BUBBLE TREE OF BRANCHED CONFORMAL IMMERSIONS AND APPLICATIONS TO THE WILLMORE FUNCTIONAL

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Abstract. We develop a bubble tree construction and prove compactness results for $W^{2,2}$ branched conformal immersions of closed Riemann surfaces, with varying conformal structures whose limit may degenerate, in $\mathbb{R}^n$ with uniformly bounded areas and Willmore energies. The compactness property is applied to construct Willmore type surfaces in compact Riemannian manifolds. This includes (a) existence of a Willmore 2-sphere in $\mathbb{S}^n$ with at least 2 nonremovable singular points (b) existence of minimizers of the Willmore functional with prescribed area in a compact manifold N provided (i) the area is small when genus is 0 and (ii) the area is close to that of the area minimizing surface of Schoen-Yau and Sacks-Uhlenbeck in the homotopy class of an incompressible map from a surface of positive genus to N and $\pi_2(N)$ is trivial (c) existence of smooth minimizers of the Willmore functional if a Douglas type condition is satisfied.

1. Introduction

Let Σ be a smooth Riemann surface and $f : \Sigma \to \mathbb{R}^n$ be a smooth immersion. The Willmore functional of f is defined by

$$W(f) = \frac{1}{4}\int_\Sigma |H_f|^2 d\mu_f$$

where $H_f = \Delta_{g_f} f$ denotes the mean curvature vector of f, and Δ_{g_f} is the Laplace operator in the induced metric g_f and $d\mu_f$ the induced area element on Σ.

For a sequence of immersions f_k of a compact surface Σ in a bounded set of $\mathbb{R}^n$ with uniformly bounded areas $\mu(f_k)$ and Willmore functionals $W(f_k)$, a subsequence of the image varifolds converges, as Radon measures, to a two dimensional integral varifold, by Allard's integral compactness theorem. The second fundamental forms A_{f_k} are uniformly bounded in the L^2-norm as

$$\int_\Sigma |A_{f_k}|^2 d\mu_{f_k} = 4W(f_k) - 4\pi\chi(\Sigma)$$

from the Gauss equation and the Gauss-Bonnet formula. In general $\|f_k\|_{W^{2,2}}$ are not uniformly bounded: we can find diffeomorphisms ϕ_k from Σ to Σ such that $f_k = f \circ \phi_k$ diverge in C^0, while a uniform bound on $\|f_k\|_{W^{2,2}}$ would imply sequential convergence in C^0 (in fact $C^\alpha, 0 < \alpha < 1$) norm by the Rellich-Kondrachov embedding theorem.

A recent advance in understanding the limit process is given in [12], where each f_k is a conformal immersion from a Riemann surface (Σ, h_k) into $\mathbb{R}^n$ and h_k is the smooth metric

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of constant curvature:

$$h_k \text{ has Gauss curvature } \pm 1, \text{ or } (\Sigma, h_k) = \mathbb{C}/\{1, a+bi\} \text{ with } -\tfrac{1}{2} < a \le \tfrac{1}{2}, \quad b \ge 0, a^2+b^2 \ge 1 \text{ and } a \ge 0 \text{ whenever } a^2+b^2=1. \tag{1.1}$$

There are two reasons to use conformal immersions. One is that the conformal diffeomorphism group of (Σ, h_k) is rather small comparing with the group of diffeomorphisms. Secondly, if we set $g_{f_k} = e^{2u_k} g_{euc}$ in an isothermal coordinate system, then we can estimate $\|u_k\|_{L^\infty}$ from the compensated compactness property of $K_{f_k} e^{2u_k}$. Thus it is possible to get an upper bound of $\|f_k\|_{W^{2,2}}$ via the equation $\Delta_{h_k} f_k = H_{f_k}$. When the conformal structures determined by f_k do not go to the boundary of the moduli space, convergence of f_k is treated in [12]: if the conformal classes induced by f_k converge in the moduli space, then there exist Möbius transformations σ_k, such that $\sigma_k \circ f_k$ converge locally in weak $W^{2,2}$ sense on Σ minus finitely many concentration points. The weak limit f_0 is a $W^{2,2}$ branched conformal immersion.

The $W^{2,2}$ conformal immersions and $W^{2,2}$ branched conformal immersions are as follows:

Definition 1. Let (Σ, h) be a connected Riemann surface. A map $f \in W^{2,2}(\Sigma, h, \mathbb{R}^n)$ is called a *conformal immersion of* (Σ, h), if

$$df \otimes df = e^{2u} h \quad \text{with} \quad \|u\|_{L^\infty(\Sigma)} < +\infty.$$

We denote the set of all such immersions by $W^{2,2}_{conf}(\Sigma, h, \mathbb{R}^n)$. It can be shown that for $f \in W^{2,2}_{conf}(\Sigma, h, \mathbb{R}^n)$ the corresponding u is continuous. When $f \in W^{2,2}_{loc}(\Sigma, h, \mathbb{R}^n)$ with $df \otimes df = e^{2u} h$ and $u \in L^\infty_{loc}(\Sigma)$, we say $f \in W^{2,2}_{conf,loc}(\Sigma, h, \mathbb{R}^n)$.

Obviously, when Σ is compact, $W^{2,2}_{conf}(\Sigma, h, \mathbb{R}^n)$ depends only on the conformal class of (Σ, h), not the choice of h.

Definition 2. We say f is a $W^{2,2}$ *branched conformal immersion of* (Σ, h) *with possible branch points* $x_1, \dots, x_m$, if $f \in W^{2,2}_{conf,loc}(\Sigma \backslash \{x_1, \dots, x_m\}, h, \mathbb{R}^n)$ and

$$\int_{\Sigma \backslash \{x_1, \dots, x_m\}} (1 + |A_f|^2) d\mu_f < +\infty.$$

The set of $W^{2,2}$ branched conformal immersions of (Σ, h_k) is denoted by $W^{2,2}_{b,c}(\Sigma, h, \mathbb{R}^n)$. For compact Σ, $W^{2,2}_{b,c}(\Sigma, h, \mathbb{R}^n)$ depends only on the conformal class of (Σ, h), not the choice of h. When Σ is compact, we say $f \in \widetilde{W}^{2,2}(\Sigma, \mathbb{R}^n)$, if there is a smooth metric h satisfying (1.1) on Σ, such that $f \in W^{2,2}_{b,c}(\Sigma, h, \mathbb{R}^n)$. In other words,

$$\widetilde{W}^{2,2}(\Sigma, \mathbb{R}^n) = \bigcup_h W^{2,2}_{b,c}(\Sigma, h, \mathbb{R}^n).$$

The first part of the paper is a study of a sequence of $W^{2,2}$ branched conformal immersions and the main goal is to establish compactness in Hausdorff distance for such immersions with uniformly bounded areas and Willmore functionals (cf. Theorem 1).

Our compactness result holds not only when h_k converges smoothly in the moduli space $\mathcal{M}_g$, but also when the conformal classes c_k of h_k converge to a degenerated one in the boundary of $\mathcal{M}_g$. Bubbles develop near points where the Willmore energy concentrates, and if c_k goes to a point in the boundary $\overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ additional complication arises as the topology of the limit may be different from that of Σ and stratified surfaces are used as possible limits. The main idea to deal with degenerating conformal structures in the limit process is as follows. First, we compose f_k with diffeomorphisms from an exhausting sequence of domains V_k of the regular part of the limiting (possibly degenerate) surface Σ_0 of (Σ_k, h_k) to a sequence of domains in Σ. Then we study convergence of f_k (composed with the diffeomorphisms) and construct bubble trees at the energy concentration points and collars, and investigate behavior between bubbles. In particular, we will prove that there is no loss of measure in the limit and there are no necks between the bubbles. Then the limit f_0 of f_k is a union of conformal maps from some components $\overline{\Sigma_0^1}, \dots, \overline{\Sigma_0^m}$ of Σ_0 (we delete those components whose images are points) and finitely many 2-spheres $S_1, \dots, S_l$ into $\mathbb{R}^n$. "no neck" means that we can glue $\overline{\Sigma_0^i}$'s and S_j's to form a stratified surface Σ_∞ (see definition below), and f_0 is a continuous map from Σ_∞ into $\mathbb{R}^n$. Then we will apply a result of Hélein [8] and a removable singularity theorem in [12] to show that for a sequence of branched conformal immersions with uniformly bounded measures and Willmore functionals, the limit we get in section 2 is in fact a branched conformal immersion of a stratified surface.

We point out that the "no loss of measure" and "no neck" phenomenon are proved whenever the following two equations hold:

$$-\Delta f_k = \frac{1}{2}|\nabla f_k|^2 H_k, \quad \text{with} \quad \sup_k \int |\nabla f_k|^2(1+|H_k|^2) < \infty, \tag{1.2}$$

$$f_{k,x} \cdot f_{k,x} = f_{k,y} \cdot f_{k,y} \text{ and } f_{k,x} \cdot f_{k,y} = 0, \quad \text{(weakly conformal)} \tag{1.3}$$

where Δ, ∇ are the operators in h_k and x, y are the isothermal coordinates on (Σ, h_k). Note that (1.2) is defined even at non-immersed points of a branched conformal immersion, so it can be applied to study branched immersions. In section 2, we study the blow-up behavior of a sequence of maps which satisfy (1.2) and (1.3).

Equation (1.2) looks similar to the equation of harmonic maps

$$-\Delta u = A(u)(du, du).$$

In fact, the arguments in section 2 are originated from the "energy identity" and "no neck" arguments of harmonic maps [4, 24, 26] (also see [1, 10, 20, 16]). When conformal structures go to the boundary of $\mathcal{M}_g$, non-trivial necks exist for harmonic map ([1, 24, 32]); in our case, however, there is no non-trivial neck due to conformality although (1.2) is much weaker than the harmonic map equation.

Definition 3. Let (Σ, d) be a connected compact metric space. We call Σ a *stratified surface with singular set P* if $P \subset \Sigma$ is a finite set such that

(1) $(\Sigma\backslash P, d)$ is a smooth Riemann surface without boundary (possibly disconnected) and d is a smooth metric $h = d|_{\Sigma\backslash P}$, and

(2) For each $p \in P$, there is δ such that $B_\delta(p) \cap P = \{p\}$ and $B_\delta(p)\backslash\{p\} = \bigcup_{i=1}^{m(p)} \Omega_i$, where $1 < m(p) < +\infty$, and each Ω_i is topologically a disk with its center deleted. Moreover, on each Ω_i, h can be extended to be a smooth metric on the disk.

In [1], the genus of Σ is defined by

$$g(\Sigma) = \frac{2 - \chi(\Sigma) + \sum\limits_{p\in P}(m(p)-1)}{2}.$$

When $g(\Sigma) = 0$, Σ is called a stratified sphere. A stratified surface with singular set $P = \emptyset$ is a smooth Riemann surface.

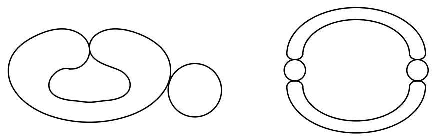

FIGURE 1. Stratified torus

For a stratified surface Σ with singular set P, we can write $\Sigma\backslash P = \bigcup_i \Sigma^i$ where Σ^i's are the disjoint connected components of Σ, and each Σ^i is a punctured Riemann surface when there are more than one components. The topological closure of Σ^i is denoted by Σ_i, so as a point-set $\Sigma = \bigcup_i \Sigma_i$. By (2) in Definition 3, each component Σ^i can be extended to a closed Riemann surface $\overline{\Sigma^i}$ by adding finitely many points. To illustrate the difference of these notations, take, for example, the stratified torus on the left in Figure 1: P contains two points, Σ^1 is the "torus" with two points deleted and Σ^2 is a 2-sphere with one point removed, Σ_1 is the "torus" and Σ_2 is the 2-sphere, while $\overline{\Sigma^1}$ is a Riemann sphere (adding 3 points at the punctures) and $\overline{\Sigma^2}$ is also a Riemann sphere (adding 1 point at the puncture).

When Σ is a stratified surface we define $f \in W^{2,2}_{b,c}(\Sigma,\mathbb{R}^n)$ if f is a $W^{2,2}$ non-trivial branched conformal immersion on each $\overline{\Sigma^i}$.

We now state the main result in the first part of the paper:

Theorem 1. *Suppose that $\{f_k\}$ is a sequence of $W^{2,2}$ branched conformal immersions of closed Riemann surfaces (Σ, h_k) in $\mathbb{R}^n$ and h_k satisfies* (1.1)*. If $f_k(\Sigma)\cap B_{R_0} \neq \emptyset$ for some fixed R_0 and*

$$\sup_k \{\mu(f_k) + W(f_k)\} < +\infty$$

then either $\{f_k\}$ converges to a point, or there is a stratified surface Σ_∞ with $g(\Sigma_\infty) \le g(\Sigma)$, a map $f_0 \in W^{2,2}_{b,c}(\Sigma_\infty,\mathbb{R}^n)$, such that a subsequence of $\{f_k(\Sigma)\}$ converges to $f_0(\Sigma_\infty)$ in Hausdorff distance with

$$\mu(f_0) = \lim_{k\to+\infty}\mu(f_k) \ \ and \ \ W(f_0) \le \lim_{k\to+\infty} W(f_k).$$

For any $\eta \in C^\infty_0(\mathbb{R}^n)$, we have

$$\lim_{k\to+\infty}\int_\Sigma \eta(f_k)d\mu_{f_k} = \int_{\Sigma_\infty}\eta(f_0)d\mu_{f_0}.$$

Moreover, if $y_1,\ldots,y_m \in f_k(\Sigma)$ for all k, then $y_1,\ldots,y_m \in f_0(\Sigma_\infty)$.

