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Dynamic stability of ferromagnetic plate under transverse magnetic field and in-plane periodic compression

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Abstract

The dynamic stability of a soft ferromagnetic rectangular and simply supported plate immersed in an applied transverse magnetic field, as well as subjected to an in-plane periodic compression is presented in this paper. The fundamental equations involving magnetoelastic interaction and magnetic damping effect for the ferromagnetic plate are developed. In the theoretical model, the expression of induced magnetic force is based on a generalized magnetoelastic variational model, and the magnetic damping is due to the Lorentz body force arising from eddy current in the ferromagnetic material. By means of a linearized magnetoelastic theory and perturbation technique, the motion equation of the ferromagnetic plate is reduced to a damped Mathieu's equation and solved. The dynamic stability of the magnetoelastic system without in-plane compression is theoretically analyzed first, to show that there exist two stable states: magnetic damped stable oscillation, and over-damped asymptotically stable motion before static divergence instability of the ferromagnetic plate occurs. The dynamic instability and stability regions for the parametric excitation of the ferromagnetic plate due to the harmonically excited in-plane compression are obtained next. The effects of magnetic damping and excitation frequency of the in-plane periodic compression on the stability regions are discussed in detail.

Keywords: Ferromagnetic plate; Magnetic damping; Magnetoelastic interaction; Dynamic stability

1. Introduction

In modern electromagnetic equipments, such as magnetically levitated vehicles, electromagnetic energy-storage devices, fusion reactors and magnetic propulsion devices, etc, ferromagnetic structures like beams, plates and shells are widely used. Different from conventional structures experienced to mechanical loadings, the electromagnetic structures immersed in strong magnetic fields usually are subjected to mighty magnetic force arising from the mutual influence between the applied magnetic field and the magnetization of ferromagnetic materials. The strong magnetic force leads to deformation of ferromagnetic structures even to losing stability [1].

The analyses for behaving magnetoelastic interaction of ferromagnetic structures can be traced to the 1960s [2,3]. Moon and Pao [4] were the first to conduct experiment of magnetoelastic buckling of a thin ferromagnetic plate in a uniform transverse magnetic field. The observations showed that the ferromagnetic plate might lose its stability when the magnetic-field intensity reaches a critical value. A theoretical model called the magnetic body couple model was proposed by them to predict the experimental phenomena of magnetoelastic buckling. In subsequent investigation, Moon and Pao [5] studied a ferromagnetic beam-plate vibrating in a transverse magnetic field and obtained that the natural frequency of the plate decreases with an increasing magnetic-field intensity and finally tends to zero as the magnetic field attains a critical value, which causes the same plate to buckle. However, since there is sometimes a big discrepancy between the theoretical predictions and the experimental data for the critical magnetic fields, many investigators, Wallerstein and Peach [6], Miya et al. [7,8], Peach et al. [9], etc., devoted their attentions to finding an explanation for this reason. Based

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on a generalized magnetoelastic variational principle, Zhou et al. [10,11] proposed a new theoretical model for magnetoelastic buckling and bending of ferromagnetic plates in a transverse and/or oblique magnetic field. They showed that the predictions from their model were much more close to the experimental data for ferromagnetic plates buckling in a transverse field [12], and at the same time the model can theoretically simulated and explained the experiment result of natural frequency increasing with magnetic-field intensity for a cantilever beam-plate being in-plane magnetic field [11]. Recently, the authors [13] have expanded the generalized magnetoelastic variational principle to the magneto-thermo-elasticity of soft ferromagnetic bodies in applied magnetic and thermal fields.

Nevertheless, the analyses by Moon and Pao [4,5] and the other investigators always neglected the eddy current retaining but only the coupling between magnetization and deformation of the ferromagnetic structures. In some ferromagnetic structures, for example, the first walls and blankets of fusion power reactors, there commonly coexist magnetization and eddy currents, as well as the electromagnetic structures subjected to mechanical loadings, such problems of the electro–magneto–elastic interaction should be paid more attention both in research and in engineering design of electromagnetic equipments. Adopting the magnetic couple model by Moon and Pao [4,5] for expression of the magnetic force, Lu et al. [14] studied a magnetoelastic buckled beam subjected to an external axial periodic force in a periodic transversal magnetic field with the effect of induced currents. Zheng and Liu [15] numerically simulated the dynamic behaviors of a ferromagnetic conducting beam in uniform transversal magnetic fields with the magnetoelastic variational model proposed by Zhou and Zheng [10]. As for the conducting strips and plates, Lee [16,17] studied their dynamic stability taking into account magnetic damping effect of eddy currents in the structures, and obtained an explicit expression for the destabilizing effect as well as the effect of relative magnetic permeability on instability.

In this paper, the dynamic stability of a soft ferromagnetic, rectangular, simply supported thin plate under a uniform transverse magnetic field and an in-plane periodic compression is carried out with the magnetic damping effect. The analysis of dynamic magnetoelasticity of the ferromagnetic plate is restricted within low frequencies and the assumption of quasi-static magnetic fields is adopted. The theoretical model proposed by Zhou and Zheng [10] is employed to express the equivalent magnetic force of magnetoelastic interaction for the ferromagnetic plate. The magnetic damping is described by the Lorentz body force arising from eddy current induced in the ferromagnetic plate. Based on a linearized magnetoelastic theory and perturbation method, the governing equation of the magnetoelastic system is reduced to a damped Mathieu's equation. The dynamic stability for free vibration of the ferromagnetic plate in absence of in-plane compression is analyzed first. The expression for instability as a result of magnetization and magnetic damping of the ferromagnetic plate is explicitly obtained. A parametric excitation of the system with the harmonically excited in-plane compression is considered next, and the corresponding stability regions are simulated in detail.

2. Fundamental governing equation

Consider an isotropic, homogeneous, soft ferromagnetic rectangular thin plate with the length a, width b, and thickness h in an applied transverse uniform magnetic field $\mathbf{B}_0$ (as shown Fig. 1). The plate is simply supported on four sides, an in-plane periodic compression $P(t) = P_0 \cos \omega_0 t$ acts along one pair of opposite sides of the plate in the x-direction.

