, L_3, L_4, L_5, L_6)^T.\\
&L_1 = a_1\phi_3^2(0) + a_2\phi_3^3(0) + a_3\phi_4^2(0) + a_4\phi_4^3(0) + h.o.t.\\
&L_2 = b_1\phi_3^2(0) + b_2\phi_3^3(0) + b_3\phi_4^2(0) + b_4\phi_4^3(0) + h.o.t.\\
&L_3 = c_1\phi_1^2(-1) + c_2\phi_1^3(-1) + c_3\phi_2^2(-1) + c_4\phi_2^3(-1)\\
&\qquad + c_5\phi_5^2(-1) + c_6\phi_5^3(-1) + c_7\phi_6^2(-1) + c_8\phi_6^3(-1) + h.o.t.\\
&L_4 = d_1\phi_1^2(-1) + d_2\phi_1^3(-1) + d_3\phi_2^2(-1) + d_4\phi_2^3(-1)\\
&\qquad + d_5\phi_5^2(-1) + d_6\phi_5^3(-1) + d_7\phi_6^2(-1) + d_8\phi_6^3(-1) + h.o.t.\\
&L_5 = e_1\phi_3^2(0) + e_2\phi_3^3(0) + e_3\phi_4^2(0) + e_4\phi_4^3(0) + h.o.t.\\
&L_6 = f_1\phi_3^2(0) + f_2\phi_3^3(0) + f_3\phi_4^2(0) + f_4\phi_4^3(0) + h.o.t.
\end{aligned}
$$

and $\phi(\theta)$=$(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta), \phi_5(\theta), \phi_6(\theta))^T \in C$, $a_i = c_{1m}f_{1m}^k(0)/k!, b_i = c_{2m}f_{2m}^k(0)/k!,$ $c_j = c_{3n}f_{3n}^k(0)/k!, d_j = c_{4n}f_{4n}^k(0)/k!, e_i = c_{5m}f_{5m}^k(0)/k!, f_i = c_{6m}f_{6m}^k(0)/k!.i = 1, ..., 4;$ $j = 1, ..., 8; k = 2, 3; m = 3, 4; n = 1, 2, 5, 6.$

Appendix B

$$E = \mid F_1, F_2, F_3, F_4, F_5, \mid, E_1 = \mid \Delta, F_2, F_3, F_4, F_5, \mid, \tag{65}$$

$$E_2 = \mid F_1, \Delta, F_3, F_4, F_5, \mid, E_3 = \mid F_1, F_2, \Delta, F_4, F_5, \mid, \tag{66}$$

$$E_4 = \mid F_1, F_2, F_3, \Delta, F_5, \mid, E_5 = \mid F_1, F_2, F_3, F_4, \Delta, \mid . \tag{67}$$

where

$$F_1 = (-k - \mathrm{i}\omega_0, \alpha_{32}e^{-\mathrm{i}\omega_0\tau_0}, \alpha_{42}e^{-\mathrm{i}\omega_0\tau_0}, 0, 0)^T, F_2 = (\alpha_{23}, -k - \mathrm{i}\omega_0, 0, \alpha_{53}, \alpha_{63})^T,$$
$$F_3 = (\alpha_{24}, 0, -k - \mathrm{i}\omega_0, \alpha_{54}, \alpha_{64})^T, F_4 = (0, 0, 0, -k - \mathrm{i}\omega_0, , 0)^T,$$
$$F_5 = (0, 0, 0, 0, -k - \mathrm{i}\omega_0)^T, \Delta = (0, -\alpha_{31}e^{-\mathrm{i}\omega_0\tau_0}, -\alpha_{41}e^{-\mathrm{i}\omega_0\tau_0}, 0, 0)^T.$$

Appendix C

$$E^* = \mid F_1^*, F_2^*, F_3^*, F_4^*, F_5^*, \mid, E_1^* = \mid \Delta^*, F_2^*, F_3^*, F_4^*, F_5^*, \mid,$$
$$E_2^* = \mid F_1^*, \Delta^*, F_3^*, F_4^*, F_5^*, \mid, E_3^* = \mid F_1^*, F_2^*, \Delta^*, F_4^*, F_5^*, \mid,$$
$$E_4^* = \mid F_1^*, F_2^*, F_3^*, \Delta^*, F_5^*, \mid, E_5^* = \mid F_1^*, F_2^*, F_3^*, F_4^*, \Delta^*, \mid .$$