In fact, we will prove that f_k converges to f_0 in the sense of bubble tree: for each k, we can find a domain U_k of Σ and a domain V_k of Σ_∞, such that

1) $V_k \subset V_{k+1}$, and $P = \Sigma_\infty \backslash \bigcup_k V_k$ is a finite set which contains all singular points of Σ_∞. Moreover, $\Sigma_\infty \backslash V_k$ is a union of topological disks with finitely many small disks removed, and $H_1^1(\Sigma_\infty \backslash V_k) \to 0$, where H_1^1 is the Hausdorff measure:

$$H_1^1(S) = \inf \left\{ \sum_{i=1}^\infty \mathrm{diam}(\Omega_i) : S \subset \bigcup_{i=1}^\infty \Omega_i, \quad \mathrm{diam}(\Omega_i) < 1 \right\}.$$

2) $\Sigma \backslash U_k$ is a smooth surface with boundary, possibly disconnected, $H_1^1(f_k(\Sigma \backslash U_k)) \to 0$. Moreover, $f_k(\Sigma \backslash U_k)$ converges to P in Hausdorff distance.

3) There is a sequence of diffeomorphisms $\phi_k : V_k \to U_k$, such that for any $\Omega \subset\subset \Sigma_\infty \backslash P$, $f_k \circ \phi_k$ converges in $W^{2,2}(\Omega, \mathbb{R}^n)$ weakly.

In Theorem 1, the singular points of Σ_∞ arise in three ways: (a) the limit point to which a sequence of closed geodesics that are not null-homotopic in $f_k(\Sigma)$ pinches, (b) a bubble point of f_k, so belonging to a 2-sphere (the bubble), (c) a point where both (a) and (b) happen.

In the second part of the paper, we apply Theorem 1 to obtain several existence results of Willmore surfaces in compact Riemannian manifolds. Here we note that Theorem 1 is applicable for surfaces immersed in a compact Riemannian manifold N. To see this, for Σ immersed in N which is isometrically embedded in $\mathbb{R}^n$, direct calculation shows that the Willmore functional of Σ in $\mathbb{R}^n$ is dominated by its Willmore functional in N together with the area $\mu(\Sigma)$, see Lemma 4.1.

We first consider 2-spheres immersed in the round unit sphere $\mathbb{S}^n, n \geq 3$. Fix at least two distinct points $y_1, \ldots, y_m, m \geq 2$ on $\mathbb{S}^n$. Define

$$\beta_0^n(y_1, \ldots, y_m) = \inf \left\{ W_n(f) : f \in W_{conf}^{2,2}(S^2, \mathbb{S}^n), y_1, \ldots, y_m \in f(S^2) \right\}$$

where $W_n(f) = \int_{S^2} \left(1 + \frac{1}{4}|H_f|^2\right) d\mu_f$ and H_f is the mean curvature vector of $f(S^2)$ in $\mathbb{S}^n$.

We show

Theorem 2. *If $\beta_0^n(y_1, \ldots, y_m) < 8\pi$, then there is a $W^{2,2}$ conformal immersion of S^2 in $\mathbb{S}^n$ without self-intersections realizing $\beta_0^n(y_1, \ldots, y_m)$. For any $\epsilon > 0$, there exists a Willmore sphere in $\mathbb{S}^n$ with $W_n(f) < 4\pi + \epsilon$, which has at least 2 nonremovable singularities.*

By results in [15], [27], a singular point of a Willmore surface with density $\theta^2 < 2$ in $\mathbb{R}^n$ can be removed if its residue is 0. Kuwert and Schätzle also point out that the removability can not be true generally, for example, 0 is the true singular point of an inverted half catenoid ([15], P. 337). The second statement in Theorem 2 provides examples of embedded Willmore surface which has a nonremovable singular point with density $\theta^2 = 1$, and it is an application of the first statement with five points prescribed in $\mathbb{S}^n$. In fact, by a very recent result of T. Lamm and H. T. Nguyen [18], the Willmore spheres in Theorem 2 have at least 4 nonremovable singular points. A Gap Lemma in [14] asserts existence of a constant $\epsilon_0(n)$, such that any closed smooth Willmore surfaces immersed in $\mathbb{R}^n$ with $W(f) < 4\pi + \epsilon_0$ must be round spheres. However, in light of Theorem 2, such a gap result is no longer true if we allow the surfaces have singular points.

We then consider minimizers of the Willmore functional subject to area constraint. A fundamental existence result for incompressible minimal surfaces due to Schoen-Yau [30] and Sacks-Uhlenbeck [29] asserts: If $\varphi : \Sigma \to N$ induces an injection from the fundamental group of Σ to that of N, then there is a branched minimal immersion $f : \Sigma \to N$ so that

f induces the same action on the fundamental groups as φ and f has least area among all such maps. We denote the area of the minimizer by a_φ.

Theorem 3. *Let N be a compact Riemannian manifold and let Σ be a closed surface of genus g. Then*

(1) For $\beta_0(N,a) = \inf\{W(f) : \mu(f) = a > 0, f \in W^{2,2}_{conf}(S^2, N)\}$, $\lim_{a\to 0}\beta_0(N,a) = 4\pi$, and there is an embedding realizing $\beta_0(N,a)$ for all sufficiently small a.

(2) Suppose $\varphi : \Sigma \to N$ induces an injection $\varphi_\# : \pi_1(\Sigma) \to \pi_1(N)$ and $\pi_2(N) = 0$. Let $\beta_g(N,a,\varphi) = \inf\{W(f) : f \in \widetilde{W}^{2,2}(\Sigma, N), \mu(f) = a, f$ is homotopic to $\varphi\}$. Then there is $\delta > 0$, such that for any $a \in [a_\varphi, a_\varphi + \delta)$ there is a branched conformal immersion f of (Σ, h) attaining $\beta_g(N,a,\varphi)$. Moreover, when $\dim N = 3$, f is an immersion for small δ.

For $\beta_0(N,a)$, Lamm and Metzger showed in [17] that if it is attained by a surface with positive mean curvature in the sufficiently small geodesic ball around a point p, then the scalar curvature of N must have a critical point at p.

When N has negative sectional curvature, the area of an immersed surface is dominated by the Willmore functional. We now describe a sufficient condition of Douglas type for existence. Let $S(g)$ be the set of connected stratified Riemann surfaces $\Sigma = \bigcup_i \Sigma_i$ satisfying (a) genus of $\Sigma_i < g$ if $g > 0$ and (b) $i > 1$ if $g = 0$. Note that a surface in $S(g)$ has genus at most g and smooth surfaces of genus g are not in $S(g)$. Isometrically embed N into $\mathbb{R}^n$. Define

$$\alpha^*(g) = \inf\{W(f) : f \in W^{2,2}_{b,c}(\Sigma, \mathbb{R}^n), \Sigma \in S(g)\}$$

$$\alpha(g) = \inf\{W(f) : f \in W^{2,2}_{b,c}(\Sigma, \mathbb{R}^n), \Sigma \text{ is a smooth surface of genus } g\}.$$

Similarly, for $0 < a < \infty$, define

$$\begin{aligned}\gamma^*(g,a) &= \inf\{W(f,\Sigma,\mathbb{R}^n) : f \in W^{2,2}_{b,c}(\Sigma,\mathbb{R}^n), f(\Sigma) \subset N, \Sigma \in S(g), \mu(f(\Sigma)) \le a\}\\ \gamma(g,a) &= \inf\{W(f,\Sigma,\mathbb{R}^n) : f \in W^{2,2}_{b,c}(\Sigma,\mathbb{R}^n), f(\Sigma) \subset N, \Sigma \in \mathcal{M}_g, \mu(f(\Sigma)) \le a\}.\end{aligned}$$

Theorem 4. *Let N be a compact Riemannian manifold. If $0 < \alpha(g) < \alpha^*(g)$ and N has negative sectional curvature, then there is a $W^{2,2}$ branched conformal immersion f from a closed Riemann surface of genus g with $W(f) = \alpha(g)$. If $0 < \gamma(g,a) < \gamma^*(g,a)$ then there is a $W^{2,2}$ branched conformal immersion f from a closed Riemann surface of genus g with $W(f) = \gamma(g,a)$.*

We should mention that, the Willmore type functionals of immersed 2-spheres in Riemannian manifolds are also studied by Kuwert, Mondino and Schygulla recently [13]. They proved that for a 3-dimensional compact Riemannian manifold M, (i) if the sectional curvature $K^M > 0$, then there exists a smooth minimizer for $E = \frac{1}{2}\int_{S^2}|A|^2$; (ii) if the sectional curvature $K^M \le 2$ and the scalar curvature $R^M(\overline{x}) > 6$ for some $\overline{x} \in M$, then there exists a smooth minimizer for $W_1 = \int_{S^2}(1 + \frac{1}{4}|H|^2)d\mu$. For higher codimension, Mondino and Riviere have obtained, among other results, existence of a branched conformal immersion of S^2 minimizing E among weak branched immersions of S^2 with finite total curvature [21]. For a closed surface Σ in $\mathbb{R}^3$ whose traceless part of the second fundamental form is small in L^2, De Lellis and Müller have shown, by estimating the conformal metric, that Σ is $W^{2,2}$-close to S^2 and the induced metric is C^0-close to the standard metric of S^2, [2, 22].

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2. BLOW-UP ANALYSIS - ENERGY IDENTITY AND ABSENCE OF NECK

Let (Σ, h) be a Riemann surface which may not be compact, where h is a smooth metric compatible with the complex structure of Σ. For given $p > 1$ and $R > 0$, let $\mathcal{F}^p(\Sigma, h, R)$ be the set of mappings $f : \Sigma \to \mathbb{R}^n$ which satisfy

(1) $f \in W^{2,p}_{loc}(\Sigma, h)$;
(2) $f(\Sigma)$ is contained in the closed ball centered at the origin with radius R in $\mathbb{R}^n$;
(3) $\Delta_h f = F(f)$ with $|F(f)| \leq \beta\, |\nabla_h f|^2$ a.e. on Σ, where β is a nonnegative measurable function on Σ with

$$\int_\Sigma \beta^2 |\nabla_h f|^2 d\mu_h < +\infty.$$

We note that (2) is needed only for the compactness arguments.

When $f \in \mathcal{F}^p(\Sigma, h, R)$, we introduce a notation by

$$H(f) = \begin{cases} 2\frac{\Delta_h f}{|\nabla_h f|^2}, & \text{if } |\nabla_h f| \neq 0 \\ 0, & \text{if } |\nabla_h f| = 0. \end{cases}$$

By (3), $|\Delta_h f| \leq \beta |\nabla f|^2$, almost everywhere on $\{x : \nabla_h f = 0\}$.

We define $W(f)$ to be

$$W(f) = \frac{1}{8}\int_\Sigma |H(f)|^2 |\nabla_h f|^2 d\mu_h.$$

Then $W(f) < \infty$ for $f \in \mathcal{F}^p(\Sigma, h, R)$ follows from (3) as

$$\Delta_h f = F(f) = \frac{1}{2} H(f) |\nabla_h f|^2.$$

We denote by $\mathcal{F}^p_{conf}(\Sigma, h, R)$ the set of $f \in \mathcal{F}^p(\Sigma, h, R)$ and f is weakly conformal a.e., i.e. $\partial f \otimes \partial f = 0$ almost everywhere on Σ, where $\partial f = \frac{\partial f}{\partial z} dz$ in a local complex coordinate system on Σ.

Note that when f is a smooth conformal immersion $H(f), \frac{1}{2}|\nabla_h f|^2 d\mu_h, W(f)$ are the mean curvature vector, the area element and locally the Willmore functional of $f(\Sigma)$, respectively.

By the Kondrachov embedding theorem, functions in $\mathcal{F}^p(\Sigma, h, R)$ are also locally in $W^{1,2}$. The right hand side of the equation $\Delta_h f = F(f)$ is not necessarily in L^2 under the assumption (3).

We point out that $H(f)$, $\mathcal{F}^p, \mathcal{F}^p_{conf}$ are conformally invariant, in the sense that if $h' = e^{2u} h$ for some smooth function u on Σ, we always have

$$H_h(f) = H_{h'}(f), \quad \mathcal{F}^p(\Sigma, h, R) = \mathcal{F}^p(\Sigma, h', R), \quad \mathcal{F}^p_{conf}(\Sigma, h, R) = \mathcal{F}^p_{conf}(\Sigma, h', R).$$

Thus we may select preferred metrics h, e.g. the ones with constant curvature.

In this section, we will study regularity, compactness and the blow-up behavior of a sequence $\{f_k\} \subset \mathcal{F}^p$.

2.1. **ϵ-regularity, removable singularity and weak limit.** In this subsection, we will show that some well-known results for harmonic maps still hold for mappings in $\mathcal{F}^p$.

Let D be the unit 2-disk centered at 0. For simplicity, write $\mathcal{F}^p(D, dx^2+dy^2, R)$ as $\mathcal{F}^p(D,R)$.