2.1. Equations of magnetic fields

Here, we restrict our attention to the quasi-static magnetic field. The electric field, charge distribution and conduction current in the ferromagnetic medium, as well as the electric field induced by eddy current, are neglected on account of the low vibration frequency of the magnetoelastic system [1]. As a consequence, the magnetic fields both inside and outside regions of the ferromagnetic plate are governed by Maxwell's equations as follows:

$$\nabla \cdot \mathbf{B}^+ = 0, \quad \nabla \times \mathbf{H}^+ = \mathbf{0} \quad \text{in } \Omega^+(\mathrm{u}), \tag{1}$$

$$\nabla \cdot \mathbf{B}^- = 0, \quad \nabla \times \mathbf{H}^- = \mathbf{0} \quad \text{in } \Omega^-(\mathrm{u}) \tag{2}$$

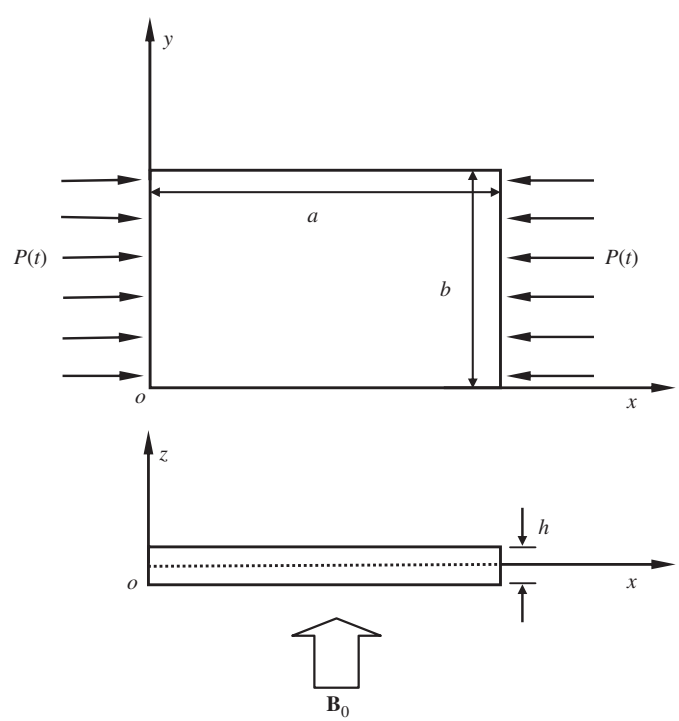

Fig. 1. The sketch of rectangular ferromagnetic plate.

and the corresponding connecting and boundary conditions for magnetic fields are

$$\mathbf{n}\cdot(\mathbf{B}^+ - \mathbf{B}^-) = 0,$$
$$\mathbf{n}\times(\mathbf{H}^+ - \mathbf{H}^-) = \mathbf{0} \quad \text{on } \mathrm{S} = \Omega^+ \cap \Omega^-(\mathrm{u}), \tag{3}$$

$$\mathbf{B}^- = \mathbf{B}_0 = B_0\mathbf{k} \quad \text{on } \mathrm{S}_0 \text{ or at } \infty, \tag{4}$$

where $\Omega^+(\mathbf{u})$ and $\Omega^-(\mathbf{u})$, respectively, represent inside and outside regions of the deformed ferromagnetic plate with displacement vector denoted by $\mathbf{u}$; $\mathbf{n}$ is a unit vector outward normal to the surface S of the ferromagnetic plate; S_0 denotes a closed surface which surrounds and is far away from the ferromagnetic medium; $\nabla = \partial/\partial x\mathbf{i} + \partial/\partial y\mathbf{j} + \partial/\partial z\mathbf{k}$ is the gradient operator in the space, where the Cartesian coordinate system xyz is chosen, without loss of generality, and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the x-, y- and z-axis, respectively.

For linear ferromagnetic medium, the magnetic constitutive relationships, between magnetic field vector $\mathbf{H}$ and magnetic induction $\mathbf{B}$, can be written by

$$\mathbf{B}^+ = \mu_0\mu_r\mathbf{H}^+ \quad \text{in } \Omega^+(\mathrm{u}), \tag{5}$$

$$\mathbf{B}^- = \mu_0\mathbf{H}^- \quad \text{in } \Omega^-(\mathrm{u}), \tag{6}$$

in which μ_r and μ_0 are the relative magnetic permeability of the ferromagnetic plate and the permeability of a vacuum, respectively; χ is the susceptibility of material of the plate, given by $\chi = \mu_\mathrm{r} - 1$.

2.2. Motion equations of ferromagnetic plate

On the basis of the small bending theory and Kirchhoff's assumptions of thin plate, the deformation of a plate is expressed as

$$\mathbf{u} = -\frac{\partial w(x,y,t)}{\partial x}z\mathbf{i} - \frac{\partial w(x,y,t)}{\partial y}z\mathbf{j} + w(x,y,t)\mathbf{k}, \tag{7}$$

where $w(x, y, t)$ is the transverse displacement or deflection at mid-plane of the ferromagnetic plate. Omitting in-plane inertia terms, the motion equations of the rectangular simply supported plate and the boundary conditions are given by

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \int_{-h/2}^{h/2} f_x \,\mathrm{d}z = 0,$$
$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + \int_{-h/2}^{h/2} f_y \,\mathrm{d}z = 0, \tag{8}$$

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - \left(N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} + N_y\frac{\partial^2 w}{\partial y^2}\right) + \rho h\frac{\partial^2 w}{\partial t^2}$$
$$= q_z(x,y,t) + \int_{-h/2}^{h/2} f_z \,\mathrm{d}z + \frac{\partial}{\partial x}\int_{-h/2}^{h/2} f_x z \,\mathrm{d}z + \frac{\partial}{\partial y}\int_{-h/2}^{h/2} f_y z \,\mathrm{d}z, \tag{9}$$

$$w(x,y,t) = 0, \quad \frac{\partial^2 w(x,y,t)}{\partial x^2} = 0 \quad \text{at } x = 0,\ a, \tag{10a}$$

$$w(x,y,t) = 0, \quad \frac{\partial^2 w(x,y,t)}{\partial y^2} = 0 \quad \text{at } y = 0,\ b, \tag{10b}$$

where $D = Eh^3/12(1-v^2)$ is the flexural rigidity of the ferromagnetic plate; E and v are the Young's modulus and Poisson's ratio, respectively; $q_z(x,y,t)$ is the equivalent magnetic force acted on the ferromagnetic thin plate, which is expressed by the generalized magnetoelastic variational model for a soft ferromagnetic plate in complex magnetic fields in following form [10]:

$$q_z(x,y,t) = \frac{\mu_0\mu_r\chi}{2}\left\{\left[H_n^+(x,y,h/2,t)\right]^2 - \left[H_n^+(x,y,-h/2,t)\right]^2\right\} - \frac{\mu_0\chi}{2}\left\{\left[H_\tau^+(x,y,h/2,t)\right]^2 - \left[H_\tau^+(x,y,-h/2,t)\right]^2\right\}. \tag{11}$$

