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Decay and scattering of small solutions for Rosenau equations ☆

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ABSTRACT

We study the long-time behavior of small solutions of the Cauchy problem for a Rosenau equation. For a class of nonlinearity of the perturbation, the global small solution was obtained, and the decay and scattering for small amplitude solution are established.

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1. Introduction

In this paper, we study the long-time behavior of solutions and nonlinear scattering theory for a Rosenau equation

$$u_{tt} - u_{xx} + u_{xxxx} + u_{xxxxtt} = f(u)_{xx} \tag{1.1}$$

or, equivalently, the system of equations

$$\frac{d\vec{u}}{dt} = A\vec{u} + N(\vec{u}), \tag{1.2}$$

where $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}, A = \begin{pmatrix} 0 & \partial_x \\ \frac{(1-\partial_x^2)\partial_x}{1+\partial_x^4} & 0 \end{pmatrix}, N(\vec{u}) = \begin{pmatrix} 0 \\ \frac{\partial_x}{1+\partial_x^4}(f(u)) \end{pmatrix}$.

In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using the well-known KDV equation. To overcome this shortcoming of the KDV equation, Rosenau [1,2] proposed the so-called Rosenau equation. Two typical equations are

$$u_t + u_{xxxxt} + u_x + uu_x = f(u)_x \tag{1.3}$$

and

$$u_{tt} - \alpha u_{xx} + \beta u_{xxxx} + \gamma u_{xxxxtt} = f(u)_{xx}. \tag{1.4}$$

The existence and the uniqueness of the solution for (1.3) was proved by Park [3]. In [4], Barreto et al. proved the existence of solutions for the Rosenau Eq. (1.3) in domains with moving boundary. In [5], Liu discussed the asymptotic behavior of solutions to the Rosenau–Burgers equation with a periodic initial boundary. The numerical approximation to the Eq. (1.3) in one space dimension has been discussed by Chung and Ha [6] using finite element Galerkin methods, where they obtained optimal H^2-estimates and non-optimal estimates in the L^2-norm. Shubin Wang [12] considers the existence and the uniqueness of the solution for the Cauchy problem of the Rosenau Eq. (1.4).

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It should be noted that the Rosenau Eq. (1.4) becomes the "good" Boussinesq equation, if $\beta = 0$. Bona and Sachs [7] showed local well-posedness of smooth solutions for the Cauchy problem of the Boussinesq equation with a smooth nonlinear function $f(u)$ by Kato's theory. Liu Yue [8–10] investigated the conditions for the finite-time blow-up of solutions, decay and scattering of small solutions, and strong instability of a solitary wave with small speed of propagation for the Cauchy problem of the Boussinesq equation with the generalized nonlinear term $f(u)$. In [13,14], Ying Wang and Chunlai Mu considered the decay, blow-up and scattering of solution for a generalized Boussinesq equation.

Our principal aim in the present work is to study the long-time behavior of small solutions of the problem (1.1). Throughout this paper, we use the following notations. Let $|\cdot|_p$ denote the norm in the Lebesgue space L^p, $\vec{u}(t) = (u(t), v(t))$ denote the vector function $x \mapsto \vec{u}(x,t)$. Let $\|\vec{u}(t)\|_{s,p} = \|u(t)\|_{s,p} + \|v(t)\|_{s+1,p}$ denote the norm of the space $X_{s,p} = L^p_s \times L^p_{s+1}$, where $L^p_s = J^{-s}L^p$ denotes the Bessel potential space with the potential $J^s = (I - \partial^2)^{\frac{s}{2}}$ whose norm will be denoted by $\|\cdot\|_{s,p} = |J^s\cdot|_p$. When $p = 2$ we will write H^s instead of L^2_s, with norm $\|\cdot\|_s$ instead of $\|\cdot\|_{s,2}$ and $X_s = H^s \times H^{s+1}$ with the norm $\|\vec{u}\|_{s,2} = \|(u,v)\|_s = \|u\|_s + \|v\|_{s+1}$.

At first, we write (1.2) as an integral equation and treat the nonlinearity as a small perturbation of the linear part of the equation. We use an estimate of the linearized equation and contraction mapping theory to obtain the following existence theorem of global small amplitude solution.

Theorem 1.1. *Let $m \geqslant s > \frac{1}{2}$ with m an integer, and $f \in C^m(R)$ satisfy $|f(\tau)| = O(|\tau|^p)$ as $\tau \to 0$ with $p > 2$. Suppose $(p-2)\theta > 1$ with*

$$\theta = \begin{cases} \frac{2s-1}{2(3+2s)}, & \text{for } s \leqslant \frac{9}{2}, \\ \frac{4}{3+2s}, & \text{for } s \geqslant \frac{9}{2}. \end{cases}$$

Then there exist $\delta > 0$ such that for any $\vec{u}_0 = (u_0, v_0) \in X_s \bigcap X_{0,1}$ satisfying

$$\|\vec{u}_0\|_s + \|\vec{u}_0\|_{0,1} < \delta \tag{1.5}$$

and a unique solution $\vec{u} = (u, v)$ to (1.2) in $C^1([0,\infty); X_s)$ with $\vec{u}(0) = \vec{u}_0$ and

$$\sup_{0 \leqslant t < \infty} [(1+t)^\theta |u(t)|_\infty + \|\vec{u}(t)\|_s] \leqslant C\delta,$$

where the constant C only depends on f.