Proposition 2.1. *(ϵ-regularity) There is an $\epsilon_0=\epsilon_0(p)>0$ such that for any $f\in\mathcal{F}^p(D,R)$, $1<p<2$, if $W(f)<\epsilon_0^2$, then*

$$\|\nabla f\|_{W^{1,p}(D_{\frac{1}{2}})}\leq C\,\|\nabla f\|_{L^2(D)}.$$

Proof. By working with a smaller disk, without loss of generality, we may assume that $f\in L^1(D)$. Set

$$\bar f=\frac{1}{|D|}\int_D f d\sigma$$

and let η be a cut-off function which is 1 in $D_{1/2}$, 0 in $D\backslash D_{3/4}$ and $0\leq\eta\leq 1$. Then for the equation

$$\Delta\left(\eta(f-\bar f)\right)=(f-\bar f)\Delta\eta+2\nabla\eta\nabla f+\frac{1}{2}\eta H(f)|\nabla f|^2:=\phi$$

we have

$$\begin{aligned}|\phi| &\leq C_1\left(|f-\bar f|+|\nabla f|\right)+\frac{1}{2}\eta\,|H(f)|\,|\nabla f|^2\\ &\leq C_1\left(|f-\bar f|+|\nabla f|\right)+C_2\,|H(f)|\,|\nabla f|\left(|\nabla\left(\eta(f-\bar f)\right)|+|f-\bar f|\right)\end{aligned}$$

since

$$\begin{aligned}\tfrac{1}{2}\eta\,|H(f)|\,|\nabla f|^2 &= \tfrac{1}{2}\eta\,|H(f)|\,\nabla(f-\bar f)\nabla f\\ &= \tfrac{1}{2}\,|H(f)|\,\nabla\left(\eta(f-\bar f)\right)\nabla f-\tfrac{1}{2}\,|H(f)|\,(f-\bar f)\nabla\eta\nabla f\\ &\leq C_2\,|H(f)|\,|\nabla f|\left(|\nabla\left(\eta(f-\bar f)\right)|+|f-\bar f|\right).\end{aligned}$$

By the L^p estimates for elliptic equations,

$$\begin{aligned}\left\|\eta(f-\bar f)\right\|_{W^{2,p}(D)} &\leq C_3\Big(\left\|f-\bar f\right\|_{L^p(D)}+\|\nabla f\|_{L^p(D)}\\ &\quad+\left\|H(f)|\nabla f|\left(|\nabla\left(\eta(f-\bar f)\right)|+|f-\bar f|\right)\right\|_{L^p(D)}\Big).\end{aligned}$$

For $1<p<2$, the Hölder inequality and the Sobolev inequality imply

$$\begin{aligned}&\left\|H(f)|\nabla f|\left(\left|\nabla\left(\eta(f-\bar f)\right)\right|+\left|f-\bar f\right|\right)\right\|_{L^p(D)}\\ &\leq\|H(f)\nabla f\|_{L^2(D)}\left(\|\nabla\left(\eta(f-\bar f)\right)\|_{L^{\frac{2p}{2-p}}(D)}+\|f-\bar f\|_{L^{\frac{2p}{2-p}}(D)}\right)\\ &\leq\epsilon_0\,C_4\,\|\eta(f-\bar f)\|_{W^{2,p}(D)}+\epsilon_0\,C_5\,\|f-\bar f\|_{W^{1,p}(D)}\end{aligned}$$

since $W(f)<\epsilon_0$. Applying the Poincaré inequality and noting $1<p<2$, we get

$$\|f-\bar f\|_{L^p(D)}+\|\nabla f\|_{L^p(D)}+\epsilon_0\,C_5\,\|f-\bar f\|_{W^{1,p}(D)}\leq C_6\,\|\nabla f\|_{L^2(D)}.$$

Choose ϵ_0 so that $C_3C_4\,\epsilon_0<1/2$, then we get

$$\|\eta(f-\bar f)\|_{W^{2,p}(D)}<C_7\,\|\nabla f\|_{L^2(D)}$$

which completes the proof. $\square$

Proposition 2.2. *(Gap constant) Let Σ be a closed Riemann surface. There is a constant $\epsilon_1 = \epsilon_1(\Sigma, p) > 0$, such that for any $f \in \mathcal{F}^p(\Sigma, h, R)$ where $1 < p < 2$, if $W(f) < \epsilon_1^2$, then f is constant.*

Proof. Let $\bar{f} = \frac{1}{|\Sigma|}\int_\Sigma f$. It follows from the equation

$$\Delta_h(f - \bar{f}) = \frac{1}{2}H(f)|\nabla f|^2$$

that

$$\begin{aligned}\int_\Sigma |\nabla(f-\bar{f})|^2 &\leq \frac{1}{2}\int_\Sigma |f-\bar{f}|\,|H(f)|\,|\nabla f|^2 \\ &\leq \left(\int_\Sigma H(f)^2|\nabla f|^2\right)^{\frac{1}{2}}\left(\int_\Sigma |f-\bar{f}|^{\frac{2p}{2p-2}}\right)^{\frac{2p-2}{2p}}\left(\int_\Sigma |\nabla f|^{\frac{2p}{2-p}}\right)^{\frac{2-p}{2p}} \\ &\leq C_1 W(f)^{\frac{1}{2}}\|\nabla f\|_{L^2(\Sigma)}\|\nabla f\|_{L^{\frac{2p}{2-p}}(\Sigma)}\end{aligned}$$

where we used the Sobolev inequality, the Poincaré inequality and $1 < p < 2$. Then,

$$\|\nabla f\|_{L^2(\Sigma)} \leq C_1 W(f)^{\frac{1}{2}}\|\nabla f\|_{L^{\frac{2p}{2-p}}(\Sigma)}.$$

Using the Poincaré inequality and $1 < p < 2$ again, we have

$$\|f-\bar{f}\|_{L^p(\Sigma)} \leq C_2\|\nabla f\|_{L^2(\Sigma)} \leq C_1C_2\,W(f)^{\frac{1}{2}}\|\nabla f\|_{L^{\frac{2p}{2-p}}(\Sigma)}.$$

Since

$$\left\|\frac{1}{2}H(f)|\nabla f|^2\right\|_{L^p(\Sigma)} \leq \left(\int_\Sigma \frac{1}{4}H(f)^2|\nabla f|^2\right)^{\frac{1}{2}}\left(\int_\Sigma |\nabla f|^{\frac{2p}{2-p}}\right)^{\frac{2-p}{2p}} = W(f)^{\frac{1}{2}}\|\nabla f\|_{L^{\frac{2p}{2-p}}(\Sigma)},$$

it follows from the L^p estimates for elliptic equations that

$$\begin{aligned}\|f-\bar{f}\|_{W^{2,p}(\Sigma)} &\leq C_3\left(\|H(f)|\nabla f|^2\|_{L^p(\Sigma)} + \|f-\bar{f}\|_{L^p(\Sigma)}\right) \\ &\leq C_3(1+C_1C_2)W(f)^{\frac{1}{2}}\,\|\nabla f\|_{L^{\frac{2p}{2-p}}(\Sigma)} \\ &\leq C_4 W(f)^{\frac{1}{2}}\,\|f-\bar{f}\|_{W^{2,p}(\Sigma)}\end{aligned}$$

where the Sobolev inequality was used in the last step. By choosing $\epsilon_1 < 1/C_4$ we immediately have $f = \overline{f}$. □

We now derive a key estimate for later applications. For $f : S^1 \times [-t, t] \to \mathbb{R}^n$, define

$$E(f, Q(t)) = \int_{Q(t)} |\nabla f|^2, \text{ where } Q(t) = S^1 \times [-t, t],$$

and denote $\mathcal{F}^p(Q(t), dt^2 + d\theta^2, R)$ by $\mathcal{F}^p(Q(t), R)$. We will prove the following energy decay estimate:

Proposition 2.3. *(Decay estimate) Let $f \in \mathcal{F}^p_{conf}(Q(T), R)$ with $T \geq T_0$, $1 < p < 2$. Then there is a constant $\epsilon_2 < \epsilon_0$, where ϵ_0 is the constant in Proposition 2.1, such that if*

$$\sup_{t\in[-T,T-1]} W(f, S^1 \times [t, t+1]) < \epsilon^2 \leq \epsilon_2^2$$

then

$$\int_{Q(t)} |\nabla f|^2 < CE(f, Q(T))e^{-(1-C\epsilon)(T-t)}, \quad \forall T > T_0.$$

The constants T_0, ϵ_2, C depend only on p.

Proof. Define

$$f^*(t) = \frac{1}{2\pi}\int_0^{2\pi} f(t,\theta)d\theta.$$

We have

$$\begin{aligned}
\int_{Q(t)} \left|\frac{\partial f^*}{\partial t}\right|^2 &= \int_{-t}^{t}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial f}{\partial t}d\theta\right)^2 d\theta dt \\
&\leq \frac{1}{2\pi}\int_{-t}^{t}\left(\int_0^{2\pi}\left|\frac{\partial f}{\partial t}\right|^2 d\theta \int_0^{2\pi} d\theta'\right) dt \qquad (2.1)\\
&= \int_{-t}^{t}\int_0^{2\pi}\left|\frac{\partial f}{\partial t}\right|^2 dt d\theta \\
&= \int_{Q(t)}\left|\frac{\partial f}{\partial t}\right|^2 dt d\theta.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{Q(t)} \nabla(f-f^*)\nabla f &= \int_{Q(t)}|\nabla f|^2 - \int_{Q(t)}\frac{\partial f}{\partial t}\frac{\partial f^*}{\partial t} \\
&\geq \int_{Q(t)}|\nabla f|^2 - \frac{1}{2}\left(\int_{Q(t)}\left|\frac{\partial f}{\partial t}\right|^2 + \int_{Q(t)}\left|\frac{\partial f^*}{\partial t}\right|^2\right) \qquad (2.2)\\
&\geq \int_{Q(t)}|\nabla f|^2 - \int_{Q(t)}\left|\frac{\partial f}{\partial t}\right|^2 \\
&= \frac{1}{2}\int_{Q(t)}|\nabla f|^2
\end{aligned}$$

where in the last step we used the fact that $|f_t|^2 = |f_\theta|^2$ a.e. as f is conformal a.e. On the other hand,

(2.3)
$$\begin{aligned}
\int_{Q(t)} \nabla(f-f^*)\nabla f &= -\int_{Q(t)}(f-f^*)\Delta f - \int_{\partial Q(t)}\frac{\partial f}{\partial n}(f-f^*) \\
&\leq \int_{Q(t)}|f-f^*||\nabla f|^2\frac{|H(f)|}{2} + \left|\int_{S^1\times\{T\}}\frac{\partial f}{\partial t}(f-f^*)\right| + \left|\int_{S^1\times\{-T\}}\frac{\partial f}{\partial t}(f-f^*)\right|.
\end{aligned}$$

Let $m \in [t, t+1)$ be an integer. Then for each $i = -m, -m+1, \ldots, m-1$, by (2.1) and the hypothesis in the proposition

$$\sup_{t\in[-T,T-1]}\frac{1}{4}\int_{S^1\times[t,t+1]}|H(f)|^2 < \epsilon^2 \leq \epsilon_0^2$$

it follows from Proposition 2.1 that

$$\|f - f^*\|_{C^0(S^1\times[i,i+1])} \leq C\,\|\nabla f\|_{L^2(S^1\times[i-1,i+2])}. \qquad (2.4)$$

In fact, to see (2.4), denote the average of f over $S\times[i-1,i+2]$ by $\overline{f}$ and observe that from Proposition 2.1

$$\|\nabla(f-\overline{f})\|_{W^{1,p}(S^1\times[i,i+1])}=\|\nabla f\|_{W^{1,p}(S^1\times[i,i+1])}\le C\|\nabla f\|_{L^2(S^1\times[i-1,i+2])}$$

and from the Poincaré inequality

$$\|f-\overline{f}\|_{L^2(S^1\times[i,i+1])}\le\|f-\overline{f}\|_{L^2(S^1\times[i-1,i+2])}\le C\|\nabla f\|_{L^2(S^1\times[i-1,i+2])}$$

hence

$$\|f-\overline{f}\|_{W^{2,p}(S^1\times[i,i+1])}\le C\|\nabla f\|_{L^2(S^1\times[i-1,i+2])}.$$

The Sobolev embedding theorem then implies

$$\|f-\overline{f}\|_{C^0(S^1\times[i,i+1])}\le C\|f-\overline{f}\|_{W^{2,p}(S^1\times[i,i+1])}\le C\|\nabla f\|_{L^2(S^1\times[i-1,i+2])}.$$

Therefore for any $t\in[i,i+1]$

$$\left|f^*(t)-\overline{f}\right|=\left|\frac{1}{2\pi}\int_0^{2\pi}\left(f(t,\theta)-\overline{f}\right)d\theta\right|\le C\|\nabla f\|_{L^2(S^1\times[i-1,i+2])}.$$

It follows from the triangle inequality and the above:

$$\|f-f^*\|_{C^0(S^1\times[i,i+1])}\le\|f-\bar{f}\|_{C^0(S^1\times[i,i+1])}+\|f^*-\bar{f}\|_{C^0(S^1\times[i,i+1])}\le C\|\nabla f\|_{L^2(S^1\times[i-1,i+2])}.$$