The magnetic force in above form is adopted owing to its advantage to the magnetoelastic interaction from the magnetization and deformation of the ferromagnetic plate. The variational magnetoelastic model proposed in Ref. [10] is, so far, the only model which successfully predicts the distinct two kinds of experimental phenomena of magnetoelastic interaction for soft ferromagnetic plates in magnetic fields (see Refs. [4,11]). In Eq. (11), the subscripts "n" and "τ" are used to identify the normal and tangential components of the magnetic field $\mathbf{H}^+$ on surface of the ferromagnetic plate. $\mathbf{f}(x,y,z,t) = \mathrm{f}_x\mathbf{i} + \mathrm{f}_y\mathbf{j} + \mathrm{f}_z\mathbf{k}$ is the Lorentz body force in the ferromagnetic plate calculated by the motion and magnetic induction vectors of the ferromagnetic plate [1], which gives as

$$\mathbf{f}(x,y,z,t) = \sigma\left[\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{B}^+\right]\times\mathbf{B}^+, \tag{12}$$

where σ denotes the electric conductivity of the ferromagnetic material. The integration terms in Eqs. (8) and (9) are the in-plane and transverse equivalent magnetic forces in the mid-plane of the ferromagnetic plate contributed by the Lorentz body force. In addition, the distributions of magnetic fields (solutions of Eqs. (1)–(6)) are dependent upon the deformed ferromagnetic plate. The transformation between the coordinates of a point $\mathbf{x}'$ in the region $\Omega^+(\mathbf{u})$ and on the surface S of the deformed plate and the coordinates of a point $\mathbf{x}$ for the undeformed plate can be taken as

$$\mathbf{x}' = \mathbf{x} + \mathbf{u}. \tag{13}$$

From the above equation, the magnetic force and Lorentz force of Eqs. (11) and (12), one can find that the governing equations for magnetic fields (Eqs. (1)–(6)) and those for motion of the ferromagnetic plate (Eqs. (8)–(10)) are nonlinearly coupled.

3. Perturbation technique for magnetic fields and forces

Usually, it is difficult to achieve the solution of the coupled magnetoelastic equations (1)–(10) for magnetic fields and deformation field of the ferromagnetic plate. Here we will restrict ourselves to the magnetoelastic infinitesimal deformation. Based on the linearized theory of magnetoelasticity proposed by Pao and Yeh [18] and the perturbation technique, the distributions inside and outside magnetic fields of the deformable ferromagnetic plate are given as

$$\mathbf{B}^+ = \mathbf{B}_0^+ + \mathbf{b}^+(x,y,z,t),$$
$$\mathbf{H}^+ = \mathbf{H}_0^+ + \mathbf{h}^+(x,y,z,t) \quad \text{in } \Omega^+(\mathrm{u}), \tag{14}$$

$$\mathbf{B}^- = \mathbf{B}_0^- + \mathbf{b}^-(x,y,z,t),$$
$$\mathbf{H}^- = \mathbf{H}_0^- + \mathbf{h}^-(x,y,z,t) \quad \text{in } \Omega^+(\mathrm{u}), \tag{15}$$

where the base magnetic fields $\mathbf{B}_0^+$ (or $\mathbf{H}_0^+$) and $\mathbf{B}_0^-$ (or $\mathbf{H}_0^-$) for a rigid body state are constant in space and time, and the disturbed fields $\mathbf{b}^+(x,y,z,t)$ (or $\mathbf{h}^+(x,y,z,t)$) and $\mathbf{b}^-(x,y,z,t)$ (or $\mathbf{h}^-(x,y,z,t)$) are small. We further introduce the magnetic scalar potentials Φ^+, Φ^- and ϕ^+, ϕ^-, so the magnetic fields and magnetic inductions can be re-expressed as

$$\mathbf{H}_0^+ = -\nabla\Phi^+, \quad \mathbf{B}_0^+ = -\mu_0\mu_r\nabla\Phi^+ \quad \text{in } \Omega^+(\mathrm{u}=0), \tag{16a}$$

$$\mathbf{h}^+ = -\nabla\phi^+, \quad \mathbf{b}^+ = -\mu_0\mu_r\nabla\phi^+ \quad \text{in } \Omega^+(\mathrm{u}), \tag{16b}$$

$$\mathbf{H}_0^- = -\nabla\Phi^-, \quad \mathbf{B}_0^- = -\mu_0\nabla\Phi^- \quad \text{in } \Omega^-(\mathrm{u}=0), \tag{17a}$$

$$\mathbf{h}^- = -\nabla\phi^-, \quad \mathbf{b}^- = -\mu_0\nabla\phi^- \quad \text{in } \Omega^-(\mathrm{u}). \tag{17b}$$

After taking into account the influence of deformation of the ferromagnetic plate on the magnetic field boundary conditions, the normal vector of the deformed ferromagnetic plate becomes

$$\mathbf{n} = \mathbf{n}_0 + \hat{\mathbf{n}}, \tag{18}$$

where $\mathbf{n}_0$ and $\hat{\mathbf{n}}$ are the normal vectors of the undeformed plate and the perturbation term by the deformed plate on the surface S, respectively. They are expressed as

$$\mathbf{n}_0 = \pm\mathbf{k}, \quad \hat{\mathbf{n}} = \pm\left(-\frac{\partial w}{\partial x}\mathbf{i} - \frac{\partial w}{\partial y}\mathbf{j}\right) \tag{19}$$

in which the signs "$\pm$"correspond to the upper (+) and lower (−) surfaces of the plate, respectively.

Then, we can rewrite the governing equations (1) and (2) and the boundary conditions (3) and (4) with taking into account Eqs. (18) and (19), together with constitutive relationships (5) and (6) for magnetic fields by the magnetic scalar potentials as follows:

$$\nabla^2\Phi^+ = 0 \quad \text{in } \Omega^+(\mathrm{u}=0), \tag{20a}$$

$$\nabla^2\Phi^- = 0 \quad \text{in } \Omega^-(\mathrm{u}=0) \tag{20b}$$

$$\Phi^+ = \Phi^-, \quad \mu_r\frac{\partial\Phi^+}{\partial z} = \frac{\partial\Phi^-}{\partial z} \quad \text{on } z = \pm h/2, \tag{20c}$$

$$-\nabla\Phi^- = \frac{\mathbf{B}_0}{\mu_0} \quad \text{on } S_0 \text{ or at } z \to \infty, \tag{20d}$$

and

$$\nabla^2\phi^+ = 0 \quad \text{in } \Omega^+(\mathrm{u}), \tag{21a}$$

$$\nabla^2\phi^- = 0 \quad \text{in } \Omega^-(\mathrm{u}), \tag{21b}$$

$$\mu_r\frac{\partial\phi^+}{\partial z} = \frac{\partial\phi^-}{\partial z} \quad \text{on } z = \pm h/2, \tag{21c}$$

$$\frac{\partial\phi^+}{\partial x} - H_{0z}^+\frac{\partial w}{\partial x} = \frac{\partial\phi^-}{\partial x} - H_{0z}^-\frac{\partial w}{\partial x}, \ \frac{\partial\phi^+}{\partial y} - H_{0z}^+\frac{\partial w}{\partial y} = \frac{\partial\phi^-}{\partial y} - H_{0z}^-\frac{\partial w}{\partial y} \quad \text{on } z = \pm h/2, \tag{21d}$$

$$\phi^- \to 0 \quad \text{on } S_0 \text{ or at } z \to \infty. \tag{21e}$$

Solving Eqs. (20a)–(20d), the distributions of the base magnetic fields are easily obtained by

$$\mathbf{H}_0^+ = -\nabla\Phi^+ = \frac{B_0}{\mu_0\mu_r}\mathbf{k} \quad \text{in } \Omega^+(\mathrm{u}=0), \tag{22a}$$

$$\mathbf{H}_0^- = -\nabla\Phi^- = \frac{B_0}{\mu_0}\mathbf{k} \quad \text{in } \Omega^-(\mathrm{u}=0). \tag{22b}$$