Next, we shall construct the scattering operator S. Let $S(t)$ be a C_0 group of unitary operators for the linearized equation

$$\vec{u}_t - A\vec{u} = 0, \tag{1.6}$$

with $\vec{u}(0) = \vec{u}_0 = (u_0, v_0)$. We rewrite (1.2) in its integral form

$$\vec{u}(t) = S(t)\vec{g}_- + \int_s^t S(t-\tau)N(\vec{u}(\tau))\mathrm{d}\tau, \tag{1.7}$$

$$\vec{u}(t) = S(t)\vec{g}_+ + \int_s^t S(t-\tau)N(\vec{u}(\tau))\mathrm{d}\tau, \tag{1.8}$$

where $\vec{u}$ is the solution of (1.2) with the different initial value $\vec{u}_\pm(s) = S(s)\vec{g}_\pm$ at $t = s$. Formally letting $s \to \pm\infty$, we get the following integral equations which relate $\vec{u}$ to $\vec{g}_\pm$

$$\vec{u}(t) = S(t)\vec{g}_- + \int_{-\infty}^t S(t-\tau)N(\vec{u}(\tau))\mathrm{d}\tau, \tag{1.9}$$

$$\vec{u}(t) = S(t)\vec{g}_+ + \int_{+\infty}^t S(t-\tau)N(\vec{u}(\tau))\mathrm{d}\tau. \tag{1.10}$$

At last, define the space $\mathbf{U} = \{\vec{u} \in C(R, X_{1,p+1} \bigcap X_1) | \|\vec{u}\|_{\mathbf{U}} < \infty\}$ endowed with the norm

$$\|\vec{u}\|_{\mathbf{U}} = \sup_{t \in R} \left\{ (1+|t|)^{\theta\left(1-\frac{2}{p+1}\right)} \|\vec{u}(t)\|_{1,p+1} + \|\vec{u}(t)\|_1 \right\}.$$

Now, we have all the tools to get our scattering theory. We state the main result of this paper in the following.

Theorem 1.2. *Let $f \in C^1(R)$ satisfy $|f(\tau)| = O(|\tau|^p)$ and $|f'(\tau)| = O(|\tau|^{p-1})$ as $\tau \to 0$, p satisfy $\theta\left(1 - \frac{2}{p+1}\right)p > 1$. Assume $\vec{g}_- \in X_1$ and $S(\cdot)\vec{g}_- \in \mathbf{U}$, there exists $\delta > 0$ such that if $\|S(\cdot)\vec{g}_-\|_{\mathbf{U}} < \delta$, then there exists a unique solution $\vec{u}$ of (1.9) in $C(R; X_{1,p+1} \bigcap X_1)$ and*

$$\|\vec{u} - S(t)\vec{g}_-\|_1 \to 0 \tag{1.11}$$

as $t \to -\infty$. Furthermore, there exists a unique $\vec{g}_+ \in X_1$, such that

$$\|\vec{u} - S(t)\vec{g}_+\|_1 \to 0 \tag{1.12}$$

as $t \to +\infty$. In addition,

$$\|u(t)\|_1^2 + \|v(t)\|_2^2 - 2\int_{-\infty}^{+\infty} F(u(t))\mathrm{d}t = \|\vec{g}_-\|_1^2 \tag{1.13}$$

and

$$\|\vec{g}_+\|_1 = \|\vec{g}_-\|_1, \tag{1.14}$$

where F′(s) = f(s) with F(0) = 0.

The plan of this paper is as follows. In the next section we introduce some preliminaries and the estimate of linearized equation. In Section 3, *we give the proof of Theorem 1.1. Section 4 contain the proof of Theorem 1.2.*

2. Estimates for the linearized equation

In this section, we get the estimates of linear equation. Also, we give some lemmas which are needed in later sections.

Lemma 2.1 (see [15]). *Let h be either convex or concave twice differentiable function on [a, b], with $-\infty \leqslant a < b \leqslant +\infty$. Then*

$$\left|\int_a^b e^{ih(\xi)}d\xi\right| \leqslant 4\left\{\min_{[a,b]}|h''|\right\}^{-\frac{1}{2}},$$

if $h'' \neq 0$ in [a, b].

Lemma 2.2. *For n, t > 1, 0 < ε < 1, we have*

$$\sup_{A\in R}\left|\int_{|\xi|\leqslant n} e^{ith(\xi,A)}\,\mathrm{d}\xi\right| \leqslant Ct^{-\frac{1}{2}}\max\left(\varepsilon^{-\frac{1}{2}}, n^2\right) + 10\varepsilon,$$

where $h(\xi, A) = \frac{\xi\sqrt{1+\xi^2}}{\sqrt{1+\xi^4}} + A\xi$, and C is a constant.

Proof. Let "′" denote derivatives of $h(\xi,A)$ with respect of ξ, then

$$h'(\xi,A) = \frac{1+2\xi^2-\xi^4}{(1+\xi^4)^{\frac{3}{2}}(1+\xi^2)^{\frac{1}{2}}} + A,$$

$$h''(\xi,A) = \frac{\xi(3\xi^8-8\xi^6-18\xi^4-8\xi^2+3)}{(1+\xi^4)^{\frac{5}{2}}(1+\xi^2)^{\frac{3}{2}}},$$

$$h'''(\xi,A) = \frac{-3(4\xi^{14}-15\xi^{12}-62\xi^{10}-55\xi^8+24\xi^6+39\xi^4+10\xi^2+1)}{(1+\xi^4)^{\frac{7}{2}}(1+\xi^2)^{\frac{5}{2}}}.$$

Let $h''(\xi,A) = 0$, we can get five real single roots: x_1, x_2, x_3, x_4, x_5. Furthermore, we have $x_1 = 0$, $x_2 \in (-0.489, -0.488)$, $x_3 \in (0.488, 0.489)$, $x_4 \in (-2.056, -2.055)$, $x_5 \in (2.055, 2.056)$. For $\varepsilon \in (0, \frac{1}{5})$, define

$$I_i = \{\xi \in R | -\varepsilon + x_i < \xi < x_i + \varepsilon\}, \quad i = 1, \ldots, 5,$$

then

$$\left|\int_{I_i} e^{ith(\xi,A)}\mathrm{d}\xi\right| \leqslant 2\varepsilon, \quad i = 1, \ldots, 5.$$

Let $D = \{\xi \in R | \ \varepsilon + x_5 < |\xi| < n\}$, by Lemma 2.1, we have

$$\left|\int_{\xi\in D} e^{ith(\xi,A)}\mathrm{d}\xi\right| \leqslant 4\left\{t\min_{\xi\in D}\left|\frac{\partial^2 h}{\partial\xi^2}\right|\right\}^{-\frac{1}{2}} \leqslant 4t^{-\frac{1}{2}}\max\left\{\left|\frac{\partial^2 h}{\partial\xi^2}(x_5+\varepsilon)\right|^{-\frac{1}{2}}, \left|\frac{\partial^2 h}{\partial\xi^2}(n)\right|^{-\frac{1}{2}}\right\} \leqslant Ct^{-\frac{1}{2}}\max\left\{\varepsilon^{-\frac{1}{2}}, n^2\right\}.$$