Then

$$\begin{aligned}
&\int_{S^1\times[i,i+1]}|f-f^*||\nabla f|^2\frac{|H(f)|}{2}\\
&\le \|f-f^*\|_{L^\infty(S^1\times[i,i+1])}\times\left(W(f,S^1\times[i,i+1])\int_{S^1\times[i,i+1]}|\nabla f|^2\right)^{\frac{1}{2}}\\
&\le C\epsilon\left(\int_{S^1\times[i-1,i+2]}|\nabla f|^2\int_{S^1\times[i,i+1]}|\nabla f|^2\right)^{\frac{1}{2}}\\
&\le C\epsilon\int_{S^1\times[i-1,i+2]}|\nabla f|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{Q(t)}|f-f^*||\nabla f|^2H(f) &\le \sum_{i=-m}^{m-1}\int_{S^1\times[i,i+1]}|f-f^*||\nabla f|^2H(f)\\
&\le C\epsilon\sum_{i=-m}^{m-1}\int_{S^1\times[i-1,i+2]}|\nabla f|^2\\
&\le 3C\epsilon\int_{Q(t)}|\nabla f|^2\\
&\quad +C\epsilon\left(\int_{S^1\times[-m-1,-m]}|\nabla f|^2+\int_{S^1\times[m,m+1]}|\nabla f|^2\right)\\
&\le 3C\epsilon\int_{Q(t+2)}|\nabla f|^2.
\end{aligned}\tag{2.5}$$

From (2.2), (2.3), (2.5), we have

$$\frac{1}{2}\int_{Q(t)}|\nabla f|^2\le\frac{1}{2}C'\epsilon\int_{Q(t+2)}|\nabla f|^2+\left|\int_{S^1\times\{T\}}(f-f^*)\frac{\partial f}{\partial t}\right|+\left|\int_{S^1\times\{-T\}}(f-f^*)\frac{\partial f}{\partial t}\right|\tag{2.6}$$

for some constant C'. Moreover,

$$\begin{array}{rcl}\left|\int_{S^1\times\{t\}}\frac{\partial f}{\partial t}(f-f^*)\right| & \leq & \left(\int_0^{2\pi}(f(\theta,t)-f^*(t))^2d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left|\frac{\partial f}{\partial t}(\theta,t)\right|^2d\theta\right)^{\frac{1}{2}} \\ & \leq & \left(\int_0^{2\pi}\left|\frac{\partial f}{\partial \theta}(\theta,t)\right|^2d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left|\frac{\partial f}{\partial t}(\theta,t)\right|^2d\theta\right)^{\frac{1}{2}} \\ & = & \frac{1}{2}\int_{S^1\times\{t\}}|\nabla f|^2d\theta\end{array} \tag{2.7}$$

here we used the Poincaré inequality on S^1 and the fact that $|\frac{\partial f}{\partial t}|^2=|\frac{\partial f}{\partial \theta}|^2$ a.e. Let

$$\varphi(t)=\frac{1}{2}\int_{Q(t)}|\nabla f|^2.$$

By (2.6) and (2.7), we have

$$\varphi(t)\leq\varphi'(t)+\epsilon'\varphi(t+2),$$

where $\epsilon'=C'\epsilon$. Then

$$-(e^{-t}\varphi(t))'\leq\epsilon'\varphi(t+2)e^{-t},$$

and integrating the inequality from t to $T-2$ leads to

$$\begin{array}{rcl}e^{-t}\varphi(t) & \leq & e^{-T+2}\varphi(T-2)+\epsilon'\int_t^{T-2}\varphi(s+2)e^{-s}ds \\ & = & e^{-T+2}\varphi(T-2)+\epsilon'\int_{t+2}^{T}\varphi(s)e^{-s+2}ds \\ & = & e^{-T+2}\varphi(T-2)+\epsilon'e^2\int_{t+2}^{T-2}\varphi(s)e^{-s}ds+\epsilon'e^2\int_{T-2}^{T}\varphi(s)e^{-s}ds \\ & \leq & e^{-T+2}\varphi(T)+\epsilon'e^2\int_t^{T-2}\varphi(s)e^{-s}ds+\epsilon'e^2\varphi(T)\left(e^{-T+2}-e^{-T}\right)\end{array} \tag{2.8}$$

as $\varphi(t)$ is increasing in t. Let

$$F(t)=\int_t^{T-2}\varphi(s)e^{-s}ds$$

and $\epsilon_2=\epsilon'e^2<1$. Now (2.8) leads to

$$-F'(t)\leq 2\,\varphi(T)e^{-T+2}+\epsilon_2F(t),$$

equivalently,

$$\left(e^{\epsilon_2t}F(t)\right)'+2\varphi(T)e^{-T+2}e^{\epsilon_2t}\geq 0.$$

Integrating over $[t,T-2]$ and noting $F(T-2)=0$, we have

$$F(t)\leq\frac{2\varphi(T)}{\epsilon_2}e^{2-T}\left(e^{\epsilon_2(T-2)}-e^{\epsilon_2t}\right)e^{-\epsilon_2t}. \tag{2.9}$$

Substitute (2.9) into (2.8):

$$\begin{array}{rcl}\varphi(t) & \leq & e^{2-T+t}\varphi(T)+2\varphi(T)e^{\epsilon_2(T-t-2)}e^{2-T+t}+\epsilon_2\varphi(T)e^{t-T+2} \\ & \leq & C\varphi(T)e^{(1-\epsilon_2)(T-t)}\end{array}$$

$$= \ C\varphi(T)e^{(1-C\epsilon)(T-t)}$$

for some positive constant C independent of T and f. □

Proposition 2.4. *(Removability of point singularity) Let $f \in \mathcal{F}^p_{conf}(D\backslash\{0\}, R)$, where $1 < p < 2$. If $\int_D |\nabla f|^2 < +\infty$, then $f \in \mathcal{F}^{p'}_{conf}(D, R)$ for any $p' \in (1, \frac{4}{3}) \cap (1, p]$.*

Proof. We may assume that $W(f) < \epsilon^2 < \epsilon_2^2$, otherwise, we can replace f with $f(\lambda x)$ for some $\lambda < 1$. Let $\phi : \mathbb{R}^1 \times S^1 \to \mathbb{R}^2$ be the conformal mapping given by $r = e^{-t}, \theta = \theta$. Then $f' = f(\phi)$ is a map from $[0, +\infty) \times S^1$ into $\mathbb{R}^n$. By translating $S^1 \times [t-1, t+1] \subset S^1 \times [0, 2t]$ to $S^1 \times [-1, 1] \subset S^1 \times [-t, t]$, from Proposition 2.3 we conclude

$$\int_{S^1\times[t-1,t+1]} |\nabla f'|^2 \leq C_1 e^{-\delta t}, \quad \text{where} \quad \delta = 1 - C\epsilon.$$

Then for any $r_k = e^{-k}$, we have $t_k = k$ and

$$\int_{D_{r_{k-1}}\backslash D_{r_{k+1}}} |\nabla f|^2 < C_1 r_k^\delta. \tag{2.10}$$

Set $f_k(x) = f(r_k x)$. Applying Proposition 2.1 and (2.10), we get

$$\|\nabla f_k\|_{W^{1,p}(D_1\backslash D_{e^{-1}})} \leq C_2 \|\nabla f_k\|_{L^2(D_e\backslash D_{e^{-2}})} \leq C_3\, r_k^{\frac{\delta}{2}}.$$

By the Sobolev inequality, we have

$$\left(\int_{D_1\backslash D_{e^{-1}}} |\nabla f_k|^q\right)^{\frac{1}{q}} \leq C_4 \|\nabla f_k\|_{W^{1,p}(D_1\backslash D_{e^{-1}})} \leq C_5\, r_k^{\frac{\delta}{2}}, \quad \text{where} \quad q \leq \frac{2p}{2-p}.$$

Then

$$\int_{D_1\backslash D_{e^{-1}}} |\nabla f_k|^q \leq C_6\, e^{-qk\frac{\delta}{2}}.$$

Since

$$r_k^{2-q}\int_{D_1\backslash D_{e^{-1}}} |\nabla f_k|^q = \int_{D_{r_k}\backslash D_{r_{k+1}}} |\nabla f|^q,$$

we have

$$\int_{D_{r_k}\backslash D_{r_{k+1}}} |\nabla f|^q \leq C_6\, e^{-qk\frac{\delta}{2}+(q-2)k} = C_6\, e^{k(-2+q(1-\frac{\delta}{2}))}.$$

When $q < 4$, we can choose ϵ suitably such that $q(1-\frac{\delta}{2}) < 2$, which yields

$$\int_D |\nabla f|^q \leq C_6 \sum_k 2^{-qk\frac{\delta}{2}+(q-2)k} < C_7 < \infty.$$

For any $p' \in (1, \frac{4}{3})$, set $q = \frac{2p'}{2-p'}$, so $q \in (2, 4)$. We have

$$\int_D H(f)^{p'}|\nabla f|^{2p'} \leq \left(\int_D H(f)^2|\nabla f|^2\right)^{\frac{p'}{2}} \left(\int_D |\nabla f|^q\right)^{\frac{p'}{q}} < C_8.$$

Therefore, $F(f) \in L^{p'}(D)$ with $p' > 1$ and then there exists v which solves the equation

$$-\Delta v = F(f), \quad v|_{\partial D} = 0,$$

and $v \in W^{2,p'}(D)$. Obviously, $f - v$ is a harmonic function on $D\backslash\{0\}$ with

$$\|\nabla(f-v)\|_{L^2(D)} + \|f-v\|_{L^2(D)} < +\infty.$$

Then $f - v$ is smooth on D. Now $f \in \mathcal{F}^{p'}_{conf}(D,R)$ is evident for $p' \leq p$ and $1 < p' < \frac{4}{3}$. $\square$

We now consider weak compactness of a sequence $\{f_k\} \subset \mathcal{F}^p(D,R)$ with $W(f_k) < \Lambda$. The blow-up set of $\{f_k\}$ is defined to be

$$\mathcal{C}(\{f_k\}) = \left\{ z \in D : \lim_{r\to 0} \varliminf_{k\to+\infty} W(f_k, D_r(z)) > \epsilon_2^2 \right\}.$$

We can always find a subsequence of $\{f_k\}$, whose blow-up set is a finite set. To see this, we let $z_1 \in \mathcal{C}(\{f_k\})$. Choose a subsequence $\{f_k^1\}$ of $\{f_k\}$, and a sequence $r_k^1 \to 0$ such that

$$W(f_k^1, D_{r_k^1}(z_1)) > \frac{\epsilon_2^2}{2}.$$

If $\mathcal{C}(\{f_k^1\}) \neq \{z_1\}$, then we can find $z_2 \neq z_1$ and a subsequence $\{f_k^2\}$ of $\{f_k^1\}$ and a sequence $r_k^2 \to 0$, such that

$$W(f_k^2, D_{r_k^2}(z_2)) > \frac{\epsilon_2^2}{2}.$$

Obviously, we have

$$\varlimsup_{k\to+\infty} W(f_k, D) \geq \varliminf_{k\to+\infty} W(f_k^2, D_{r_k^{1'}}(z_1)) + \varliminf_{k\to+\infty} W(f_k^2, D_{r_k^2}(z_2)) \geq \epsilon_2^2,$$

where $\{r_k^{1'}\}$ is the corresponding subsequence of $\{r_k^1\}$. Similarly, we can find $\{f_k^3\}$, $\cdots$, $\{f_k^m\}$. However, we must have $m < 2\Lambda/\epsilon^2$, for

$$\Lambda \geq \varlimsup_{k\to+\infty} W(f_k, D) \geq m\frac{\epsilon_2^2}{2}.$$

Without loss of generality, we always assume that $\mathcal{C}(\{f_k\})$ is a finite set. Then for any $z \in D\backslash\mathcal{C}(\{f_k\})$, we can find r and a subsequence of $\{f_k\}$ which is still denoted by $\{f_k\}$ for simplicity, such that

$$\lim_{k\to+\infty} W(f_k, D_r(z)) < \epsilon_0^2.$$

Then from Proposition 2.1

$$\|f_k\|_{W^{2,p}(D_{r/2}(z))} < C(r,p)\|\nabla f_k\|_{L^2(D_r(z))}.$$

Thus we may assume f_k converges weakly in $W^{2,p}_{loc}(D\backslash\mathcal{C}(\{f_k\}))$.