Since the disturbed magnetic fields are coupled with the deformation of the ferromagnetic plate, it is generally not easy to solve the disturbed fields independently. Here, we take the deflection solution of the rectangular simply supported plate in the series expression form

$$w(x,y,t) = \sum_m\sum_n A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}f(t), \tag{23}$$

where m and n denote positive integers, A_{mn} are the plate deflection amplitude coefficients. It is obvious that Eq. (23) satisfies the boundary conditions of the simply supported plate. Substitution of Eqs. (22) and (23) into the boundary conditions of Eq. (21d) of magnetic fields, the distributions of the disturbed fields can be solved from Eqs. (21a)–(21e) to obtain in following forms:

$$\begin{aligned}\mathbf{h}^+ &= -\nabla\phi^+ \\ &= \frac{B_0\chi}{\mu_0\mu_r}\sum_m\sum_n\frac{A_{mn}}{\Delta_{mn}} \\ &\quad\times\left\{\left[\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cosh(k_{mn}z)\right]\mathbf{i}\right. \\ &\quad+\left[\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cosh(k_{mn}z)\right]\mathbf{j} \\ &\quad\left.+\left[k_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sinh(k_{mn}z)\right]\mathbf{k}\right\}f(t),\end{aligned} \tag{24a}$$

$$\begin{aligned}\mathbf{h}^- &= -\nabla\phi^- \\ &= -\frac{B_0\chi}{\mu_0}\sum_m\sum_n\frac{A_{mn}}{\Delta_{mn}}\sinh\frac{k_{mn}h}{2} \\ &\quad\times\left\{\left[\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]\mathbf{i}\right. \\ &\quad-\left[\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right]\mathbf{j} \\ &\quad\left.-\left[\mathrm{sgn}(z)k_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]\mathbf{k}\right\} \\ &\quad\times e^{k_{mn}(h/2-|z|)}f(t),\end{aligned} \tag{24b}$$

here sgn(z) is the signum function, and

$$k_{mn}=\pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2},\quad \Delta_{mn}=\mu_r\sinh\frac{k_{mn}h}{2}+\cosh\frac{k_{mn}h}{2}. \tag{25}$$

Taking the foregoing distributions of magnetic fields into the equivalent magnetic force of Eq. (11), we have

$$\begin{aligned}q_z(x,y,t) &= \frac{2B_0^2\chi^2}{\mu_0\mu_r}\sum_m\sum_n\frac{A_{mn}k_{mn}}{\Delta_{mn}}\sinh\frac{k_{mn}h}{2} \\ &\quad\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}f(t)\end{aligned} \tag{26}$$

and the Lorentz body force of Eq. (12), with ignoring the infinitesimal terms of high order, becomes

$$\begin{aligned}\mathbf{f}(x,y,z,t) &= \sigma B_0^2\frac{\mathrm{d}f(t)}{\mathrm{d}t}\left(\sum_m\sum_n A_{mn}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{m\pi y}{b}z\,\mathbf{i}\right. \\ &\quad\left.+\sum_m\sum_n A_{mn}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{m\pi y}{b}z\,\mathbf{j}\right).\end{aligned} \tag{27}$$

Furthermore, the other magnetic forces are expressed as

$$\int_{-h/2}^{h/2}f_x\,\mathrm{d}z=\int_{-h/2}^{h/2}f_y\,\mathrm{d}z=0, \tag{28}$$

$$\begin{aligned}&\int_{-h/2}^{h/2}f_z\,\mathrm{d}z+\frac{\partial}{\partial x}\int_{-h/2}^{h/2}f_x z\,\mathrm{d}z+\frac{\partial}{\partial y}\int_{-h/2}^{h/2}f_y z\,\mathrm{d}z \\ &\quad=-\frac{\sigma B_0^2h^3}{12}\sum_m\sum_n A_{mn}k_{mn}^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{\mathrm{d}f(t)}{\mathrm{d}t}.\end{aligned} \tag{29}$$

Substituting Eq. (28) into the equilibrium equations of membrane resultants of Eq. (8) and taking into account the in-plane excitation compression along x-direction, one can get the membrane resultants of the thin plate as follows:

$$N_x=-P_0\cos\omega_0 t,\quad N_y=0,\quad N_{xy}=0. \tag{30}$$

Thus, with substitution of the force exerted on the ferromagnetic plate including the equivalent transverse magnetic force, Lorentz force and the above membrane resultants as well as the serial expansion of the deflection of Eq. (23) into Eq. (9), the governing equation for flexural motion of the ferromagnetic plate is reduced into the form

$$\begin{aligned}&\rho h\frac{\mathrm{d}^2f(t)}{\mathrm{d}t^2}+\frac{\sigma B_0^2h^3}{12}k_{mn}^2\frac{\mathrm{d}f(t)}{\mathrm{d}t} \\ &\quad+\left[Dk_{mn}^4-\frac{2B_0^2\chi^2}{\mu_0\mu_r}\frac{k_{mn}}{\Delta_{mn}}\sinh\frac{k_{mn}h}{2}-P_0\left(\frac{m\pi}{a}\right)^2\cos\omega_0 t\right]f(t)=0.\end{aligned} \tag{31}$$

Before the following study, some nondimensional variables are introduced here

$$\begin{aligned}&\bar{t}=\frac{t}{2a^2/h\sqrt{\frac{3\rho(1-\nu^2)}{E}}}, \\ &B^2=\frac{B_0^2}{\mu_0E}\times10^4,\eta=\frac{a}{b},\gamma=\frac{a}{h}, \\ &\bar{k}_{mn}=k_{mn}a=\pi\sqrt{m^2+n^2\eta^2}.\end{aligned} \tag{32}$$