In other intervals, such as $\{\xi \in R | \ \varepsilon < \xi < x_3 - \varepsilon\}$, the estimate is similar. Hence

$$\sup_{A\in R}\left|\int_{|\xi|\leqslant n} e^{ith(\xi,A)}\,\mathrm{d}\xi\right| \leqslant Ct^{-\frac{1}{2}}\max(\varepsilon^{-\frac{1}{2}}, n^2) + 10\varepsilon. \qquad \square$$

Lemma 2.3. *If $\vec{u}_0 \in X_s \cap X_{0,1}, s > \frac{1}{2}, \vec{u}(t)$ be the solution to the linearized Eq. (1.6) with $\vec{u}(0) = \vec{u}_0 = (u_0, v_0)$, then $\vec{u}(t) \in L^\infty \times L^\infty$, and*

$$|\vec{u}(t)|_\infty \leqslant C(1+t)^{-\theta}(\|\vec{u}_0\|_s + \|\vec{u}_0\|_{0,1}), \tag{2.1}$$

where $\theta = \begin{cases} \frac{2s-1}{2(3+2S)} & \text{if } s \leqslant \frac{9}{2}, \\ \frac{4}{3+2s}, & \text{if } s \geqslant \frac{9}{2}. \end{cases}$

Proof. Taking Fourier transform with (1.6), we can get

$$\begin{cases} \hat{u}_t = i\xi\hat{v} \\ \hat{v}_t = \frac{i(1+\xi^2)\xi}{1+\xi^4}\hat{u} \end{cases} \quad \text{and} \quad \begin{cases} \hat{u}(0) = \hat{u}_0 \\ \hat{v}(0) = \hat{v}_0. \end{cases}$$

Solve this ordinary differential equations, we have

$$\vec{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = S(t)\vec{u}_0 = \int_{-\infty}^{+\infty} e^{ix\xi} \begin{pmatrix} \cos\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) & i\frac{(1+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\sin\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) \\ i\frac{\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}\sin\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) & \cos\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix} \mathrm{d}\xi,$$

where $\langle\xi\rangle = (1+\xi^2)^{1/2}$. To estimate $|u(t)|_\infty$, it suffices to estimate

$$\left|\int_{-\infty}^{+\infty}\left(\hat{u}_0 + \left|\frac{(1+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\hat{v}_0\right|\right)e^{\pm it\left(\xi,\pm\frac{x\xi}{t}\right)}\mathrm{d}\xi\right| \leqslant \left|\int_{|\xi|>n}(\hat{u}_0 + |\langle\xi\rangle\hat{v}_0|)\mathrm{d}\xi\right| + C\left|\int_{|\xi|\leqslant n}(\hat{u}_0 + |\langle\xi\rangle\hat{v}_0|)e^{\pm it\left(\xi,\pm\frac{x\xi}{t}\right)}\mathrm{d}\xi\right|$$

$$\leqslant C\|\vec{u}_0\|_s\left(\int_{|\xi|>n}\langle\xi\rangle^{-2s}\,\mathrm{d}\xi\right)^{\frac{1}{2}} + C(|u_0|_1 + |Jv_0|_1)\left|\int_{|\xi|\leqslant n}e^{\pm it\left(\xi,\pm\frac{x\xi}{t}\right)}\mathrm{d}\xi\right|.$$

Using Lemma 2.2, we obtain

$$|u(t)|_\infty \leqslant C\|\vec{u}_0\|_s n^{\frac{1}{2}-s} + C\|\vec{u}_0\|_{0,1}\left(\varepsilon + t^{-\frac{1}{2}}\max\left(\varepsilon^{-\frac{1}{2}}, n^2\right)\right). \tag{2.2}$$

For $t > 1$, choosing $\varepsilon = (5t)^{-\alpha}$ and $n = (5t)^{\frac{\alpha}{4}}$, we can get

$$|u(t)|_\infty \leqslant C\left(\|\vec{u}_0\|_s + \|\vec{u}_0\|_{0,1}\right)\left(t^{-\left(s-\frac{1}{2}\right)\frac{\alpha}{4}} + t^{-\alpha} + t^{-\frac{1-\alpha}{2}}\right).$$

Let $\left(\frac{1}{2}-s\right)\frac{\alpha}{4} = \frac{\alpha-1}{2}$, it follows that

$$|u(t)|_\infty \leqslant C(\|\vec{u}_0\|_s + (|u_0|_1 + |Jv_0|_1))t^{-\theta}, \tag{2.3}$$

where $\theta = \begin{cases} \frac{2s-1}{2(3+2S)}, & \text{if } s \leqslant \frac{9}{2}, \\ \frac{4}{3+2s}, & \text{if } s \geqslant \frac{9}{2}. \end{cases}$ For $t \leqslant 1$,

$$|u(t)|_\infty \leqslant C\int_{-\infty}^{+\infty}(|\hat{u}_0| + |\langle\xi\rangle\hat{v}_0|)\mathrm{d}\xi \leqslant C\|\vec{u}_0\|_s. \tag{2.4}$$

Combining (2.3)with (2.4) yield

$$|u(t)|_\infty \leqslant C(1+t)^{-\theta}(\|\vec{u}\|_s + \|\vec{u}\|_{0,1}). \tag{2.5}$$

The estimate of $|v(t)|_\infty$ is similar. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *Under the assumption of Lemma 2.3, if* $\vec{u}_0 \in X_{m,1} \cap X_{m+s}$ *for some* $s > \frac{1}{2}$ *and* $m \in R$, *then* $\vec{u}(t) \in X_{m,\infty}$ *and satisfies*

$$\|\vec{u}(t)\|_{m,\infty} \leqslant C(1+t)^{-\theta}(\|\vec{u}_0\|_{m,1} + \|\vec{u}_0\|_{m+s}). \tag{2.6}$$