Corollary 2.5. *Let $\{f_k\} \subset \mathcal{F}^p_{conf}(D,R)$ with*

$$\sup_k\{E(f_k,D) + W(f_k,D)\} < \Lambda < \infty$$

and let f_0 be the weak limit of f_k in $W^{2,p}_{loc}(D\backslash\mathcal{C}(\{f_k\}))$. If $p \in (1, \frac{4}{3})$, then $f_0 \in \mathcal{F}^p_{conf}(D,R)$ and

$$W(f_0, D) \leq \varliminf_{k\to+\infty} W(f_k, D). \tag{2.11}$$

Proof. Set $F_k = \Delta f_k, k \in \mathbb{N}$. For any $\Omega \subset\subset D \backslash \mathcal{C}(f_k)$, we have $\|f_k\|_{W^{2,p}(\Omega)} < C(\Omega)$. Then by the Hölder inequality and the Sobolev inequality

$$\|F_k\|_{L^p(\Omega)} \leq \left\| \frac{1}{2} H(f_k) \nabla f_k \right\|_{L^2(\Omega)} \|\nabla f_k\|_{L^{\frac{2p}{2-p}}(\Omega)} \leq C \Lambda^{\frac{1}{2}} \|\nabla f_k\|_{W^{1,p}(\Omega)} < C'(\Omega, \Lambda).$$

We may assume, by selecting subsequences if necessary, that

$$F_k \rightharpoonup F_0 \text{ locally in } L^p(\Omega) \text{ and } |H(f_k)||\nabla f_k| \rightharpoonup \alpha \text{ locally in } L^2(\Omega).$$

Since we may also assume $\nabla f_k \to \nabla f_0$ in $L^2(\Omega)$ because $f_k \rightharpoonup f_0$ in $W^{2,p}(\Omega)$, we have

$$|H(f_k)||\nabla f_k|^2 \rightharpoonup \alpha |\nabla f_0|$$

in the sense of measures in Ω. Define

$$\beta_0 = \begin{cases} \frac{\alpha}{|\nabla f_0|} & \text{when } |\nabla f_0| \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\beta_0 |\nabla f_0|^2 = \alpha |\nabla f_0|$. Let $F_k^+ = \max\{F_k, 0\}$ and $F_k^- = -\min\{F_k, 0\} \geq 0$. Then $F_k = F_k^+ - F_k^-$ and $|F_k| = F_k^+ + F_k^-$. We may assume that

$$F_k^+ \rightharpoonup F_0^1 \text{ and } F_k^- \rightharpoonup F_0^2 \text{ in } L^p(\Omega).$$

Obviously $F_0 = F_0^1 - F_0^2$. Then for any nonnegative function $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \varphi |F_0| \leq \int_\Omega \varphi (F_0^1 + F_0^2) = \lim_{k \to +\infty} \int_\Omega \varphi |F_k| \leq \lim_{k \to +\infty} \int_\Omega \frac{1}{2} \varphi |H(f_k)||\nabla f_k|^2 = \int_\Omega \frac{1}{2} \varphi \beta_0 |\nabla f_0|^2.$$

Hence we conclude

$$|F_0| \leq \frac{1}{2} \beta_0 |\nabla f_0|^2, \quad \text{a.e. } z \in D.$$

Then, we have

$$\int_\Omega \beta_0^2 |\nabla f_0|^2 \leq \int_\Omega \alpha^2 \leq \varliminf_{k \to +\infty} \int_\Omega |H_k(f_k)|^2 |\nabla f_k|^2.$$

Moreover, since f_k converges in $L^2(\Omega)$, it follows from $\partial f_k \otimes \partial f_k = 0$ a.e. in D that $\partial f_0 \otimes \partial f_0 = 0$ a.e. in D as Ω is arbitrary. Since $\sup_k \{E(f_k) + W(f_k)\} < \infty$, there are at most finitely many points in $\mathcal{C}(\{f_k\})$. Then we conclude that $f_0 \in \mathcal{F}^p_{conf}(D, R)$ if $p \in (1, \frac{4}{3})$ by removing the point singularity across $\mathcal{C}(f_k)$ ensured by Proposition 2.4. Furthermore, we have $H(f_0) \leq \beta_0$ whenever $|\nabla f_0| \neq 0$, hence we get (2.11). □

2.2. **A criterion for absence of bubbles along cylinders.** Let $f_k \in \mathcal{F}^p_{conf}(Q(T_k), R)$, with

$$\sup_k \{E(f_k) + W(f_k)\} < \Lambda < \infty.$$

Given a sequence $t_k \in (-T_k, T_k)$ with

$$T_k - t_k \to +\infty \text{ and } t_k - (-T_k) \to +\infty, \tag{2.12}$$

we say the limit f_0 of a subsequence of $f_k(\theta, t + t_k)$, as in Corollary 2.5, is nontrivial if $E(f_0) > 0$. When f_0 is nontrivial, it is a bubble of $\{f_k\}$.

Proposition 2.6. *Let* $f_k \in \mathcal{F}^p_{conf}(S^1 \times (-T_k, T_k), R)$ *with*

$$\sup_k \{E(f_k) + W(f_k)\} = \Lambda < \infty.$$

Let ϵ_2 *be the constant in Proposition 2.3. If*

$$\lim_{T \to +\infty} \varliminf_{k \to +\infty} \sup_{t \in [-T_k+T, T_k-T]} W(f_k, S^1 \times [t, t+1]) < \epsilon_2^2,$$

then we have the following

(1) $\{f_k\}$ *has no bubble;*
(2) *there is no loss of the Dirichlet energy, i.e.*

$$\lim_{T \to +\infty} \lim_{k \to +\infty} \int_{S^1 \times [-T_k+T, T_k-T]} |\nabla f_k|^2 = 0, \tag{2.13}$$

(3) *there is no neck, i.e.*

$$\lim_{t \to +\infty} \lim_{k \to +\infty} f_k(\theta, -T_k + t) = \lim_{t \to +\infty} \lim_{k \to +\infty} f_k(\theta, T_k - t). \tag{2.14}$$

Proof. First note that (1) follows from (2).

We may assume that $f_k(\theta, -T_k + t)$ and $f_k(\theta, T_k - t)$ converge to $f_0^+(\theta, t)$ and $f_0^-(\theta, t)$ weakly in $W^{2,p}_{loc}(S^1 \times [0, +\infty))$, respectively. Then $f_0^+ \circ \phi$, $f_0^- \circ \phi \in \mathcal{F}^p(D \backslash \{0\}, R)$ with

$$E(f_0^\pm \circ \phi) + W(f_0^\pm \circ \phi) \le \Lambda$$

where ϕ is the conformal diffeomorphism between $D \backslash \{0\}$ and $S^1 \times (0, +\infty)$. By removability of point singularities asserted in Proposition 2.4, they are in $\mathcal{F}^{p'}(D, R)$ for some $p' > 1$. It then follows from the compact embedding $W^{2,p'} \subset L^2$:

$$\lim_{T \to \infty} \int_{S^1 \times [T, T+1]} \left(|\nabla f_0^+|^2 + |\nabla f_0^-|^2 \right) = 0.$$

Define $f_k^*(t) = \frac{1}{2\pi} \int_0^{2\pi} f_k(\theta, t) d\theta$. It is easy to check that

$$\lim_{t \to +\infty} \lim_{k \to +\infty} \left(\left| \int_{S^1 \times \{T_k - t\}} (f_k - f_k^*) \frac{\partial f_k}{\partial t} \right| + \left| \int_{S^1 \times \{-T_k + t\}} (f_k - f_k^*) \frac{\partial f_k}{\partial t} \right| \right) = 0.$$

In fact, this can be seen as follows:

$$\sup_{S^1 \times \{\pm(T_k - T)\}} |f_k - f_k^*| \le \operatorname*{osc}_{S^1 \times \{\pm(T_k - T)\}} f_k$$

which will converge to $\mathrm{osc}_{S^1 \times \{T\}} f_0^\pm$ as $k \to \infty$. By removability of singularity,

$$\lim_{T \to \infty} \mathrm{osc}_{S^1 \times \{T\}} f_0^\pm = 0.$$

By the Sobolev trace embedding,

$$\int_{S^1 \times \{T_k - T\}} |\nabla f_k| \le C \|\nabla f_k\|_{W^{1,p}(S^1 \times [T_k - T - 1, T_k - T + 1])}.$$

By ϵ-regularity,

$$\|\nabla f_k\|_{W^{1,p}(S^1 \times [T_k - T - 1, T_k - T + 1])} \le C \|\nabla f_k\|_{L^2(S^1 \times [T_k - T - 2, T_k - T + 2])} < C.$$

Then (2.13) follows from (2.6).

Let m_k be the integer in $[T_k - T, T_k - T + 1)$. For $0 \leq i \leq m_k - 2$, applying Proposition 2.3 on $S^1 \times [i - m_k, m_k]$ (by shifting the center circle to $S^1 \times \{i\}$, and the same below), we have

$$\int_{S^1\times[i-2,i+2]} |\nabla f_k|^2 < CE(f_k, Q(T_k - T))e^{-\delta(m_k - i)}, \quad \delta = 1 - C\epsilon_2.$$

Then from (2.4)

$$\operatorname*{osc}_{S^1\times[i-1,i+1]} f_k \leq C\sqrt{E(f_k, Q(T_k - T))}e^{-\frac{\delta}{2}(m_k - i)}.$$

When $-m_k + 2 \leq i \leq 0$, applying Proposition 2.3 on $S^1 \times [-m_k, m_k + i]$, we get

$$\int_{S^1\times[i-2,i+2]} |\nabla f_k|^2 < CE(f_k, Q(T_k - T))e^{-\frac{\delta}{2}(m_k - |i|)},$$

then we obtain

$$\operatorname*{osc}_{S^1\times[i-1,i+1]} f_k \leq C\sqrt{E(f_k, Q(T_k - T))}e^{-\frac{\delta}{2}(m_k - |i|)}.$$

Hence,

$$\operatorname*{osc}_{Q(T_k-T)} f_k \leq 2C\sqrt{E(f_k, Q(T_k - T))}\sum_{i=1}^{m_k} e^{-\frac{\delta}{2}(m_k - i)} \leq C'\sqrt{E(f_k, Q(T_k - T))}.$$

Then (2.14) can be deduced from (2.13). □

2.3. **Bubble trees for a sequence of maps from the disk D.** Let $f_k \in \mathcal{F}^p_{conf}(D, R)$ with

$$\sup_k \{E(f_k, D) + W(f_k, D)\} = \Lambda < \infty.$$

We assume 0 is the only blow-up point of $\{f_k\}$, i.e. the only point such that

$$\lim_{r\to 0} \varliminf_{k\to+\infty} W(f_k, D_r(0)) \geq \epsilon_2^2.$$

We assume that f_k converges to f_∞ weakly in $W^{2,p}_{loc}(D\backslash\{0\})$. The construction of the bubble tree at 0 will be divided into the following steps:

Step 1. Construct the first level of the bubble tree.

There exists a sequence of points $z_k \in D$ and a sequence of radii $r_k \to 0$ such that

$$W(f_k, D_{r_k}(z_k)) = \frac{\epsilon_2^2}{2} \tag{2.15}$$

and $W(f_k, D_r(z)) < \epsilon_2^2/2$ for any $r < r_k$ and $D_r(z) \subset D$. It is easy to check that $z_k \to 0$ as 0 is the only blow-up point of $\{f_k\}$.

We set $f_k'(z) = f_k(z_k + r_k z)$. Since $\mathcal{C}(\{f_k'\}) = \emptyset$, $f_k'(z)$ converges weakly in $W^{2,p}_{loc}(\mathbb{C})$. Denote the limit by f^F, which may be a trivial mapping.

Let (r, θ) be the polar coordinates centered at z_k, and set $T_k = -\ln r_k$. Let ϕ_k : $S^1 \times [0, T_k] \to \mathbb{R}^2$ be the conformal mapping given by $\phi_k(\theta, t) = z_k + (e^{-t}, \theta)$. Then

$$\phi_k^*(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) = \frac{1}{r^2}(dt^2 + d\theta^2).$$

Thus $f_k \circ \phi_k \in \mathcal{F}^p_{conf}(S^1 \times [0, T_k], R)$. We will also denote $f_k \circ \phi_k$ by f_k for simplicity of notations.

Lemma 2.7. *There exists a subsequence of* $\{f_k\}$ *and* $0 = d_k^0 < d_k^1 < \cdots < d_k^l = T_k$ *with* $l < \Lambda/\epsilon_2^2 + 1$, *such that*

$$\lim_{k\to+\infty} d_k^j - d_k^{j-1} = \infty, \tag{2.16}$$

$$W(f_k, S^1 \times [d_k^j, d_k^j + 1]) \geq \epsilon_2^2, \quad j \neq 0, l \tag{2.17}$$

and

$$\lim_{T\to+\infty} \varliminf_{k\to+\infty} \sup_{t\in[d_k^{j-1}+T, d_k^j - T]} W(f_k, S^1 \times [t, t+1]) \leq \epsilon_2^2, \quad j = 1, ..., l. \tag{2.18}$$

Proof. Suppose

$$(m-1)\epsilon_2^2 < W(f_k, S^1 \times [0, T_k]) \leq \epsilon_2^2 m,$$

where m is a positive integer. We prove the lemma by induction on m.