where

$$F_1^* = (-k + \mathrm{i}\omega_0, \alpha_{23}, \alpha_{24}, 0, 0)^T, F_2^* = (\alpha_{32}e^{-\mathrm{i}\omega_0\tau_0}, -k + \mathrm{i}\omega_0, 0, \alpha_{35}e^{-\mathrm{i}\omega_0\tau_0}, \alpha_{36}e^{-\mathrm{i}\omega_0\tau_0})^T,$$
$$F_3^* = (\alpha_{42}e^{-\mathrm{i}\omega_0\tau_0}, 0, -k + \mathrm{i}\omega_0, \alpha_{45}e^{-\mathrm{i}\omega_0\tau_0}, \alpha_{46}e^{-\mathrm{i}\omega_0\tau_0})^T, F_4^* = (0, \alpha_{53}, \alpha_{54}, -k + \mathrm{i}\omega_0, 0)^T,$$
$$F_5^* = (0, \alpha_{63}, \alpha_{64}, 0, -k + \mathrm{i}\omega_0)^T, \Delta^* = (0, -\alpha_{13}, -\alpha_{14}, 0, 0)^T.$$

Appendix D

$$f_1(0, x_t) = a_1x_{3t}^2(0) + a_2x_{3t}^3(0) + a_3x_{4t}^2(0) + a_4x_{4t}^3(0) + h.o.t.$$
$$f_2(0, x_t) = b_1x_{3t}^2(0) + b_2x_{3t}^3(0) + b_3x_{4t}^2(0) + b_4x_{4t}^3(0) + h.o.t.$$
$$f_3(0, x_t) = c_1x_{1t}^2(-1) + c_2x_{1t}^3(-1) + c_3x_{2t}^2(-1) + c_4x_{2t}^3(-1)$$
$$+c_6x_{5t}^2(-1) + c_7x_{5t}^3(-1) + c_7x_{6t}^2(-1) + c_8x_{6t}^3(-1) + h.o.t.$$
$$f_4(0, x_t) = d_1x_{1t}^2(-1) + d_2x_{1t}^3(-1) + d_3x_{2t}^2(-1) + d_4x_{2t}^3(-1)$$
$$+d_6x_{5t}^2(-1) + d_7x_{5t}^3(-1) + d_7x_{6t}^2(-1) + d_8x_{6t}^3(-1) + h.o.t.$$
$$f_5(0, x_t) = e_1x_{3t}^2(0) + e_2x_{3t}^3(0) + e_3x_{4t}^2(0) + e_4x_{4t}^3(0) + h.o.t.$$
$$f_6(0, x_t) = f_1x_{3t}^2(0) + f_2x_{3t}^3(0) + f_3x_{4t}^2(0) + f_4x_{4t}^3(0) + h.o.t.$$