Proof. Since

$$J^m\vec{u}(t) = S(t)J^m\vec{u}_0 = \int_{-\infty}^{+\infty} e^{ix\xi} \begin{pmatrix} \cos\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) & i\frac{(1+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\sin\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) \\ i\frac{\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}\sin\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) & \cos\left(\frac{\xi\langle\xi\rangle}{(1+\xi^4)^{\frac{1}{2}}}t\right) \end{pmatrix} \begin{pmatrix} \widehat{J^m u_0} \\ \widehat{J^m v_0} \end{pmatrix} \mathrm{d}\xi,$$

it follows that

$$\|\vec{u}(t)\|_{m,\infty} \leqslant C\sum\left|\int_{-\infty}^{+\infty}\left(\widehat{J^m u_0} \pm \frac{(1+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\widehat{J^m v_0}\right)e^{it\left(\xi,\pm\frac{x\xi}{t}\right)}\mathrm{d}\xi\right|$$

$$\leqslant C\sum\left|\int_{|\xi|>n}\left(\widehat{J^m u_0} \pm \frac{(1+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\widehat{J^m v_0}\right)e^{it\left(\xi,\pm\frac{x\xi}{t}\right)}\mathrm{d}\xi\right| + C\sum\left|\int_{|\xi|<n}\left(\widehat{J^m u_0} \pm \frac{(1+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\widehat{J^m v_0}\right)e^{it\left(\xi,\pm\frac{x\xi}{t}\right)}\mathrm{d}\xi\right|$$

$$\leqslant C\|\vec{u}_0\|_{m+s}\left(\int_{|\xi|>n}(1+\xi^2)^{-s}\mathrm{d}\xi\right)^{\frac{1}{2}} + C\sum\left(\left|J^m u_0\right|_1 + \left|J^{m+1}v_0\right|_1\right)\left|\int_{|\xi|\leqslant n}e^{it\left(\xi,\pm\frac{x\xi}{t}\right)}\,\mathrm{d}\xi\right|,$$

where the summation $(\sum)$ is over all two sign combinations. The rest of this proof is similar to Lemma 2.3, we omit it. □

Lemma 2.5. *Under the assumption of Lemma 2.3, if $\vec{u}_0 \in X_{m+s'+\frac{1}{2},1}$ for some $s' > \frac{1}{2}$ and $m + s' > 1$, $s' > s$, then $\vec{u}(t) \in X_{m,\infty}$ and satisfies*

$$\|\vec{u}(t)\|_{m,\infty} \leqslant C(1+t)^{-\theta}\|\vec{u}_0\|_{m+s'+\frac{1}{2},1}. \tag{2.7}$$

Moreover, the following inequality is valid,

$$\|\vec{u}(t)\|_{m,p+1} \leqslant C(1+t)^{-\theta\left(1-\frac{2}{p+1}\right)}\|\vec{u}_0\|_{\mu(s',m),q}, \tag{2.8}$$

where $\frac{1}{p+1}+\frac{1}{q}=1, \mu(s',m)=(1-\lambda)\left(s'+m+\frac{1}{2}\right)+\lambda m, \lambda=\frac{2}{p+1}$.

Proof. For $s' > s$, $m + s' > 1$, by Sobolev's embedding theorem, we have $L^1_{m+s'+\frac{1}{2}} \hookrightarrow L^2_{m+s}$. via Lemma 2.4, (2.7) is true. Since the linear operator $S(t)$ is unitary in X_m for $m \in R$, applying the interpolation theorem for the operator $S(t)$ to (2.7), we get (2.8). The proof of Lemma 2.5 is complete. □

Lemma 2.6 (see [11]). *If $f \in C^m(R)$, $f(0) = 0$, m is an integer and $1 \leqslant s \leqslant m$, then*

$$\|\partial_x f(u)\|_s \leqslant \tilde{f}(\|u\|_{s-1}+\|u\|_1)\|u\|_s \quad \forall u \in H^s(R), \tag{2.9}$$

where $\tilde{f}$ is a continuous function that depends only on f.

Lemma 2.7 (see [16]). *If $f, g \in H^s(R), s > \frac{1}{2}$, then*

$$\|fg\|_s \leqslant C(|f|_\infty\|g\|_s + |g|_\infty\|f\|_s). \tag{2.10}$$

3. Existence of global small amplitude solutions

In this section, we consider the existence of the global small amplitude solution for (1.2), and give the proof of Theorem 1.1 using the contraction mapping theorem.

Proof of Theorem 1.1. Define a metric space

$$X = \{\vec{u} = (u, v) \in C([0,\infty); X_s);\ |||\vec{u}||| \leqslant M\delta\},$$

equipped with the norm

$$|||\vec{u}||| = \sup_{t\geqslant 0}\left[(1+t)^\theta|\vec{u}(t)|_\infty + \|\vec{u}(t)\|_s\right],$$

where $\delta > 0$ satisfies (1.5) and $M > 0$ is a constant to be determined later. It is easy to see that X is a complete metric space. Consider a map as follows

$$\Phi(\vec{u}) = S(t)\vec{u}_0 + \int_0^t S(t-\tau)N(\vec{u}(\tau))\mathrm{d}\tau.$$

Using Lemmas 2.4 and 2.7, it follows easily that

$$\begin{aligned}
|\Phi(\vec{u})|_\infty &\leqslant C(1+t)^{-\theta}\delta + C\int_0^t (1+t-\tau)^{-\theta}\left\{\left\|\frac{\partial_x}{I+\partial_x^4}f(u(\tau))\right\|_{s+1} + \left|\frac{\partial_x J}{I+\partial_x^4}f(u(\tau))\right|_1\right\}\mathrm{d}\tau \\
&\leqslant C(1+t)^{-\theta}\delta + C\int_0^t (1+t-\tau)^{-\theta}\left(|u|_\infty^{p-1}\|u\|_s + |u|_\infty^{p-2}|u\|u_x|_2\right)\mathrm{d}\tau \\
&\leqslant C(1+t)^{-\theta}\delta + \int_0^t (1+t-\tau)^{-\theta}\left[(1+\tau)^{-\theta(p-1)} + (1+\tau)^{-\theta(p-2)}\right]|||\vec{u}|||^p\mathrm{d}\tau \leqslant C(1+t)^{-\theta}(\delta + |||\vec{u}|||^p),
\end{aligned}$$

since $(p-2)\theta > 1 > \theta > 0$. This implies that

$$(1+t)^{-\theta}\|\Phi(t)\|_\infty \leqslant C\delta + C|||\vec{u}|||^p. \tag{3.1}$$