When $m = 1$, the lemma is obvious by taking $d_k^0 = 0, d_k^1 = T_k$ and (2.17) is vacuous. Assuming the lemma is true for $m-1$, we will prove it also true for m. First of all, if

$$\lim_{T\to+\infty} \varliminf_{k\to+\infty} \sup_{t\in[T, T_k - T]} W(f_k, S^1 \times [t, t+1]) \leq \epsilon_2^2, \tag{2.19}$$

then the lemma follows since $[d_k^{j-1} + T, d_k^j - T] \subset [T, T_k - T]$. If (2.19) does not hold, we can find t_k such that

$$t_k \to +\infty, \quad T_k - t_k \to +\infty,$$

and

$$W(f_k, S^1 \times [t_k, t_k + 1]) \geq \epsilon_2^2.$$

Then

$$W(f_k, S^1 \times [0, t_k]) \leq \epsilon_2^2 (m-1) \quad \text{and} \quad W(f_k, S^1 \times [t_k + 1, T_k]) \leq \epsilon_2^2 (m-1).$$

Using the induction hypothesis on $[0, t_k]$ and $[t_k + 1, T_k]$, we can find

$$0 = \bar{d}_k^0 < \bar{d}_k^1 < \cdots < \bar{d}_k^{\bar{l}} = t_k, \quad \text{and} \quad t_k + 1 = \hat{d}_k^0 < \hat{d}_k^1 < \cdots < \hat{d}_k^{\hat{l}} = T_k,$$

such that

$$\bar{d}_k^i - \bar{d}_k^{i-1} \to +\infty, \qquad \hat{d}_k^i - \hat{d}_k^{i-1} \to +\infty,$$
$$W(f_k, S^1 \times [\bar{d}_k^j, \bar{d}_k^j + 1]) \geq \epsilon_2^2, \quad W(f_k, S^1 \times [\hat{d}_k^j, \hat{d}_k^j + 1]) \geq \epsilon_2^2,$$

and

$$\lim_{T\to+\infty} \varliminf_{k\to+\infty} \sup_{t\in[\bar{d}_k^{j-1}+T, \bar{d}_k^j - T]} W(f_k, S^1 \times [t, t+1]) \leq \epsilon_2^2,$$
$$\lim_{T\to+\infty} \varliminf_{k\to+\infty} \sup_{t\in[\hat{d}_k^{j-1}+T, \hat{d}_k^j - T]} W(f_k, S^1 \times [t, t+1]) \leq \epsilon_2^2.$$

Put

$$d_k^i = \begin{cases} \bar{d}_k^i & i \leq \bar{l}, \\ \hat{d}_k^{i-\bar{l}} & i > \bar{l}. \end{cases}$$

The induction is complete. $\square$

We now start to construct the bubble tree at the first level. In Lemma 2.7, if $l = 1$, in view of Proposition 2.6, we do not do anything as there is no bubble developing in $S^1 \times [0, T_k]$ when $k \to \infty$. If $l > 1$, we set $f_k^i(\theta, t) = f_k(\theta, d_k^i + t)$. We may assume $\{f_k^i\}$

converges weakly in $W^{2,p}$ to a bubble f_∞^i in any compact set outside the blow-up points of $\{f_k^i\}$. By Proposition 2.6, there are no other bubbles of f_k between f_∞^i and f_∞^{i+1} and $f_\infty^i \cup f_\infty^{i+1}$ is connected.

Clearly, $\{f_k^0\}$ and $\{f_k^l\}$ have no blow-up points. Moreover f_∞^0 is $f_\infty \circ \phi_0|_{S^1\times[0,+\infty)}$ and f_∞^l is $f^F \circ \phi_0|_{S^1\times(-\infty,0]}$, where $\phi_0 : S^1 \times \mathbb{R} \to \mathbb{C}$ is given by $(\theta, t) \mapsto (e^{-t}, \theta)$. Removing the point singularity by Proposition 2.4, $f_\infty^1, \cdots, f_\infty^{l-1}$ and f^F can be considered as conformal mappings from S^2 into $\mathbb{R}^n$.

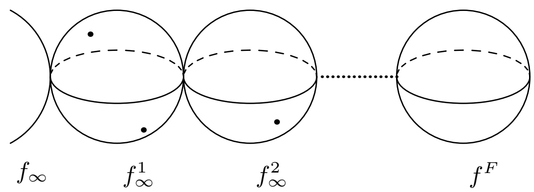

FIGURE 2. Bubble tree: First level (dots denote concentration points)

For a stratified sphere, we can define a *dual graph* as following: 1) Associate one vertex for each component of the stratified sphere; 2) Vertices are connected by edges if the corresponding components meet at a point.

Let S_1 be the stratified sphere with l components whose dual graph is an open path (i.e. a tree such that each vertex has at most 2 edges attached). We define F^1 to the continuous map from S_1 into $\mathbb{R}^n$, such that F^1 is f_∞^i on the i-th component when $i < l$ and f^F on the l-th component. We call F^1 *the first level of bubble tree of* $\{f_k\}$.

We define $E(F^1)$ and $W(F^1)$ by

$$E(F^1) = \sum_{i=1}^{l-1} \int_{S^1\times\mathbb{R}} |\nabla f_\infty^i|^2 + \int_{S^2} |\nabla f^F|, \quad W(F^1) = \sum_{i=1}^{l-1} W(f_\infty^i) + W(f^F).$$

Then

$$\lim_{\delta\to 0}\lim_{k\to+\infty} \int_{D_\delta} |\nabla f_k|^2 = E(F^1) + \sum_i \sum_{p\in\mathcal{C}(\{f_k^i\})} \lim_{r\to 0}\lim_{k\to+\infty} \int_{B_r(p)} |\nabla f_k^i|^2$$

and

$$\lim_{\delta\to 0}\lim_{k\to+\infty} W(f_k, D_\delta) \geq W(F^1) + \sum_i \sum_{p\in\mathcal{C}(\{f_k^i\})} \lim_{r\to 0}\lim_{k\to+\infty} W(f_k^i, B_r(p)).$$

To show convergence of $|\nabla f_k|^2$ in the sense of distributions, we take a test function $\eta \in C_0^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned}
\int_{D_\delta(z_k)} \eta(f_k)|\nabla f_k|^2 &= \int_{D_\delta(z_k)\setminus D_{e^T r_k}(z_k)} \eta(f_k)|\nabla f_k|^2 + \int_{D_{e^T r_k}(z_k)} \eta(f_k)|\nabla f_k|^2 \\
&= \left(\int_{S^1\times[-\ln\delta, T]} + \sum_{i=1}^{l-1} \int_{S^1\times[d_k^i - T, d_k^i + T]} + \sum_{i=0}^{l-1} \int_{S^1\times[d_k^i+T, d_k^{i+1}-T]} \right) \eta(f_k)|\nabla f_k|^2 \\
&\quad + \int_{D_{e^T r_k}(z_k)} \eta(f_k)|\nabla f_k|^2
\end{aligned}$$

$$= \int_{S^1\times[-\ln\delta,T]}\eta(f_k^0)|\nabla f_k^0|^2+\sum_{i=1}^{l-1}\int_{S^1\times[-T,T]}\eta(f_k^i)|\nabla f_k^i|^2 + \sum_{i=0}^{l-1}\int_{S^1\times[d_k^i+T,d_k^{i+1}-T]}\eta(f_k)|\nabla f_k|^2+\int_{D_{e^T}}\eta(f_k')|\nabla f_k'|^2.$$

Since $|\eta(f_k)|$ is bounded, and

$$\lim_{T\to+\infty}\lim_{k\to+\infty}\int_{S^1\times[d_k^i+T,d_k^{i+1}-T]}|\nabla f_k|^2=0,$$

we have

$$\lim_{T\to+\infty}\lim_{k\to+\infty}\int_{S^1\times[d_k^i+T,d_k^{i+1}-T]}\eta(f_k)|\nabla f_k|^2=0.$$

Recalling that f_k^i and f_k' converge weakly in $W^{2,p}$ on any compact sets which contain no concentration points, we get

$$\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{D_\delta}\eta(f_k)|\nabla f_k|^2 = \sum_{i=1}^{l-1}\int_{S^1\times\mathbb{R}}\eta(f_\infty^i)|\nabla f_\infty^i|^2+\int_{\mathbb{C}}\eta(f^F)|\nabla f^F|^2 + \sum_i\sum_{p\in\mathcal{C}(\{f_k^i\})}\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(p)}\eta(f_k^i)|\nabla f_k^i|^2.$$

Step 2. We consider convergence of $\{f_k^i\}$ near its blow-up points.

For each $p\in\mathcal{C}(\{f_k^i\})$, we find a small r such that $B_r(p)\subset S^1\times\mathbb{R}$ contains only one blow-up point. Then for each p, using the arguments in Step 1, we have the first level of bubble tree of $\{f_k^i\}$, which is a map F_p from a stratified sphere S_p into $\mathbb{R}^n$. Each S_p is attached to S_1 at p. Taking union over $p\in\mathcal{C}(\{f_k^i\})$ gives us a continuous map F^2 from S_2, which is a union of stratified spheres, into $\mathbb{R}^n$. We call F^2 *the second level of the bubble tree of* $\{f_k\}$.

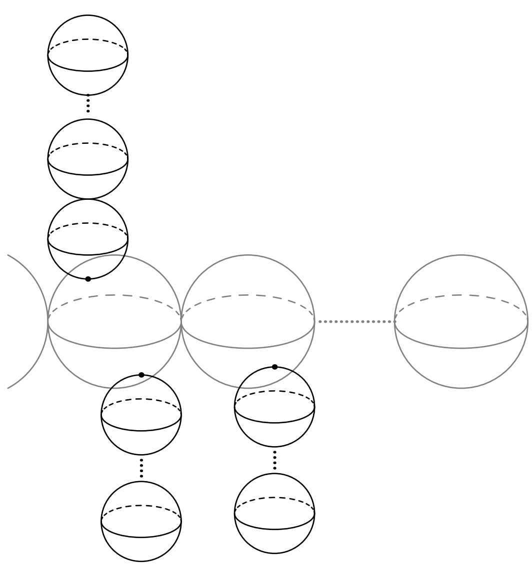

FIGURE 3. Bubble tree: Second level

Step 3. In the same way, we can build the third and higher levels of the bubble tree. Since each step will take away at least ϵ_2^2 from the Willmore functional, the construction will stop after finite many steps. In the end we get a stratified surface S which is the union of all levels and a mapping F from S into $\mathbb{R}^n$. We collapse the components of S on which F maps into points, *i.e.* deleting the ghost bubbles, then we get a new stratified surface S' and a continuous map F' from S' into $\mathbb{R}^n$, such that F' is nontrivial on each component of S'. We call F' is *the bubble tree of* $\{f_k\}$ *at* 0. Moreover, we have

$$\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{D_\delta}\eta(f_k)|\nabla f_k|^2=\int_{S'}\eta(F')|\nabla F'|^2,$$

and

$$W(F')\leq\lim_{\delta\to 0}\lim_{k\to+\infty}W(f_k,D_\delta).$$

2.4. **Bubble trees for a sequence of maps from cylinders $Q(T_k)$ with $T_k\to+\infty$.** In this subsection, we develop the analysis needed in subsection 2.5 when we deal with degeneration of conformal structures.

Let $f_k\in\mathcal{F}^p_{conf}(Q(T_k),R)$ with $T_k\to+\infty$. We assume $f_k(\theta,t+T_k)$ and $f_k(\theta,t-T_k)$ weakly converge in $W^{2,p}_{loc}(S^1\times(-\infty,0])$ and $W^{2,p}_{loc}(S^1\times[0,+\infty))$, respectively. In light of Proposition 2.6, we only need to consider the case that the following happens:

$$\lim_{T\to+\infty}\underline{\lim}_{k\to+\infty}\sup_{t\in[-T_k+T,T_k-T]}W(f_k,S^1\times[t,t+1])\geq\epsilon_2^2 \tag{2.20}$$

since otherwise there will be no bubbles, no necks and no energy loss. When (2.20) holds, there exist $t_k\in(-T_k,T_k)$ such that $T_k-t_k\to+\infty$, $T_k+t_k\to+\infty$ as $k\to\infty$ and

$$W(f_k,S^1\times[t_k,t_k+1])\geq\epsilon_2^2.$$

By Lemma 2.7, we can find (by translations)

$$-T_k=d_k^0<d_k^1<\cdots<d_k^l=T_k$$

which satisfy (2.16), (2.17) and (2.18). Recall that l is independent of k. We may assume $f_k^i(t,\theta)=f_k(d_k^i+t,\theta)$ converges weakly to f_∞^i in $W^{2,p}$ outside the blow-up points $\mathcal{C}(\{f_k^i\})$ of $\{f_k^i\}$. Let Σ_∞^1 be the stratified surface with $l-1$ components whose dual graph is an open path. Then we get a continuous map F^1 from Σ^1_∞ into $\mathbb{R}^n$, and F^1 is f^i_∞ on the i-th component for $i=1,2,\cdots,l-1$. Moreover, we have

$$\begin{aligned}\lim_{T\to+\infty}\lim_{k\to+\infty}\int_{S^1\times[-T_k+T,T_k-T]}\eta(f_k)|\nabla f_k|^2 &= \int_{\Sigma^1_\infty}\eta(F^1)|\nabla F^1|^2\\ &\quad+\sum_{i=1}^{l-1}\sum_{p\in\mathcal{C}(\{f_k^i\})}\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(p)}\eta(f_k^i)|\nabla f_k^i|^2\end{aligned}$$

and

$$\lim_{T\to+\infty}\lim_{k\to+\infty}W(f_k,Q(T_k-T))\geq W(F^1)+\sum_{i=1}^{l-1}\sum_{p\in\mathcal{C}(\{f_k^i\})}\lim_{r\to 0}\lim_{k\to+\infty}W(f_k^i,B_r(p)).$$

The first level of the bubble tree of $\{f_k\}$ is F^1 in this case. Then we use the arguments in section 2.3 to construct the second level of the bubble tree at $\bigcup_{i=1}^{l-1}\mathcal{C}(\{f_k^i\})$, and similarly the third level and so on. The construction stops in finitely many steps. In the end, we get a stratified sphere S'', and a map F'' from S'' to N, such that

$$\lim_{T\to+\infty}\lim_{k\to+\infty}\int_{S^1\times[-T_k+T,T_k-T]}\eta(f_k)|\nabla f_k|^2=\int_{S''}\eta(F'')|\nabla F''|^2$$

and

$$\lim_{T\to+\infty}\lim_{k\to+\infty}W(f_k,Q(T_k-T))\geq W(F'').$$

2.5. **Convergence in Hausdorff distance.** In this subsection we prove:

Theorem 2.8. *Assume that $\{(\Sigma,h_k)\}$ is a sequence of closed Riemann surfaces of genus g, where h_k satisfies (1.1). Suppose that $f_k\in\mathcal{F}^p_{conf}(\Sigma,h_k,R)$ with $p\in(1,\frac{4}{3})$ and*

$$\sup_p\{E(f_k)+W(f_k)\}<\Lambda<\infty. \tag{2.21}$$

Then either $\{f_k\}$ converges to a point, or there is a stratified surface Σ_∞ with $g(\Sigma_\infty)\leq g$, an $f_0\in\mathcal{F}^p_{conf}(\Sigma_\infty,R)$, such that a subsequence of $f_k(\Sigma_k)$ converges to $f_0(\Sigma_\infty)$ in Hausdorff distance with

$$E(f_0)=\lim_{k\to+\infty}E(f_k)\quad\textit{and}\quad W(f_0)\leq\lim_{k\to+\infty}W(f_k).$$

For any $\eta\in C^\infty_0(\mathbb{R}^n)$, we have

$$\lim_{k\to+\infty}\int_{\Sigma_k}\eta(f_k)|\nabla f_k|^2d\mu_{h_k}=\int_{\Sigma_\infty}\eta(f_0)|\nabla f_0|^2d\mu_{\Sigma_\infty}.$$

Remark. Here $f_0\in\mathcal{F}^p_{conf}(\Sigma_\infty,R)$ means that $f_0\in C^0(\Sigma_\infty,\mathbb{R}^n)$, and for any component Σ^i_∞ of Σ_∞, f_0 is nontrivial on Σ^i_∞ and $f_0|_{\Sigma^i_\infty}\in\mathcal{F}^p(\overline{\Sigma^i_\infty},h_i,R)$.