It should be noted that the nondimensional variable B^2 is gained from B_0^2/μ_0E multiplied by number 10^4 so that the value is not too small due to the Young's modulus about the magnitude of 10^{11} for most ferromagnetic materials. This treatment will not distort the solution of the problem. Furthermore, Eq. (31) in the nondimensional form is rewritten as

$$\begin{aligned}&\frac{\mathrm{d}^2f}{\mathrm{d}\bar{t}^2}+B^2G\bar{k}_{mn}^2\frac{\mathrm{d}f}{\mathrm{d}\bar{t}}+\left[\bar{k}_{mn}^4-B^2H_{mn}\bar{k}_{mn}\right. \\ &\quad\left.-P_0K_m\cos\left(\frac{2a^2}{h}\sqrt{\frac{3\rho(1-\nu^2)}{E}}\omega_0\bar{t}\right)\right]f=0\end{aligned} \tag{33}$$

in which

$$\begin{aligned}&G=\frac{\mu_0\sigma}{2\times10^4}\sqrt{\frac{Eh^2(1-\nu^2)}{3\rho}},\quad K_m=\frac{12(1-\nu^2)(m\pi)^2\gamma^2}{Eh}, \\ &H_{mn}=\frac{24(1-\nu^2)\chi^2\gamma^3}{\mu_r\times10^4}\frac{\sinh(\bar{k}_{mn}/2\gamma)}{\mu_r\sinh(\bar{k}_{mn}/2\gamma)+\cosh(\bar{k}_{mn}/2\gamma)}.\end{aligned} \tag{34}$$

4. Dynamic stability analysis

4.1. Magnetoelastic stability without compression

We firstly perform free vibration of the ferromagnetic plate under an applied transverse magnetic field without in-plane compression, i.e., $P_0=0$. Eq. (33) is re-expressed as the simple one

$$\frac{\mathrm{d}^2f}{\mathrm{d}\bar{t}^2}+\lambda_0\frac{\mathrm{d}f}{\mathrm{d}\bar{t}}+\alpha_0 f=0, \tag{35}$$

where

$$\lambda_0=B^2G\bar{k}_{mn}^2,\quad \alpha_0=\bar{k}_{mn}^4-B^2H_{mn}\bar{k}_{mn}. \tag{36}$$

Here, we investigate it in a manner of a normal mode and take

$$f=A\mathrm{e}^{\bar{\omega}\bar{t}}. \tag{37}$$

Substitution of Eq. (37) into Eq. (35) gives

$$\bar{\omega}^2 + \lambda_0\bar{\omega} + \alpha_0 = 0 \tag{38}$$

From the equation above, one can easily get the complex frequencies as

$$\bar{\omega}_{mn} = \frac{-\lambda_0 \pm \mathrm{i}\sqrt{4\alpha_0 - \lambda_0^2}}{2} = -\frac{B^2 G\bar{k}_{mn}^2}{2} \pm \mathrm{i}\frac{1}{2}\sqrt{4(\bar{k}_{mn}^4 - B^2 H_{mn}\bar{k}_{mn}) - (B^2 G\bar{k}_{mn}^2)^2}. \tag{39}$$

It shows that the ferromagnetic plate oscillates with the characteristic damping ratio of $-\lambda_0/2$ which is proportional to the square of magnetic field intensity B^2, as well vibrating with the frequency of $\sqrt{4\alpha_0 - \lambda_0^2}/2$. From the imaginary parts of $\bar{\omega}_{mn}$, i.e. $\mathrm{Im}(\bar{\omega}_{mn})$, the frequencies of the ferromagnetic plate decrease with an increasing of magnetic field B^2. The contributions for the frequency decreasing are from two ways. The first one, related to the first term with minus sign in Eq. (39), is due to magnetization of the ferromagnetic material and the coupling between magnetic force (or field) and deformation of ferromagnetic structure, which is so-called negative magnetic stiffness effect. The other, related to the second term with minus sign in Eq. (39), is from the damping, which is so-called magnetic damping effect. When the frequencies (i.e. $\mathrm{Im}(\bar{\omega}_{mn})$) approach to zero, the oscillation ceases and the plate is critically damped. The critical values are given by

$$(B_{mn}^2)_{\mathrm{c}} = \frac{2[\sqrt{H_{mn}^2 + G^2\bar{k}_{mn}^6} - H_{mn}]}{G^2\bar{k}_{mn}^3}. \tag{40}$$

The critical magnetic fields, $(B_{mn})_{\mathrm{c}}$, as the functions of the relative magnetic permeability μ_{r}, magnetic damping parameter G, geometrical ratio parameters γ and η for the few lower modes of the ferromagnetic plate are plotted in Figs. 2a–d. For the demonstration in these figures, some material and geometrical parameters of the ferromagnetic plate are chosen as listed in Table 1. We can find that, from Fig. 2, the critical value will be the smallest one when $m = n = 1$, that is

$$B_{\mathrm{c}}^2 = \frac{2\left[\sqrt{H_{11}^2 + G^2\bar{k}_{11}^6} - H_{11}\right]}{G^2\bar{k}_{11}^3}. \tag{41}$$

Furthermore, we take a view of two special cases: (a) when the magnetization effect of the ferromagnetic plate is tiny even to be neglectable, i.e. the susceptibility approaching to zero ($\chi \to 0$ or nonferrous conducts $\mu_{\mathrm{r}} \to 1$) or H_{mn} being zeros, the magnetic field for stability of the ferromagnetic plate will reach the critical value only caused by the magnetic damping, and from Eq. (39) the critical field becomes $B_{\mathrm{c}}^2 = 2/G$ which is independent of normal modes; (b) when the magnetic damping effect is small even tends to zero, i.e. $G \to 0$ (non-conducting ferromagnetic medium $\sigma \to 0$, or extremely thin plate), the magnetic field for stability of the plate will reach the critical value only due to the magnetization or negative magnetic stiffness, and from Eq. (39) the smallest critical value gives $B_{\mathrm{c}}^2 = \bar{k}_{11}^3/H_{11}$ which is independent of the damping factor G (e.g., $G < 0.1$). These characteristic features can also be clearly observed from Fig. 3. Fig. 3 also shows that the critical values for different relative magnetic permeabilities of the ferromagnetic plate approach to same value especially for the larger permeability. Actually, for the case of high permeability (e.g. $\mu_r \gg 1$ for iron, nickel, and cobalt, etc.) of the very thin soft ferromagnetic plate (i.e., $1/\gamma = h/a \ll 1$), there are following approximations:

$$\mu_r \approx \chi \gg 1, \quad \sinh\frac{\pi\sqrt{1+\eta^2}}{2\gamma} \approx \frac{\pi\sqrt{1+\eta^2}}{2\gamma}, \quad \cosh\frac{\pi\sqrt{1+\eta^2}}{2\gamma} \approx 1, \quad \mu_r\frac{\pi\sqrt{1+\eta^2}}{2\gamma} \gg 1. \tag{42}$$

Thus, the critical value of magnetic field in this case is reduced to the form

$$B_{\mathrm{c}}^2 = \frac{\bar{k}_{11}^3}{H_{11}} = \frac{(\pi\sqrt{1+\eta^2})^3}{(24(1-\nu^2)\chi^2\gamma^3)/(\mu_r \times 10^4)\left(\sinh(\pi\sqrt{1+\eta^2}/2\gamma)\right)/\left(\mu_r\sinh(\pi\sqrt{1+\eta^2}/2\gamma) + \cosh(\pi\sqrt{1+\eta^2}/2\gamma)\right)} \approx \frac{\pi^3(1+\eta^2)^{3/2}}{24\times 10^{-4}(1-\nu^2)}\gamma^{-3}. \tag{43}$$

It shows that the critical field B_{c} decreases with $-3/2$ power linearly depending on the geometrical parameter γ (the ratio of width to thickness of the plate) for a fixed η. The results are similar to the theoretical predictions and experimental data obtained for a cantilevered ferromagnetic beam-plate in transverse fields [4,5]. The curves of critical magnetic fields versus the geometrical parameters of the ferromagnetic thin plate with different magnetic damping factors are plotted in Fig. 4. From Fig. 4, we can find that the critical value decreases with increasing geometrical parameter, γ, the ratio of width to thickness of the plate. For a small γ, the magnetic damping obviously affects the critical field, as the higher is the magnetic damping, the lower is the critical value. This feature is attenuated for a large γ.