Since linear operator $S(t)$ is unitary in X_s for any $s \in R$, it follows that

$$\|\vec{u}(t)\|_s = \|S(t)\vec{u}_0\|_s = \|\vec{u}_0(t)\|_s.$$

On the other hand, we have

$$\|\Phi(\vec{u})\|_s \leqslant \|\vec{u}_0\|_s + \int_0^t \|N(\vec{u}(\tau))\|_s \mathrm{d}\tau \leqslant \delta + \int_0^t \left\| \frac{1}{I+\partial_x^4} \partial_x f(u)(\tau) \right\|_{s+1} \mathrm{d}\tau \leqslant \delta + C\int_0^t |u(\tau)|_\infty^{p-1} \|u(\tau)\|_s \mathrm{d}\tau$$
$$\leqslant \delta + C|||\vec{u}|||^p \int_0^t (1+\tau)^{-\theta(p-1)} \mathrm{d}\tau \leqslant \delta + C|||\vec{u}|||^p. \tag{3.2}$$

Therefore, combining (3.1) with (3.2) yields

$$|||\Phi(\vec{u})||| \leqslant (C+1)\delta + C|||\vec{u}|||^p. \tag{3.3}$$

Taking that $M = 2C + 1$ and δ is enough small such that $M^p\delta^{p-1} < 1$, from (3.5) we know that Φ maps X into X. By using of Lemma 2.7, for any $\vec{u}, \vec{v} \in X$, we have the following estimate

$$|\Phi(\vec{u}) - \Phi(\vec{w})|_\infty \leqslant \int_0^t (1+t-\tau)^{-\theta} \left(\left\| \frac{\partial_x(f(u)-f(w))}{I+\partial_x^4} \right\|_{s+1} + \left| \frac{\partial_x J(f(u)-f(w))}{I+\partial_x^4} \right|_1 \right) \mathrm{d}\tau$$
$$\leqslant C\int_0^t (1+t-\tau)^{-\theta} \Big\{ \left(|u(\tau)|_\infty + |w(\tau)|_\infty\right)^{p-1} \|(u-w)(\tau)\|_s + (\|u(\tau)\|_s + \|w(\tau)\|_s)\Big(|u(\tau)|_\infty^{p-2}$$
$$+|w(\tau)|_\infty^{p-2}\Big)|(u-w)(\tau)|_\infty + |u|_\infty^{p-2}|u|_2\|(u-w)(\tau)\|_1 + |\partial_x w|_2\Big(|u|_\infty^{p-2} + |w|_\infty^{p-2}\Big)|u-w|_2\Big\}\mathrm{d}\tau$$
$$\leqslant C(1+t)^{-\theta}(|||\vec{u}||| + |||\vec{w}|||)^{p-1}|||\vec{u}-\vec{w}|||,$$

therefore

$$(1+t)^\theta |\Phi(\vec{u}) - \Phi(\vec{w})|_\infty \leqslant C(|||\vec{u}||| + |||\vec{w}|||)^{p-1}|||\vec{u}-\vec{w}|||. \tag{3.4}$$

On the other hand, we have

$$\|\Phi(\vec{u}) - \Phi(\vec{w})\|_s \leqslant C\int_0^t \left\| \frac{\partial_x(f(u)-f(w))}{I+\partial_x^4}(\tau) \right\|_{s+1} \mathrm{d}\tau \leqslant C\int_0^t \Big\{ \left(|u(\tau)|_\infty + |w(\tau)|_\infty\right)^{p-1} \|(u-w)(\tau)\|_s + (\|u(\tau)\|_s$$
$$+\|w(\tau)\|_s)\Big(|u(\tau)|_\infty^{p-2} + |w(\tau)|_\infty^{p-2}\Big)|(u-w)(\tau)|_\infty\Big\}\mathrm{d}\tau \leqslant C(|||\vec{u}||| + |||\vec{w}|||)^{p-1}|||\vec{u}$$
$$-\vec{w}||| \int_0^t (1+\tau)^{-\theta(p-1)}\, \mathrm{d}\tau \leqslant C(|||\vec{u}||| + |||\vec{w}|||)^{p-1}|||\vec{u}-\vec{w}|||. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$|||\Phi(\vec{u}) - \Phi(\vec{w})||| \leqslant C(|||\vec{u}||| + |||\vec{w}|||)^{p-1}|||\vec{u}-\vec{w}|||. \tag{3.6}$$

Taking that δ is enough small such that $C(2M\delta)^{p-1} < \frac{1}{2}$, we know that Φ is strictly contractive. Using the contraction mapping principle, the problem (1.2) have a unique solution $u(x,t)$ on X. The proof of Theorem 1.1 is completed.

4. Decay and scattering of small solutions

In this section, we consider the scattering the small solution for (1.2), and give the proof of Theorem 1.2.

Proof of Theorem 1.2. We shall prove this theorem by four steps below.