Proof of Theorem 2.8: The proof will consist of three cases according to the genus of Σ.

Spherical case. When Σ is a sphere, as there is only one conformal structure on a 2-sphere, we may let $h_k\equiv h$. Let $\mathcal{C}(\{f_k\})=\{p_1,\ldots,p_m\}$. We can choose δ, such that $B_\delta(p_i)\cap B_\delta(p_j)=\emptyset$. Using isothermal coordinates, each $B_\delta(p_i)$ with metric h is conformal to a Euclidean disk, the results can be deduced from subsection 2.3 directly.

Toric case. Suppose that (Σ,h) is induced by lattice $\{1,a+bi\}$ in $\mathbb{C}$, where $-\frac{1}{2}<a\leq\frac{1}{2}$, $b>0$, $a^2+b^2\geq1$, and $a\geq0$ whenever $a^2+b^2=1$. Then the conformal map f from (Σ,h) into $\mathbb{R}^n$ can be composed with the projection $\mathbb{C}\to\Sigma$ to yield a conformal map $\widetilde{f}$ from $\mathbb{C}$ into $\mathbb{R}^n$ which satisfies

$$\widetilde{f}(z+\lambda)=\widetilde{f}(z),\quad\text{for all}\quad\lambda\in\mathbb{Z}\oplus\mathbb{Z}(a+bi).$$

Let $\Pi:\mathbb{C}\to S^1\times\mathbb{R}$ defined by $x+yi\to(2\pi x,2\pi y)$ be the conformal covering map, where $2\pi x$ and $2\pi(x+m)$ are the same point in S^1 for $m\in\mathbb{Z}$. Then (Σ,h) is conformal to $(S^1\times\mathbb{R})/G$, where $G\cong\mathbb{Z}$ is the transformation group of $S^1\times\mathbb{R}$ generated by the mapping $(\theta,t)\to(\theta+2\pi a,t+2\pi b)$. Then $\tilde{f}$ descents to a conformal map $f':S^1\times\mathbb{R}\to\mathbb{R}^n$, which satisfies $f'\circ\Pi=\widetilde{f}$.

Now we assume $(\Sigma_k,h_k)=S^1\times\mathbb{R}/G_k$, where G_k is generated by

$$(\theta,t)\to(\theta+\theta_k,t+b_k),\quad\text{where}\quad b_k\geq\sqrt{\pi^2-\theta_k^2},\quad\text{and}\quad\theta_k\in[-\frac{\pi}{2},\frac{\pi}{2}].$$

In the moduli space $\mathcal{M}_1$ of genus 1 surfaces, (Σ,h_k) diverges if and only if $b_k\to+\infty$.

For $f_k \in \mathcal{F}^p_{conf}(\Sigma, h_k, R)$ with (2.21), we lift each f_k to a mapping $f_k' : S^1 \times \mathbb{R} \to \mathbb{R}^n$ which satisfies

$$f_k'(\theta, t) = f_k'(\theta + \theta_k, t + b_k).$$

After translations, we may assume that $f_k'(\theta, t + \frac{b_k}{2})$ and $f_k'(\theta, t - \frac{b_k}{2})$ have no blow-up points as $k \to \infty$. Then f_k' satisfies the conditions in subsection 2.4 for $T_k = b_k/2$. Since $f_k'(\theta, -T_k + t) = f_k'(\theta + \theta_k, T_k + t)$, the weak limit of $f_k'(\theta, -T_k + t)$ in $W^{2,p}_{loc}(S^1 \times [0, +\infty))$ and the weak limit of $f_k'(\theta, T_k + t)$ in $W^{2,p}_{loc}(S^1 \times (-\infty, 0])$ yield a conformal map from $S^1 \times \mathbb{R}$ into $\mathbb{R}^n$. So the Hausdorff limit of $f_k(\Sigma)$ is the image of a continuous map F from a stratified surface S of genus 1 into $\mathbb{R}^n$ with

$$\int_S \eta(F)|\nabla F|^2 d\mu_S = \lim_{k \to +\infty} \int_{\Sigma_k} \eta(f_k)|\nabla f_k|^2 d\mu_{h_k}, \quad W(F) \leq \lim_{k \to +\infty} W(f_k).$$

Hyperbolic case. For the hyperbolic case, we first briefly review the compactness of moduli space.

Let Σ_0 be a stable surface in $\overline{\mathcal{M}_g}$ with nodal points $\mathcal{N} = \{a_1, \dots, a_{m'}\}$. Geometrically, Σ_0 is obtained by pinching m' non null homotopy curves in a surface with genus $g > 1$ to points $a_1, \dots, a_{m'}$, thus $\Sigma_0 \backslash \mathcal{N}$ can be divided to finite components $\Sigma_0^1, \dots, \Sigma_0^s$. For each Σ_0^i, we can extend Σ_0^i to a smooth closed Riemann surface $\overline{\Sigma_0^i}$ by adding a point at each puncture. Moreover, the complex structure of Σ_0^i can be extended smoothly to a complex structure of $\overline{\Sigma_0^i}$.

We say h_0 determines a hyperbolic structure on Σ_0 if h_0 is a smooth complete metric on $\Sigma_0 \backslash \mathcal{N}$ with finite volume and Gauss curvature -1. We define

$$\Sigma_0(h_0, \delta) = \left\{ p \in \Sigma_0 \backslash \mathcal{N} : \mathrm{injrad}^{h_0}_{\Sigma_0 \backslash \mathcal{N}}(p) < \delta \right\} \cup \mathcal{N}.$$

Around each nodal point a_j in Σ_0, let $\Sigma_0(a_j, h_0, \delta)$ be the component of $\Sigma_0(h_0, \delta)$ which contains a_j. Let h_0^i be the metric on $\overline{\Sigma_0^i}$ which has Gauss curvature ± 1 or curvature 0, and is conformal to h_0 on Σ_0^i.

Now, we let $\{\Sigma_k\}$ be a sequence of closed Riemann surfaces of fixed genus g with hyperbolic structures h_k, such that $\Sigma_k \to \Sigma_0$ in the moduli space $\overline{\mathcal{M}_g}$. By Proposition 5.1 in [9], there exists a maximal collection $\Gamma_k = \{\gamma_k^1, \dots, \gamma_k^{m'}\}$ of pairwise disjoint, simple closed geodesics in Σ_k with $\ell_k^j = L(\gamma_k^j) \to 0$, such that after passing to a subsequence the following holds:

1. There are maps $\varphi_k \in C^0(\Sigma_k, \Sigma_0)$, such that $\varphi_k : \Sigma_k \backslash \Gamma_k \to \Sigma_0 \backslash \mathcal{N}$ is diffeomorphic and $\varphi_k(\gamma_k^j) = a_j$ for $j = 1, \dots, m'$.
2. For the inverse diffeomorphisms $\psi_k : \Sigma_0 \backslash \mathcal{N} \to \Sigma_k \backslash \Gamma_k$, we have $\psi_k^*(h_k) \to h_0$ in $C^\infty_{loc}(\Sigma_0 \backslash \mathcal{N})$, where h_0 determine a hyperbolic structure on $\Sigma_0 \backslash \mathcal{N}$.
3. Let c_k be the complex structure over Σ_k, and c_0 be the complex structure on $\Sigma_0 \backslash \mathcal{N}$. Then
$$\psi_k^*(c_k) \to c_0 \ \text{ in } \ C^\infty_{loc}(\Sigma_0 \backslash \mathcal{N}).$$

For the transformation

$$(\theta, t) \to \left(\theta, t - \frac{\pi^2}{l_k^j} \right),$$

we have the cylindrical version of the Collar Lemma (cf (4.3) and (4.5) in [32]):

Lemma 2.9. *For each γ_k^j as above, there is a collar U_k^j containing γ_k^j, which is isometric to the cylinder $Q_k^j = Q(\frac{\pi^2}{l_k^j} - \tau_k)$ with metric*

$$h_k^j = \left(\frac{l_k^j}{2\pi\cos(\frac{l_k^j}{2\pi}t)}\right)^2 (dt^2 + d\theta^2), \tag{2.22}$$

where $\tau_k = \frac{2\pi}{l_k^j}\arctan(\sinh(\frac{l_k^j}{2}))$. Moreover, for any $(\theta, t) \in S^1 \times (-\frac{\pi^2}{l_k^j} + \tau_k, \frac{\pi^2}{l_k^j} - \tau_k)$, we have

$$\sinh(injrad_{\Sigma_k}(\theta, t))\cos(\frac{l_k^j t}{2\pi}) = \sinh\frac{l_k^j}{2}. \tag{2.23}$$

Let ϕ_k^j be the isometry between Q_k^j and U_k^j. Then $\varphi_k \circ \phi_k^j(\theta, \frac{\pi^2}{l_k^j} - \tau_k + t) \bigcup \varphi_k \circ \phi_k^j(\theta, -\frac{\pi^2}{l_k^j} + \tau_k + t)$ converges in $C^\infty_{loc}(S^1 \times (-\infty, 0) \cup S^1 \times (0, \infty))$ to an isometry from $S^1 \times (-\infty, 0) \cup S^1 \times (0, +\infty)$ to $\Sigma_0(a_j, h_0, 1)\backslash\{a_j\}$.

We need the following local existence and compactness of conformal diffeomorphisms.

Theorem 2.10. [3] *Let h_k, h_0 be smooth Riemannian metrics on a surface M, such that $h_k \to h_0$ in $C^{s,\alpha}(M)$, where $s \in \mathbb{N}$, $\alpha \in (0,1)$. Then for each point $z \in M$ there exist neighborhoods U_k, U_0 of z and smooth conformal diffeomorphisms $\vartheta_k : D \to U_k, \vartheta_0 : D \to U$, such that $\vartheta_k \to \vartheta_0$ in $C^{s+1,\alpha}(\overline{D}, M)$.*

Proof of Theorem 2.8 (continued): For a sequence of $f_k \in \mathcal{F}^p_{conf}(\Sigma, h_k, R)$ satisfying the energy bound (2.21), let

$$\widetilde{f}_k = f_k \circ \psi_k$$

which is a mapping from $\Sigma_0\backslash\mathcal{N}$ to $\mathbb{R}^n$. It is easy to check that $\widetilde{f}_k \in \mathcal{F}^p_{conf}(\Sigma_0\backslash\mathcal{N}, \psi_k^*(h_k), R)$.