Appendix E

$$
\begin{aligned}
&g(z,\overline{z}) = \overline{q}^{*T} f_0(z,\overline{z})\\
&= \overline{D}[f_1(0,x_t) + \overline{\rho}_1^* f_2(0,x_t) + \overline{\rho}_2^* f_3(0,x_t) + \overline{\rho}_3^* f_4(0,x_t) + \overline{\rho}_4^* f_5(0,x_t) + \overline{\rho}_5^* f_6(0,x_t)]\\
&= \frac{1}{2}\overline{D}\{[a_1\rho_2^2 + a_3\rho_3^2] + \overline{\rho}_1^*[b_1\rho_2^2 + b_3\rho_3^2] + \overline{\rho}_2^*[(c_1 + c_3\rho_1^2 + c_5\rho_4^2 + c_7\rho_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}]\\
&+\overline{\rho}_3^*[(d_1 + d_3\rho_1^2 + d_5\rho_4^2 + d_7\rho_5^2)e^{-2\mathrm{i}\omega_0\tau^{(j)}}] + \overline{\rho}_4^*[e_1\rho_2^2 + e_3\rho_2^2] + \overline{\rho}_5^*[f_1\rho_2^2 + f_3\rho_2^2]\}z^2\\
&+\overline{D}\{[a_1|\rho_2|^2 + a_3|\rho_3|^2] + \overline{\rho}_1^*[b_1|\rho_2|^2 + b_3|\rho_3|^2] + \overline{\rho}_2^*[c_1 + c_3|\rho_1|^2 + c_5|\rho_4|^2 + c_7|\rho_5|^2]\\
&+\overline{\rho}_3^*[d_1 + d_3|\rho_1|^2 + d_5|\rho_4|^2 + d_7|\rho_5|^2] + \overline{\rho}_4^*[e_1|\rho_2|^2 + e_3|\rho_3|^2] + \overline{\rho}_5^*[f_1|\rho_2|^2 + f_3|\rho_3|^2]\}z\overline{z}\\
&+\frac{1}{2}\overline{D}\{[a_1\overline{\rho}_2^2 + a_3\overline{\rho}_3^2] + \overline{\rho}_1^*[b_1\overline{\rho}_2^2 + b_3\overline{\rho}_2^2] + \overline{\rho}_2^*[(c_1 + c_3\overline{\rho}_1^2 + c_5\overline{\rho}_4^2 + c_7\overline{\rho}_5^2)e^{-\mathrm{i}i\omega_0\tau^{(j)}}]\\
&+\overline{\rho}_3^*[(d_1 + d_3\overline{\rho}_1^2 + d_5\overline{\rho}_4^2 + d_7\overline{\rho}_5^2)e^{-\mathrm{i}i\omega_0\tau^{(j)}}] + \overline{\rho}_4^*[e_1\overline{\rho}_2^2 + e_3\overline{\rho}_2^2] + \overline{\rho}_5^*[f_1\overline{\rho}_2^2 + f_3\overline{\rho}_2^2]\}\overline{z}^2\\
&+\frac{1}{2}\overline{D}\{[2a_1\rho_2 W_{11}^{(3)}(0) + 2a_3\rho_3 W_{11}^{(4)}(0) + a_1\overline{\rho}_2 W_{20}^{(3)}(0) + a_3\overline{\rho}_2 W_{20}^{(3)}(0)\\
&+3a_2|\rho_2|^2\rho_2 + 3a_4|\rho_3|^2\rho_3] + \overline{\rho}_1^*[2b_1\rho_2 W_{11}^{(3)}(0) + 2b_3\rho_3 W_{11}^{(4)}(0)\\
&+b_1\overline{\rho}_2 W_{20}^{(3)}(0) + b_3\overline{\rho}_2 W_{20}^{(3)}(0) + 3b_2|\rho_2|^2\rho_2 + 3b_4|\rho_3|^2\rho_3]\\
&+\overline{\rho}_2^*[2e^{-\mathrm{i}\omega_0\tau^{(j)}}(c_1 W_{11}^{(1)}(-1) + c_3 W_{11}^{(2)}(-1)\rho_1 + c_5 W_{11}^{(5)}(-1)\rho_4 + c_7 W_{11}^{(6)}(-1)\rho_5)\\
&+e^{\mathrm{i}\omega_0\tau^{(j)}}(c_1 W_{20}^{(1)}(-1) + c_3 W_{20}^{(2)}(-1)\overline{\rho}_1 + c_5 W_{20}^{(5)}(-1)\overline{\rho}_4 + c_7 W_{20}^{(6)}(-1)\overline{\rho}_5)\\
&+3e^{-\mathrm{i}\omega_0\tau^{(j)}}(c_2 + c_4|\rho_1|^2\rho_1 + c_6|\rho_4|^2\rho_4 + c_8|\rho_5|^2\rho_5)] + \overline{\rho}_3^*[2e^{-\mathrm{i}\omega_0\tau^{(j)}}(d_1 W_{11}^{(1)}(-1) + d_3 W_{11}^{(2)}(-1)\rho_1\\
&+d_5 W_{11}^{(5)}(-1)\rho_4 + d_7 W_{11}^{(6)}(-1)\rho_5) + e^{i\omega_0\tau^{(j)}}(d_1 W_{20}^{(1)}(-1) + d_3 W_{20}^{(2)}(-1)\overline{\rho}_1\\
&+d_5 W_{20}^{(5)}(-1)\overline{\rho}_4 + d_7 W_{20}^{(6)}(-1)\overline{\rho}_5) + 3e^{-\mathrm{i}\omega_0\tau^{(j)}}(d_2 + d_4|\rho_1|^2\rho_1 + d_6|\rho_4|^2\rho_4\\
&+d_8|\rho_5|^2\rho_5)] + \overline{\rho}_4^*[2e_1\rho_2 W_{11}^{(3)}(0) + 2e_3\rho_3 W_{11}^{(4)}(0) + e_1\overline{\rho}_2 W_{20}^{(3)}(0) + e_3\overline{\rho}_2 W_{20}^{(3)}(0)\\
&+3e_2|\rho_2|^2\rho_2 + 3e_4|\rho_3|^2\rho_3] + \overline{\rho}_5^*[2f_1\rho_2 W_{11}^{(3)}(0) + 2f_3\rho_3 W_{11}^{(4)}(0)\\
&+f_1\overline{\rho}_2 W_{20}^{(3)}(0) + f_3\overline{\rho}_2 W_{20}^{(3)}(0) + 3f_2|\rho_2|^2\rho_2 + 3f_4|\rho_3|^2\rho_3]\}z^2\overline{z} + \cdots
\end{aligned}
$$