Step1. First of all, we are going to show the existence of solution $\vec{u}$ for any $-\infty \leqslant s \leqslant +\infty$. Consider a complete metric subspace $\mathbf{U}(\delta_1) = \{\vec{u} \in \mathbf{U} | \|\vec{u}\|_{\mathbf{U}} \leqslant \delta_1\}$ of $\mathbf{U}$ and a map T defined as follows

$$T\vec{u}(t) = S(t)\vec{g}_- + \int_s^t S(t-\tau)N(\vec{u}(\tau))\mathrm{d}\tau. \tag{4.1}$$

We want to verify the map is contraction. It follows from Lemma 2.5, that

$$\|T\vec{u}(t)\|_{1,p+1} \leqslant \|S(t)\vec{g}_-\|_{1,p+1} + C\left| \int_s^t (1+|t-\tau|)^{-\theta} \left\| \frac{\partial_x}{I+\partial_x^4} N(\vec{u}(\tau)) \right\|_{\mu(s',1),q} \mathrm{d}\tau \right|$$
$$= \|S(t)\vec{g}_-\|_{1,p+1} + C\left| \int_s^t (1+|t-\tau|)^{-\theta} \left\| \frac{\partial_x}{I+\partial_x^4} f(u(\tau)) \right\|_{\mu(s',1)+1,q} \mathrm{d}\tau \right|$$
$$\leqslant \|S(t)\vec{g}_-\|_{1,p+1} + C\left| \int_s^t (1+|t-\tau|)^{-\theta} \|\vec{u}(\tau)\|_{1,p+1}^p \mathrm{d}\tau \right|, \tag{4.2}$$

where $\frac{1}{q} + \frac{1}{p+1} = 1$. Therefore, we get

$$(1+|t|)^{\theta\left(1-\frac{2}{p+1}\right)} \|T\vec{u}(t)\|_{1,p+1} \leqslant (1+|t|)^{\theta\left(1-\frac{2}{p+1}\right)} \|S(t)\vec{g}_-\|_{1,p+1} + CI\|\vec{u}\|_{\mathbf{U}}^p, \tag{4.3}$$

where

$$I = \sup_{t,s\in R}(1+|t|)^{\theta\left(1-\frac{2}{p+1}\right)}\left|\int_s^t (1+|t-\tau|)^{-\theta}(1+|\tau|)^{-\theta p\left(1-\frac{2}{p+1}\right)}\,\mathrm{d}\tau\right|.$$

Since $\theta\left(1-\frac{2}{p+1}\right)p > 1$, the integral is $O((1+|t|)^{-\theta})$ as $|t|\to\infty$, and I is bounded uniformly. On the other hand, we have

$$\begin{aligned}\|T\vec{u}(t)\|_1 &\leqslant \|S(t)\vec{g}_-\|_1 + \left|\int_s^t \left\|S(t-\tau)\frac{\partial_x}{I+\partial_x^4}N(\vec{u}(\tau))\right\|_1 \mathrm{d}\tau\right| \leqslant \|S(t)\vec{g}_-\|_1 + C\left|\int_s^t \left\|\frac{\partial_x}{I+\partial_x^4}f(u(\tau))\right\|_1 \mathrm{d}\tau\right| \\ &\leqslant \|S(t)\vec{g}_-\|_1 + C\left|\int_s^t \|\vec{u}(\tau)\|_{1,p+1}^p \mathrm{d}\tau\right| \leqslant \|S(t)\vec{g}_-\|_1 + C\int_{-\infty}^{+\infty}(1+|\tau|)^{-\theta\left(1-\frac{2}{p+1}\right)}\mathrm{d}\tau\|\vec{u}\|_{\mathbf{U}}^p \leqslant \|S(t)\vec{g}_-\|_1 + C\|\vec{u}\|_{\mathbf{U}}^p. \end{aligned} \tag{4.4}$$

Combining (4.3) with (4.4), we obtain

$$\|T\vec{u}\|_{\mathbf{U}} \leqslant \|S(\cdot)\vec{g}_-\|_{\mathbf{U}} + C(I+1)\|\vec{u}\|_{\mathbf{U}}^p. \tag{4.5}$$

If we choose $\delta \leqslant \frac{\delta_1}{2}$ and δ_1 so small that $C(I+1)\delta_1^{p-1} \leqslant \frac{1}{2}$, then $\|T\vec{u}\|_{\mathbf{U}} \leqslant \delta_1$ for any $\vec{u}\in\mathbf{U}(\delta_1)$. That is, T:$\mathbf{U}(\delta_1)\to\mathbf{U}(\delta_1)$.

On the other hand, if $\vec{u},\vec{v}\in\mathbf{U}(\delta_1)$, $\vec{u}=(u,u_1)$, $\vec{v}=(v,v_1)$, then

$$\begin{aligned}\|T\vec{u}(t)\|_{1,p+1} &\leqslant \left|\int_s^t (1+|t-\tau|)^{-\theta}\left\|\frac{\partial_x}{I+\partial_x^4}(f(u(\tau))-f(v(\tau)))\right\|_{\mu(s',1)+1,q}\mathrm{d}\tau\right| \\ &\leqslant C\left|\int_s^t (1+|t-\tau|)^{-\theta}\left(\|\vec{u}\|_{1,p+1}^{p-1}+\|\vec{v}\|_{1,p+1}^{p-1}\right)\|\vec{u}-\vec{v}\|_{1,p+1}\,\mathrm{d}\tau\right|.\end{aligned}$$

Hence

$$(1+|t|)^{\theta\left(1-\frac{2}{p+1}\right)}\|(T\vec{u}-T\vec{v})(t)\|_{1,p+1} \leqslant CI(\|\vec{u}\|_{\mathbf{U}}^{p-1}+\|\vec{v}\|_{\mathbf{U}}^{p-1})\|\vec{u}-\vec{v}\|_{\mathbf{U}}. \tag{4.6}$$