As the magnetic field increases beyond the critical field B_{c}, the real part of ω will increase from a negative value (damping effect) to zero or even become positive. When the real part of ω is positive (i.e., $\mathrm{Re}(\omega) > 0$), the divergence instability occurs, and the critical divergence

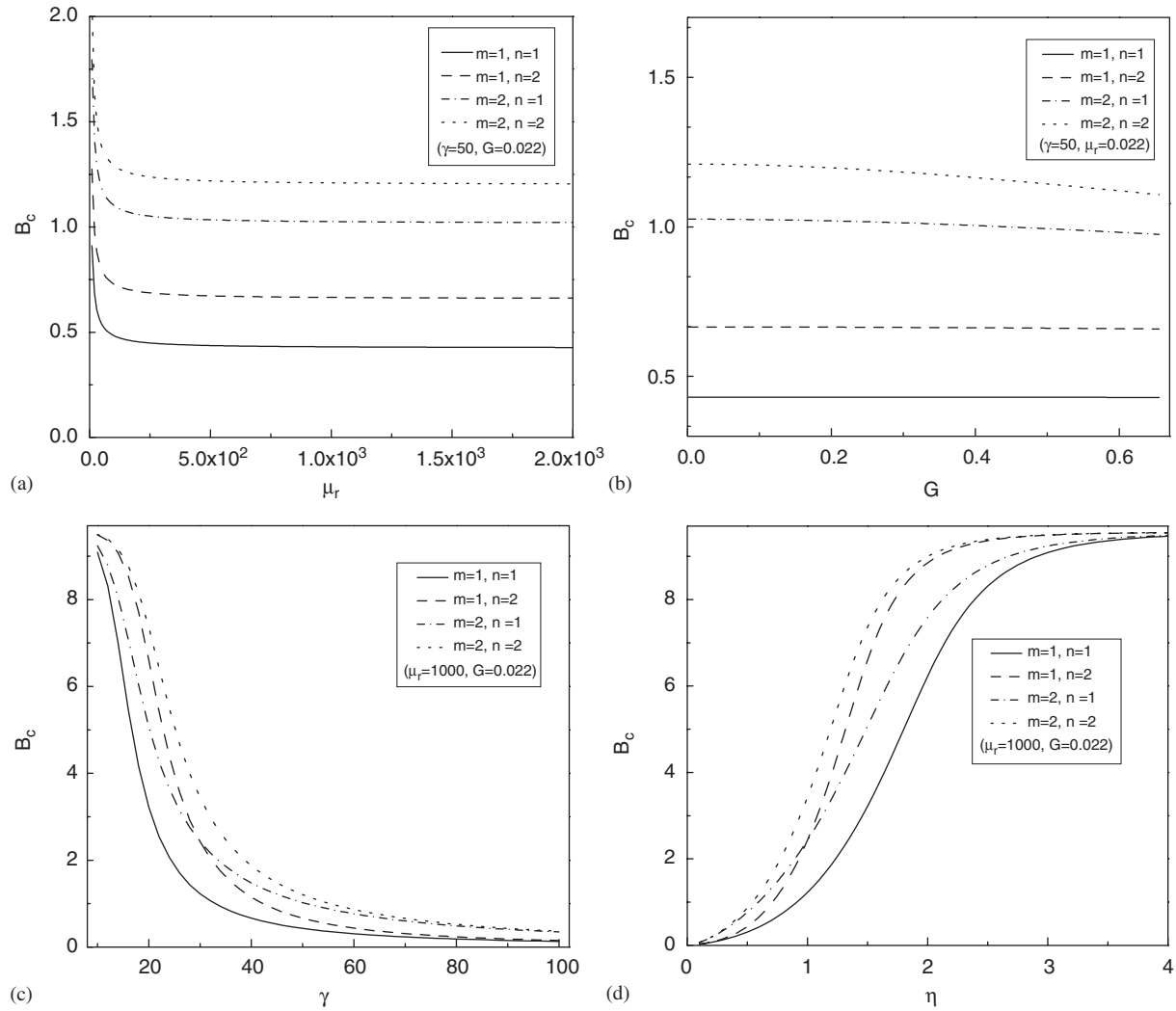

Fig. 2. The critical field as function of material and geometric parameters of ferromagnetic plate for few lower modes.

Table 1
Material and geometrical parameters of ferromagnetic plate

Parameter	Value
Density, ρ (kg/m^3)	3.0×10^3
Young's modulus, E (Mpa)	1.2×10^5
Poisson's ratio, ν	0.3
Conductivity, σ (mho/m)	1.0×10^7
Width, b (m)	8.3×10^{-1}
Thickness, h (m)	1.0×10^{-2}

condition gives

$$B_{\mathrm{d}}^2 = \frac{\bar{k}_{11}^3}{H_{11}}, \tag{44}$$

which is independent of the magnetic damping factor G, and same as the damping critical value when $G = 0$, that is $B_{\mathrm{d}}^2 = B_{\mathrm{c}}^2$. The critical divergence field B_{d}^2 is commonly higher than the critical damping field B_{c}^2 because the effect of magnetic damping can decrease B_{c}^2, specially for small geometrical parameters γ (as shown in Fig. 4). Therefore, for the ferromagnetic damping system, with an increase of the magnetic field, the ferromagnetic thin plate shows different magneto-mechanics behavior orderly by magnetic damped stable oscillation, over-damped asymptotically stable motion and static divergence instability. Correspondingly, if the magnetic damping effect is not taken into account, the ferromagnetic plate will lose stability directly from the magnetoelastic oscillation when the magnetic field reaches the critical value of B_{c}^2.