Appendix F

$$
\begin{aligned}
&L_1^{(1)} = (2i\omega_0 + k, 0, -\alpha_{31}e^{-2\mathrm{i}\omega_0\tau(j)}, -\alpha_{41}e^{-2\mathrm{i}\omega_0\tau(j)}, 0, 0)^T,\\
&L_1^{(2)} = (0, 2i\omega_0 + k, -\alpha_{32}e^{-2\mathrm{i}\omega_0\tau(j)}, -\alpha_{42}e^{-2\mathrm{i}\omega_0\tau(j)}, 0, 0)^T,\\
&L_1^{(3)} = (-\alpha_{13}, -\alpha_{23}, 2\mathrm{i}\omega_0 + k, 0, -\alpha_{53}, -\alpha_{63})^T, L_1^{(4)} = (-\alpha_{14}, -\alpha_{24}, 2\mathrm{i}\omega_0 + k, 0, -\alpha_{54}, -\alpha_{64})^T,\\
&L_1^{(5)} = (0, 0, -\alpha_{35}e^{-2\mathrm{i}\omega_0\tau(j)}, -\alpha_{45}e^{-2\mathrm{i}\omega_0\tau(j)}, 2i\omega_0 + k, 0)^T,\\
&L_1^{(6)} = (0, 0, -\alpha_{36}e^{-2\mathrm{i}\omega_0\tau(j)}, -\alpha_{46}e^{-2\mathrm{i}\omega_0\tau(j)}, 0, 2\mathrm{i}\omega_0 + k)^T.
\end{aligned}
$$

$$G_1^{(1)} = \frac{1}{L_1}|L_1^{(*)}, L_1^{(2)}, L_1^{(3)}, L_1^{(4)}, L_1^{(5)}, L_1^{(6)}|, G_1^{(2)} = \frac{1}{L_1}|L_1^{(1)}, L_1^{(*)}, L_1^{(3)}, L_1^{(4)}, L_1^{(5)}, L_1^{(6)}|,$$
$$G_1^{(3)} = \frac{1}{L_1}|L_1^{(1)}, L_1^{(2)}, L_1^{(*)}, L_1^{(4)}, L_1^{(5)}, L_1^{(6)}|, G_1^{(4)} = \frac{1}{L_1}|L_1^{(1)}, L_1^{(2)}, L_1^{(3)}, L_1^{(*)}, L_1^{(5)}, L_1^{(6)}|,$$
$$G_1^{(5)} = \frac{1}{L_1}|L_1^{(1)}, L_1^{(2)}, L_1^{(3)}, L_1^{(4)}, L_1^{(*)}, L_1^{(6)}|, G_1^{(6)} = \frac{1}{L_1}|L_1^{(1)}, L_1^{(2)}, L_1^{(3)}, L_1^{(4)}, L_1^{(5)}, L_1^{(*)}|.$$

where

$$L_1 = |L_1^{(1)}, L_1^{(2)}, L_1^{(3)}, L_1^{(4)}, L_1^{(5)}, L_1^{(6)}|.$$

Appendix G

$$L_2^{(1)} = (-k, 0, \alpha_{31}, \alpha_{41}, 0, 0)^T, L_2^{(2)} = (0, -k, \alpha_{32}, \alpha_{42}, 0, 0)^T,$$
$$L_2^{(3)} = (\alpha_{13}, \alpha_{23}, -k, 0, \alpha_{53}, \alpha_{63})^T, L_2^{(4)} = (\alpha_{14}, \alpha_{24}, 0, -k, \alpha_{54}, \alpha_{64})^T,$$
$$L_2^{(5)} = (0, 0, \alpha_{35}, \alpha_{45}, -k, 0)^T, L_2^{(6)} = (0, 0, \alpha_{36}, \alpha_{46}, 0, -k)^T.$$

$$G_2^{(1)} = \frac{1}{L_2}|L_2^{(*)}, L_2^{(2)}, L_2^{(3)}, L_2^{(4)}, L_2^{(5)}, L_2^{(6)}|, G_2^{(2)} = \frac{1}{L_2}|L_2^{(1)}, L_2^{(*)}, L_2^{(3)}, L_2^{(4)}, L_2^{(5)}, L_2^{(6)}|,$$
$$G_2^{(3)} = \frac{1}{L_2}|L_2^{(1)}, L_2^{(2)}, L_2^{(*)}, L_2^{(4)}, L_2^{(5)}, L_2^{(6)}|, G_2^{(4)} = \frac{1}{L_2}|L_2^{(1)}, L_2^{(2)}, L_2^{(3)}, L_2^{(*)}, L_2^{(5)}, L_2^{(6)}|,$$
$$G_2^{(4)} = \frac{1}{L_2}|L_2^{(1)}, L_2^{(2)}, L_2^{(3)}, L_2^{(4)}, L_2^{(*)}, L_2^{(6)}|, G_2^{(4)} = \frac{1}{L_2}|L_2^{(1)}, L_2^{(2)}, L_2^{(3)}, L_2^{(4)}, L_2^{(5)}, L_2^{(*)}|.$$

where

$$L_2 = |L_2^{(1)}, L_2^{(2)}, L_2^{(3)}, L_2^{(4)}, L_2^{(5)}, L_2^{(6)}|.$$

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