Now we estimate in X_1,

$$\begin{aligned}\|(T\vec{u}-T\vec{v})\|_1 &\leqslant C\left|\int_s^t |f_x(u)-f_x(v)|_2\mathrm{d}\tau\right| \leqslant C\int_{-\infty}^{+\infty}\left|(|u|^{p-2}+|v|^{p-2})|u-v|_{\frac{2(p+1)}{p-1}}|u_x|_{p+1}+|v|_{2(p+1)}^{p-1}|u_x-v_x|_{p+1}\right|\mathrm{d}\tau \\ &\leqslant C\int_{-\infty}^{+\infty}\left[|u-v|_\infty|u_x|_{p+1}\left(|u|_{\frac{2(p+1)(p-2)}{p-1}}^{p-2}+|v|_{\frac{2(p+1)(p-2)}{p-1}}^{p-2}\right)+|v|_{2(p+1)}^{p-1}|u_x-v_x|_{p+1}\right]\mathrm{d}\tau \\ &\leqslant C\int_{-\infty}^{+\infty}\left(\|u(\tau)\|_{1,p+1}^{p-1}+\|v(\tau)\|_{1,p+1}^{p-1}\right)\|(u-v)(\tau)\|_{1,p+1}\,\mathrm{d}\tau \leqslant C(\|\vec{u}\|_{\mathbf{U}}^{p-1}+\|\vec{v}\|_{\mathbf{U}}^{p-1})\|\vec{u}-\vec{v}\|_{\mathbf{U}}.\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\|(T\vec{u}-T\vec{v})\|_{\mathbf{U}} \leqslant C(I+1)(\|\vec{u}\|_{\mathbf{U}}^{p-1}+\|\vec{v}\|_{\mathbf{U}}^{p-1})\|\vec{u}-\vec{v}\|_{\mathbf{U}}. \tag{4.8}$$

Therefore, if we choose $\delta \leqslant \frac{\delta_1}{2}$ and δ_1 so small that $C(I+1)\delta_1^{p-1} \leqslant \frac{1}{2}$, then T is a contraction map on $\mathbf{U}(\delta_1)$ and so there exists has a unique solution of (1.9) in $\mathbf{U}(\delta_1)$.

Step 2. Now we are going to show (1.12). We rewrite (1.7) as

$$\vec{u}_s(t) = S(t)\vec{u}_{0,s} + \int_0^t S(t-\tau)\vec{f}(\vec{u}_s(\tau))\mathrm{d}\tau, \tag{4.9}$$

where $\vec{u}_{0,s} = \vec{g}_- - \int_0^s S(-\tau)\vec{f}(\vec{u}_s(\tau))\mathrm{d}\tau$. We find

$$\|\vec{u}_{0,s}\|_1 \leqslant \|\vec{g}_-\|_1 + \int_0^s |f_x(u(\tau))|_2\mathrm{d}\tau \leqslant \|\vec{g}_-\| + C\|\vec{u}\|_{\mathbf{U}}^p \leqslant C\delta_1 + C\delta_1^p, \tag{4.10}$$

where the constant C is independent of s. If δ_1 is chosen sufficiently small, by the uniqueness of solution of (1.2), we have

$$\|\vec{u}_s(t)\|_1^2 - 2\int_{-\infty}^{+\infty}F(u_s(t))\mathrm{d}x = \|\vec{u}_s(s)\|_1^2 - 2\int_{-\infty}^{+\infty}F(u_s(s))\mathrm{d}x = \|S(s)\vec{g}_-\|_1^2 - 2\int_{-\infty}^{+\infty}F(S(s)\vec{g}_-)\mathrm{d}x. \tag{4.11}$$

Using (2.8) in Lemma 2.5, we obtain

$$\left|\int_{-\infty}^{+\infty}F(S(s)\vec{g}_-)\right| \leqslant C\int_{-\infty}^{+\infty}|S(s)\vec{g}_-|^{p+1}\mathrm{d}x \leqslant C\|S(s)\vec{g}_-\|_{1,p+1}^{p+1} \leqslant C\left[(1+|t|)^{-\theta\left(1-\frac{2}{p+1}\right)}\right]^{p+1}\|\vec{g}_-\|_{1,q}^{p+1} \to 0 \tag{4.12}$$

as $t\to\pm\infty$. Hence

$$\lim_{s\to-\infty}\left(\|\vec{u}_s(t)\|_1^2-2\int_{-\infty}^{+\infty}F(\vec{u}_s(t))\mathrm{d}x\right)=\|\vec{g}_-\|_1^2. \tag{4.13}$$

On the other hand, let $\vec{u}$ be the solution of (1.7). Since $\|\vec{u}_s\|_{\mathbf{U}}\leqslant\delta_1$ for $-\infty\leqslant s\leqslant+\infty$, and δ_1 is independent of s, we have

$$\begin{aligned}\|\vec{u}_s(t)-\vec{u}(t)\|_1&\leqslant\left|\int_s^t|f_x(u_s(\tau))-f_x(u(\tau))|_2\mathrm{d}\tau\right|+\int_{-\infty}^s|f_x(u(\tau))|_2\mathrm{d}\tau\\&\leqslant C\Big(\|\vec{u}_s\|_{\mathbf{U}}^{p-1}+\|\vec{u}\|_{\mathbf{U}}^{p-1}\Big)\|\vec{u}_s-\vec{u}\|_{\mathbf{U}}+C\int_{-\infty}^s(1+|\tau|)^{-\theta(1-\frac{2}{p+1})p}\,\mathrm{d}\tau\|\vec{u}\|_{\mathbf{U}}^p\leqslant C\delta_1^{p-1}\|\vec{u}_s-\vec{u}\|_{\mathbf{U}}+C\delta_1^p\varepsilon(s),\end{aligned} \tag{4.14}$$

where $\varepsilon(s)=\int_{-\infty}^s(1+|\tau|)^{-\theta(1-\frac{2}{p+1})}\mathrm{d}\tau\to0$ as $s\to-\infty$, and

$$(1+|\tau|)^{\theta(1-\frac{2}{p+1})}\|\vec{u}_s(t)-\vec{u}(t)\|_{1,p+1}\leqslant CI\Big(\|\vec{u}_s\|_{\mathbf{U}}^{p-1}+\|\vec{u}\|_{\mathbf{U}}^{p-1}\Big)\|\vec{u}_s-\vec{u}\|_{\mathbf{U}}+C\eta(s)\|\vec{u}\|_{\mathbf{U}}^p, \tag{4.15}$$

where

$$\eta(s)=\sup_{t\in R}(1+|\tau|)^{\theta(1-\frac{2}{p+1})}\int_{-\infty}^s(1+|t-\tau|)^{-\theta(1-\frac{2}{p+1})}(1+|\tau|)^{-\theta(1-\frac{2}{p+1})p}\mathrm{d}\tau, \tag{4.16}$$

where $\eta(s)\to0$ as $s\to-\infty$. Hence we obtain

$$\|\vec{u}_s(t)-\vec{u}(t)\|_{\mathbf{U}}\leqslant2C\delta_1^{p-1}\|\vec{u}_s(t)-\vec{u}(t)\|_{\mathbf{U}}+C\delta_1^p(\varepsilon(s)+\eta(s)). \tag{4.17}$$