4.2. Magnetoelastic stability with compression

The magnetoelastic parametric excitation with the time-dependent compressive force $P(t)$ for the ferromagnetic thin plate is considered in this part. By introducing another nondimensional time (different from $\bar{t}$) for the convenience of the following analysis as

$$\tau = \frac{a^2}{h}\sqrt{\frac{3\rho(1-\nu^2)}{E}}\omega_0 \bar{t} \tag{45}$$

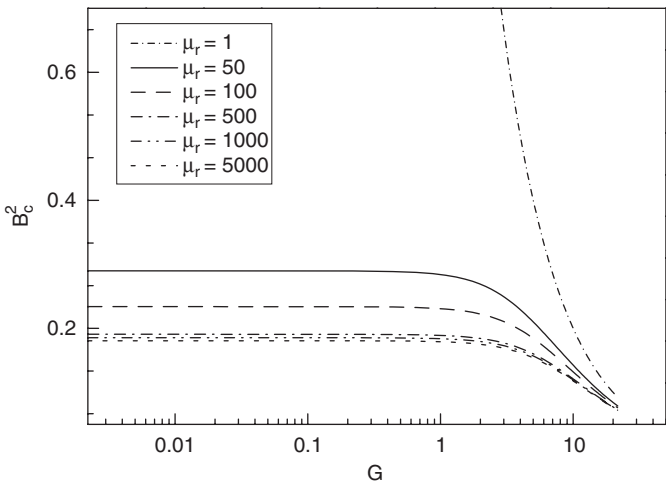

Fig. 3. The effect of magnetic damping factor G on critical field of ferromagnetic plate with different relative magnetic permeability when geometric parameter $\gamma = 50$.

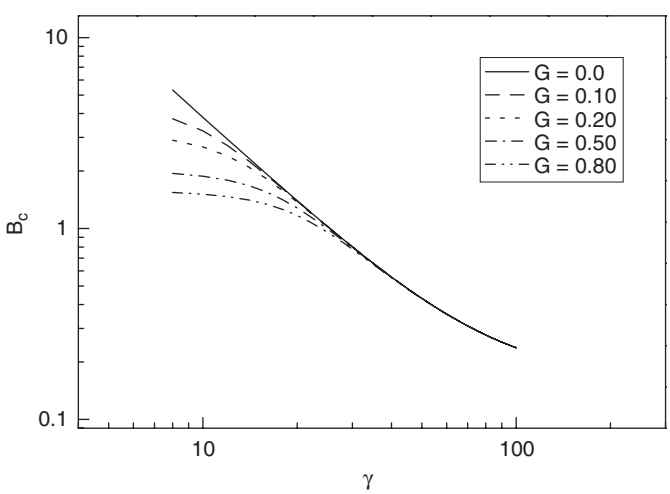

Fig. 4. The effect of geometric parameter ratio γ on critical magnetic field of ferromagnetic plate with different magnetic damping factor when relative magnetic permeability $\mu_r = 1000$.

one can easily recast Eq. (33) into the well-known damped Mathieu's equation [19]

$$\frac{d^2 f}{d\tau^2} + \lambda \frac{df}{d\tau} + (\alpha - \beta \cos 2\tau) f = 0, \tag{46}$$

where the new nondimensional parameters are defined as follows:

$$\lambda = \frac{B^2 G \bar{k}_{11}^2}{\Omega}, \quad \alpha = \frac{\bar{k}_{11}^4 - B^2 H_{11} \bar{k}_{11}}{\Omega^2},$$
$$\beta = \frac{F}{\Omega^2}, \quad F = P_0 K_1, \quad \Omega = \frac{a^2 \omega_0}{h} \sqrt{\frac{3\rho(1-\nu^2)}{E}} \tag{47}$$

and $m = n = 1$ is taken due to the critical value occurring.

Here, the spectral collocation method is employed (see [20]) to solve the damped Mathieu's equation (46), where

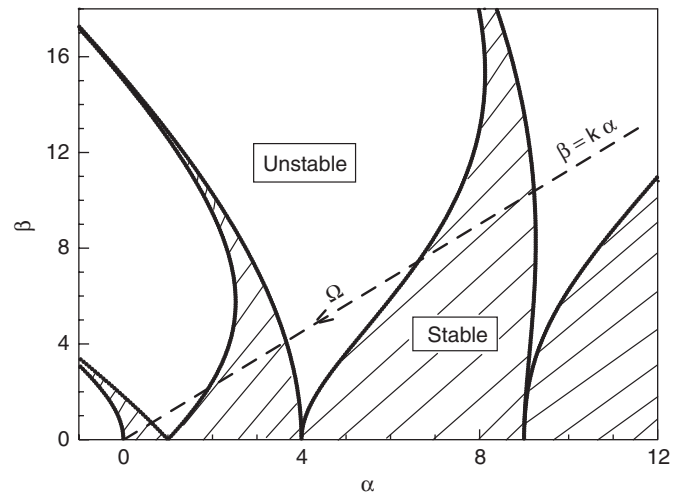

Fig. 5. The stability regions for non-damping magnetoelastic system of ferromagnetic plate with parametric excitation when $\mu_r = 1000$, $\gamma = 50$.

the unknown solution to the differential equation is expanded as a global interpolant, for example a trigonometric interpolant for period problems. The classical approach to solving the problem is to assume the Fourier series representation as

$$f(\tau) = \sum_{k=1}^{N} c_k e^{ik\tau}. \tag{48}$$

Inserting this series into (45) yields a recurrence relation for the expansion coefficients c_k in terms of (α, β), which are called characteristic values. Let $[\boldsymbol{A}]$ and $[\boldsymbol{B}]$ be the $N \times N$ Fourier second and first derivative matrixes, based on the equidistant nodes $\tau_k = 2\pi(k-1)/N$, $(k = 1, 2, \ldots, N)$. Let $[\boldsymbol{C}]$ be a $N \times N$ diagonal matrix whose diagonal elements are $C_{kk} = \cos(2\tau_k)$, $(k = 1, 2, \ldots, N)$. Then the damped Mathieu's equation (46) is approximated by

$$\{\beta[C] - [A] - \lambda[B]\}[y] = \alpha[y], \tag{49}$$

where $[\boldsymbol{y}]$ is the vector of approximate eigenfunction values $f(\tau_k)$. The eigenvalues can be found by computing determinants, or by computing continued fractions [21].

For the non-damping magnetoelastic system of the ferromagnetic plate (i.e. $\lambda = 0$ or $G = 0$), Fig. 5 shows the stability and instability regions dependent upon the characteristic values (α, β). Let us take a close view on how the properties of the parametric oscillations change with the excitation frequency Ω. Since the ratio of the parameter β to α remains constant for different Ω, the sequence of states for the magnetoelastic system is determined by the representative points on the dotted line $\beta = k\alpha$ passing through the origin of the coordinates. The slope of the line is give by the ratio

$$k = \frac{\beta}{\alpha} = \frac{F}{\bar{k}_{11}^4 - B^2 H_{11} \bar{k}_{11}} = \frac{F}{H_{11} \bar{k}_{11} \left(B_d^2 - B^2\right)} \tag{50}$$

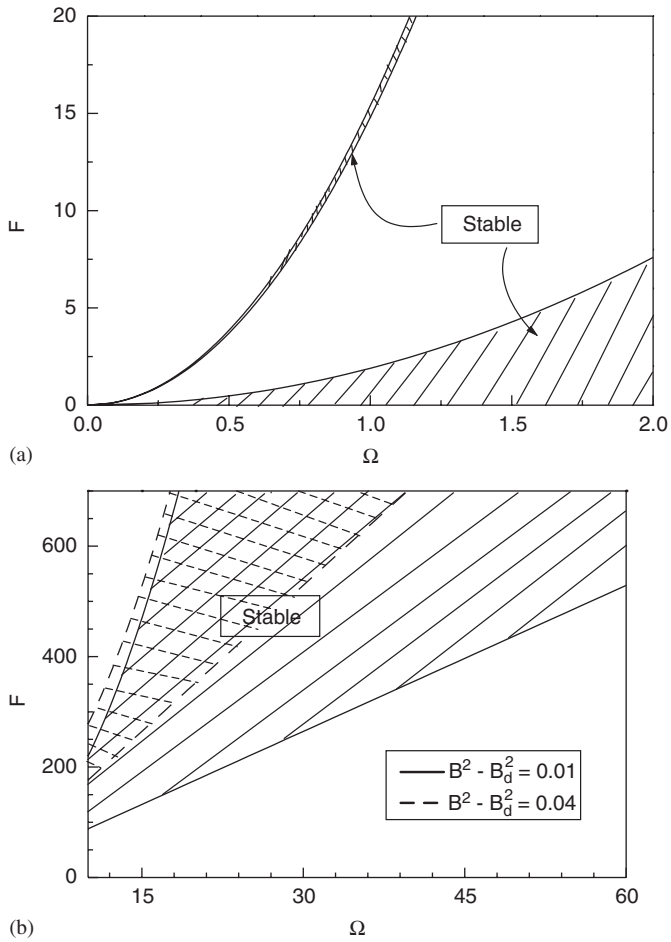

Fig. 6. The stable regions for non-damping magnetoelastic system of ferromagnetic plate with parametric excitation compression: (a) for $B^2 = B_d^2$ ($\alpha = 0$) and (b) for $B^2 > B_d^2$ (near origin), when $\mu_r = 1000$, $\gamma = 50$.

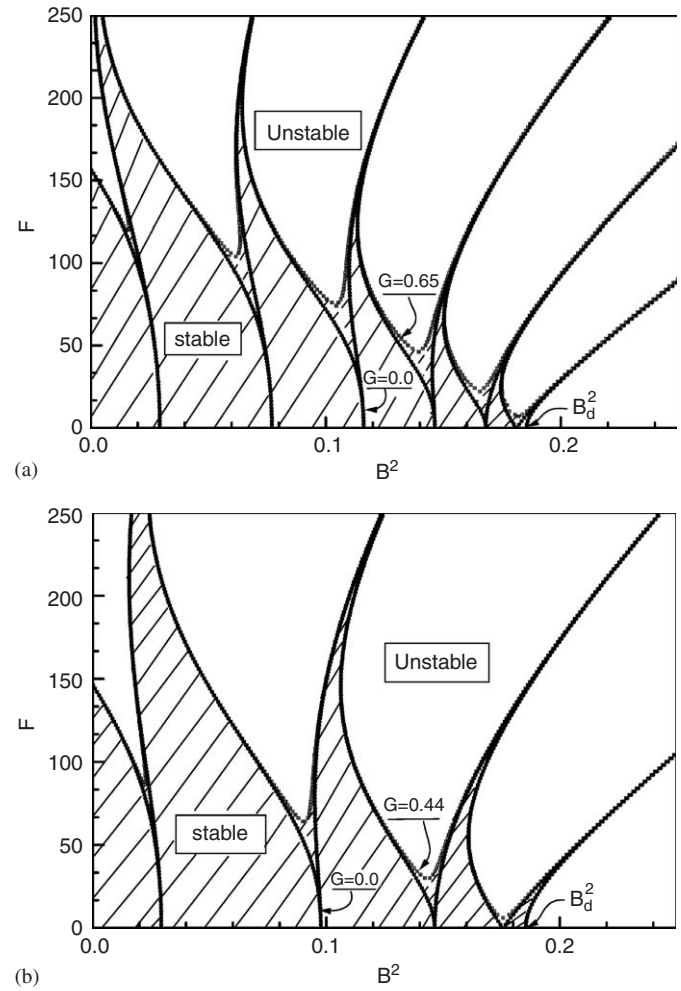

Fig. 7. The stable regions for magnetoelastic system of ferromagnetic plate for parametric excitation compression with magnetic damping: (a) for excitation frequency $\Omega = 2.05$ and (b) for $\Omega = 3.08$, when $\mu_r = 1000$, $\gamma = 50$.

in which B_d^2 is the critical divergence magnetic field for the ferromagnetic plate (see Eq. (44)). With an increment of the frequency Ω, from Fig. 5, the parameters α and β decrease proportionately, concomitantly the stable and unstable regions alternate. When the applied magnetic field B^2 approaches the critical value B_d^2, that is $k \to \infty$, the line coincides with the ordinate. In this case, the magnetoelastic system of the ferromagnetic plate may remain stable in some regions as shown in Fig. 6(a) of the plot of the frequency Ω depending on the compression F. As the applied magnetic field B^2 is beyond the critical field B_d^2, the line $\beta = k\alpha$ lies in the second quadrant. It is noted that the magnetoelastic system may remain stable in some patches. Fig. 6(b) shows the stability regains near the origin by Ω–F diagram.

The magnetic damping can also influence the stability regions of the ferromagnetic plate. Figs. 7a and b plot the stability regions of the plate with the diagram of the applied magnetic field as function of the compressive force amplitude for different excitation frequencies. We can see that the stable regions are broadened a bit by the magnetic damping effect of the ferromagnetic plate. This tendency is more obvious for the larger applied magnetic-field intensity. In the special case of $F = 0$, namely no in-plane compression acted on the ferromagnetic plate, the plate can keep its stability until the applied field reaches the divergence critical value B_d^2.

5. Conclusions

The dynamic stability of a soft ferromagnetic thin plate in a transverse magnetic field and an in-plane periodic compression is investigated. The fundamental equations for the ferromagnetic plate are developed including the effect of magnetoelastic interaction and magnetic damping. For the free vibration in absence of excitation compression, the expression for dynamic stability as a result of magnetization and magnetic damping of ferromagnetic plate together is explicitly obtained. It shows that there exist two stable states comprising magnetic damping stable

oscillation, over-damping asymptotically stable motion before the ferromagnetic plate exhibits divergence instability. The magnetic damping obviously affects the critical field, as the higher is the magnetic damping, the lower is the critical value. The dynamic stability regions for the parametric excitation of the magnetoelastic system with harmonically excited in-plane compression are discussed as well. The effect of magnetic damping and the frequency of in-plane compression on the stability characteristics are simulated numerically which shows that the magnetic damping extends the stable regions of parametric excitation to a certain extent.

Acknowledgments

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