Choosing δ_1 so small that $2C\delta_1^{p-1}\leqslant\frac{1}{2}$, we obtain $\|\vec{u}_s(t)-\vec{u}(t)\|_{\mathbf{U}}\to0$ as $s\to-\infty$. Also, we have

$$\begin{aligned}\left|\int_{-\infty}^{+\infty}(F(u_s(t))-F(u(t)))rm\mathrm{d}x\right|&\leqslant C\int_{-\infty}^{+\infty}\big(|u_s(t)|^p+|u(t)|^p\big)|u_s(t)-u(t)|\mathrm{d}x\\&\leqslant C\Big(|u_s(t)|_{p+1}^p+|u(t)|_{p+1}^p\Big)|u_s(t)-u(t)|_{p+1}\leqslant C\big(\|\vec{u}_s\|_{\mathbf{U}}^p+\|\vec{u}\|_{\mathbf{U}}^p\big)\|\vec{u}_s-\vec{u}\|_{\mathbf{U}}\\&\leqslant2C\delta_1^p\|\vec{u}_s-\vec{u}\|_{\mathbf{U}}\to0\end{aligned} \tag{4.18}$$

as $t\to-\infty$. Hence, taking the limit in (4.13) yields (1.12).

Step3. We are going to show asymptotic behavior (1.11). Since $\vec{u}\in\mathbf{U}(\delta_1)$, we have

$$\begin{aligned}\|\vec{u}(t)-S(t)\vec{g}_-\|_1&\leqslant C\int_{-\infty}^s\|\vec{f}(u(\tau))\|_1\mathrm{d}\tau\leqslant C\int_{-\infty}^t\|u(\tau)\|_{1,p+1}^p\mathrm{d}\tau\leqslant C\int_{-\infty}^t(1+|\tau|)^{-\theta(1-\frac{2}{p+1})p}\,\mathrm{d}\tau\|\vec{u}\|_{\mathbf{U}}^p\\&\leqslant C\delta_1^p\int_{-\infty}^t(1+|\tau|)^{-\theta(1-\frac{2}{p+1})p}\mathrm{d}\tau\to0\end{aligned} \tag{4.19}$$

as $t\to-\infty$, since $\theta\Big(1-\frac{2}{p+1}\Big)p>1$.

Step4. Finally, we are going to show the existence of $\vec{g}_-\in X_1$, (1.13) and (1.14). We define

$$\vec{g}_+=\vec{g}_-+\int_{-\infty}^{+\infty}S(-\tau)\vec{f}(\vec{u}(\tau))\mathrm{d}\tau. \tag{4.20}$$

Since the solution of (1.9) is in $\mathbf{U}(\delta_1)$, we have

$$\|\vec{g}_+\|_1\leqslant\|\vec{g}_-\|_1+C\int_{-\infty}^{+\infty}(1+|\tau|)^{-\theta(1-\frac{2}{p+1})p}\,\mathrm{d}\tau\|\vec{u}\|_{\mathbf{U}}^p\leqslant\|\vec{g}_-\|_1+2C\delta_1^p. \tag{4.21}$$

This implies $\vec{g}_+\in X_1$ by $\vec{g}_-\in X_1$.

To prove (1.13), first of all, we can show $\|S(\cdot)\vec{g}_+\|_{\mathbf{U}}<\frac{\delta_1}{2}$. In fact, it follows from (4.21) that

$$(1+|\tau|)^{\theta(1-\frac{2}{p+1})}\|S(t)\vec{g}_+\|_{1,p+1}\leqslant\|S(\cdot)\vec{g}_-\|_{\mathbf{U}}+CI\|\vec{u}\|_{\mathbf{U}}^p, \tag{4.22}$$

where I is defined as before. Hence by (4.21) and (4.22), we obtain

$$\|S(\cdot)\vec{g}_+\|_{\mathbf{U}}\leqslant\|S(\cdot)\vec{g}_-\|_{\mathbf{U}}+C(I+1)\delta_1^p. \tag{4.23}$$

If we choose $\delta<\frac{\delta_1}{4}$ and $C(I+1)\delta_1^{p-1}<\frac{1}{4}$, then

$$\|S(\cdot)\vec{g}_+\|_{\mathbf{U}}<\frac{\delta_1}{2}. \tag{4.24}$$

Consider the integral equation

$$\vec{u}_s(t)=S(t)\vec{g}_++\int_s^tS(t-\tau)\vec{f}(\vec{u}_s(\tau))\mathrm{d}\tau, \tag{4.25}$$

with $-\infty \leqslant s \leqslant +\infty$. Exactly in the proof of existence of $\vec{u}_s$ of (1.7), we can show that (4.25) has a unique solution $\vec{u}_s \in \mathbf{U}(\delta_1)$. We rewrite it as

$$\vec{u}_s(t) = S(t)\vec{u}_{0,s} + \int_0^t S(t-\tau)\vec{f}(\vec{u}_s(\tau))\mathrm{d}\tau, \tag{4.26}$$

where $\vec{u}_{0,s} = \vec{g}_+ - \int_0^s S(-\tau)\vec{f}(\vec{u}(\tau))\mathrm{d}\tau$. Hence exactly following the proof of (1.13), one can find

$$\|u(t)\|_1^2 + \|v(t)\|_2^2 - 2\int_{-\infty}^{+\infty} F(u(t))\mathrm{d}t = \|\vec{g}_+\|_1^2,$$

where $\vec{u}$ is the solution of (1.9). This implies (1.14). The proof of (1.11) is similar to (1.12). This completes the proof of Theorem 1.2.

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