First, we show $\widetilde{f}_k$ converges in $W^{2,p}_{loc}(\Sigma_0\backslash(\mathcal{N} \cup \mathcal{C}(\{f_k\})))$. Given a point $z \in \Sigma_0\backslash(\mathcal{N} \cup \mathcal{C}(\{\widetilde{f}_k\}))$, we choose $U_k, U, \vartheta_k, \vartheta$ as in Theorem 2.10 and U_k, $U \subset \Sigma_0\backslash(\mathcal{N} \cup \mathcal{C}(\{\widetilde{f}_k\}))$. Let

$$\widehat{f}_k = \widetilde{f}_k \circ \vartheta_k$$

and note that $\widehat{f}_k \in \mathcal{F}^p_{conf}(D, R)$. We can assume that $\widehat{f}_k$ converges to $\widehat{f}_\infty$ in $W^{2,p}_{loc}(D_{3/4})$ with $\partial\widehat{f}_\infty \otimes \partial\widehat{f}_\infty = 0$. Let $V = \vartheta(D_{1/2})$. Since ϑ_k converges to ϑ, $\vartheta_k^{-1}(V) \subset D_{3/4}$ for sufficiently large k, $\widetilde{f}_k = \widehat{f}_k(\vartheta_k^{-1})$ converges to $\widetilde{f}_\infty = \widehat{f}_\infty(\vartheta_0^{-1})$ weakly in $W^{2,p}(V, h_0)$. Then $\widetilde{f}_\infty \in \mathcal{F}^p_{conf}(V, h_0, R)$. For any nonnegative continuous function φ supported in V, by Fatou's lemma, we have

$$\lim_{k\to+\infty}\int_V \varphi|H(\widetilde{f}_k)|^2|\nabla\widetilde{f}_k|^2 = \lim_{k\to+\infty}\int_D \varphi(\vartheta_k)|H(\widehat{f}_k)|^2|\nabla\widehat{f}_k|^2 \geq \int_D \varphi(\vartheta_0)|H(\widehat{f}_0)|^2|\nabla\widehat{f}_0|^2. \tag{2.24}$$

We may thus assume $\widetilde{f}_k$ converges weakly to $\widetilde{f}_\infty$ in $W^{2,p}_{loc}(\Sigma_0\backslash(\mathcal{N} \cup \mathcal{C}(\{f_k\})))$. Then $\widetilde{f}_\infty|_{\Sigma_0^i} \in W^{2,p}_{loc}(\Sigma_0^i, h_0^i)$. So for $p \in (1, \frac{4}{3})$, $\widetilde{f}_\infty|_{\Sigma_0^i}$ extends to a map in $\mathcal{F}^p_{conf}(\overline{\Sigma_0^i}, h_0^i, R)$. Further,

$$\lim_{k\to+\infty}\int_{\Sigma_k} \eta(f_k)|\nabla f_k|^2 d\mu_{h_k} = \int_{\Sigma_0} \eta(\widetilde{f}_\infty)|\nabla\widetilde{f}_\infty|^2 d\mu_{\Sigma_0}$$

$$+\sum_j \lim_{\delta\to 0}\lim_{k\to+\infty}\int_{\varphi_k^{-1}(\Sigma_0(a_j,\delta))}\eta(f_k)|\nabla f_k|^2 d\mu_{h_k}+\sum_{z\in\mathcal{C}(\{\widetilde{f}_k\})}\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(z,h_0)}\eta(\widetilde{f}_k)|\nabla\widetilde{f}_k|^2 d\mu_{\psi_k^*(h_k)}$$

and from (2.24)

$$\lim_{k\to+\infty}W(\widetilde{f}_k)\geq W(\widetilde{f}_\infty)+\sum_{z\in\mathcal{C}(\{\widetilde{f}_k\})}\lim_{r\to 0}\lim_{k\to+\infty}W(\widetilde{f}_k,B_r(z,h_0))+\sum_j\lim_{\delta\to 0}\lim_{k\to+\infty}W(\widetilde{f}_k,\Sigma_0(a_j,\delta)).$$

Next, we construct a bubble tree at a point $z\in\mathcal{C}(\{f_k\})\backslash\mathcal{N}$. We have a bubble tree F of $\widehat{f}_k$ at z. We define it to be a bubble tree of $\widetilde{f}_k$ at z. By the arguments in subsection 2.3, we have

$$\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(z,h_0)}\eta(\widetilde{f}_k)|\nabla\widetilde{f}_k|^2 d\mu_{\psi_k^*(h_k)}=\lim_{r\to 0}\lim_{k\to+\infty}\int_{D_r}\eta(\widehat{f}_k)|\nabla\widehat{f}_k|^2=\int_{S'}\eta(F)|\nabla F|^2,$$

and

$$\lim_{r\to 0}\lim_{k\to+\infty}W(\widetilde{f}_k,B_r(z,h_0))=\lim_{r\to 0}\lim_{k\to+\infty}W(\widehat{f}_k,D_r)\geq W(F).$$

Lastly, we consider convergence of f_k on the collars. Set

$$\check{f}_k^j=f_k\circ\phi_k^j\quad\text{and}\quad T_k^j=\frac{\pi^2}{l_k^j}-T.$$

We may choose T to be sufficiently large such that the two sequences $\check{f}_k^j(T_k^j-t,\theta)$ and $\check{f}_k^j(-T_k^j+t,\theta)$ have no blow-up points in $[0,T]$ and $[-T,0]$ respectively. Then $\check{f}_k^j$ satisfies the conditions in subsection 2.4. We get a bubble tree F^j. So the convergence of $\check{f}_k^j$ is clear. Since

$$\check{f}_k^j=f_k\circ\phi_k^j=f_k\circ\psi_k\circ(\varphi_k\circ\phi_k^j)=\widetilde{f}_k(\varphi_k\circ\phi_k^j),$$

we have

$$\begin{aligned}\check{f}_k^j(T_k^j-t,\theta)&=\widetilde{f}_k(\varphi_k\circ\phi_k^j(T_k^j-t,\theta)),\\ \check{f}_k^j(t-T_k^j,\theta)&=\widetilde{f}_k(\varphi_k\circ\phi_k^j(t-T_k^j,\theta)).\end{aligned}$$

By the convergence statement in the Collar Lemma, $\varphi_k\circ\phi_k^j(t+T-\pi^2/l_k^j,\theta)$ and $\varphi_k\circ\phi_k^j(\pi^2/l_k^j-T-t,\theta)$ converge in $C^\infty_{loc}((-\infty,0)\times S^1\cup(0,\infty)\times S^1)$ to an isometry from $(-\infty,0)\times S^1\cup(0,+\infty)\times S^1$ to $\Sigma_0(a_j,1)\setminus\{a_j\}$. We conclude that the image of the limit of $\check{f}_k^j(T_k^j-t,\theta)$ and that of $\check{f}_k^j(-T_k^j+t,\theta)$ are both contained in the image of $\widetilde{f}_\infty$. As in subsection 2.4,

$$\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{\Sigma_0(a_j,\delta)}\eta(f_k)|\nabla f_k|^2 d\mu_{h_k}=\lim_{T\to+\infty}\lim_{k\to+\infty}\int_{Q(T_k-T)}\eta(\check{f})|\nabla\check{f}|^2=\int_{S''}\eta(F^j)|\nabla F^j|^2,$$

and

$$\lim_{\delta\to 0}\lim_{k\to+\infty}W(\widetilde{f}_k,\Sigma_0(a_j,\delta))=\lim_{T\to+\infty}\lim_{k\to+\infty}W(\check{f},Q(T_k-T))\geq W(F^j).$$

Thus, we complete the proof. ☐

Remark 2.11. *When $\Sigma_0 \in \mathcal{M}_p$, i.e. $\mathcal{N} = \emptyset$, ψ_k is a sequence of smooth diffeomorphisms from Σ to Σ. In this case, $g(\Sigma_\infty) = g(\Sigma)$, and*

$$\Sigma_\infty = \Sigma'_\infty \cup S_1 \cup S_2 \cdots \cup S_m,$$

where Σ'_∞ is a smooth Riemann surface of genus p, and each S_i is a sphere.

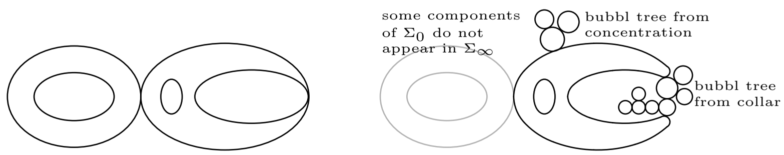

FIGURE 4. Σ_0 (limit of (Σ, h_k)) and Σ_∞

We now generalize Theorem 2.8 to surfaces with marked points. Let us briefly review the compactification of the moduli space of surfaces with marked points. Let $\overline{\mathcal{M}}_{g,m}$ be the moduli space of compact Riemann surfaces of genus g with m marked points. Let $(\Sigma_0, x_{0,1}, \dots, x_{0,m}) \in \partial\overline{\mathcal{M}}_{g,m}$ with nodal points $\mathcal{N} = \{a_1, \dots, a_{m'}\}$. Geometrically, Σ_0 is obtained by pinching some homotopically nontrivial closed curves which do not pass any of $x_{0,1}, \dots, x_{0,m}$ into the points in $\mathcal{N}$, and $\Sigma\backslash\mathcal{N}$ can be divided to connected components $\Sigma_0^1, \cdots, \Sigma_0^s$. For each Σ_0^i, we can extend Σ_0^i to a smooth closed Riemann surface $\overline{\Sigma_0^i}$ by adding a point at each puncture. Moreover, the complex structure of Σ_0^i can be extended smoothly to a complex structure of $\overline{\Sigma_0^i}$.

We say h is a hyperbolic structure on $(\Sigma, x_1, \dots, x_m) \in \mathcal{M}_{g,m}$ if h is a smooth complete metric on $\Sigma\backslash\{x_1, \dots, x_m\}$ with curvature -1 and finite volume. We say h_0 is a hyperbolic structure on $(\Sigma_0, x_{0,1}, \dots, x_{0,m}) \in \overline{\mathcal{M}}_{g,m}\backslash\mathcal{M}_{g,m}$ if h_0 is a smooth complete metric on

$$\Sigma\backslash\{a_1, \dots, a_{m'}, x_{0,1}, ..., x_{0,m}\}$$

with curvature -1 and finite volume.

For a surface Σ with hyperbolic structure h and with marked points $x_1, \dots, x_m$, we define $\Sigma^* = \Sigma\backslash\{x_1, \dots, x_m\}$, and h^* to be the hyperbolic structure on $(\Sigma, x_1, \dots, x_m)$ which is conformal to h on Σ^*.

Let $\{(\Sigma_k, x_{k,1}, \dots, x_{k,m}\}$ be a sequence of marked surfaces in $\mathcal{M}_{g,m}$ with hyperbolic structures h_k and

$$(\Sigma_k, x_{k,1}, \dots, x_{k,m}) \to (\Sigma_0, x_{0,1}, \dots, x_{0,m}) \text{ in } \overline{\mathcal{M}}_{g,m}.$$

By Proposition 5.1 in [9] again, there exists a maximal collection $\Gamma_k = \{\gamma_k^1, \dots, \gamma_k^{m'}\}$ of pairwise disjoint, simple closed geodesics in Σ_k with $\ell_k^j = L(\gamma_k^j) \to 0$ as $k \to \infty$, such that, after passing to a subsequence if necessary, the following holds:

(1) There are maps $\varphi_k \in C^0(\Sigma_k, \Sigma_0)$, such that $\varphi_k : \Sigma_k\backslash\Gamma_k \to \Sigma_0\backslash\mathcal{N}$ is diffeomorphic and $\varphi_k(\gamma_k^i) = a_i$ for $i = 1, \dots, m'$, and $\varphi_k(x_{k,j}) = x_{0,j}$ for $j = 1, \dots, m$.
(2) For the inverse diffeomorphisms $\psi_k : \Sigma_0\backslash\mathcal{N} \to \Sigma_k\backslash\Gamma_k$, we have $\psi_k^*(h_k) \to h$ in $C^\infty_{loc}(\Sigma_0^*\backslash\mathcal{N})$.
(3) Let c_k be the complex structure on Σ_k, and c_0 be the complex structure on $\Sigma_0\backslash\mathcal{N}$. Then $\psi_*(c_k) \to c_0$ in $C^\infty_{loc}(\Sigma_0\backslash\mathcal{N})$.

Moreover, the Collar Lemma also holds for the moduli space with marked points.

Theorem 2.12. *In addition to the assumptions in Theorem 2.8, we assume $y_1, \dots, y_m \in f_k(\Sigma)$ for $m \geq 2$. Then there is a stratified surface Σ_∞ with $g(\Sigma_\infty) \leq g$, and an $f_0 \in \mathcal{F}^p_{conf}(\Sigma_\infty, R)$ with $y_1, \dots, y_m \in f_0(\Sigma_\infty)$, such that a subsequence of $\{f_k(\Sigma_k)\}$ converges to $f_0(\Sigma_\infty)$ in Hausdorff distance with*

$$E(f_0) = \lim_{k\to+\infty} E(f_k) \quad \text{and} \quad W(f_0) \leq \lim_{k\to+\infty} W(f_k).$$

For any $\eta \in C_0^\infty(\mathbb{R}^n)$, we have

$$\lim_{k\to+\infty} \int_{\Sigma_k} \eta(f_k)|\nabla f_k|^2 d\mu_{h_k} = \int_{\Sigma_\infty} \eta(f_0)|\nabla f_0|^2 d\mu_{\Sigma_\infty}.$$

Proof. Let $\widetilde{f}_k = f_k \circ \psi_k$. In view of Theorem 2.8, we only need to consider convergence of $\{\widetilde{f}_k\}$ near $x_{0,j}, j = 1, \dots, m$.

Choose a complex coordinate $\{U, (x, y)\}$ on Σ_0 compatible with c_0, with $x_{0,j} = (0, 0)$. Let $c'_k = \psi_k^*(c_k)$. We set

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = c'_k(e_1),$$

and h'_k to be the metric on U defined by

$$h'_k(e_1, e_1) = h'_k(e_2, e_2) = 1, \quad h'_k(e_1, e_2) = 0.$$

Then h'_k is compatible with c'_k, and converges smoothly to a metric which is compatible with c_0 in U. Then we consider the weak convergence of $\{\widetilde{f}_k\}$ in $U\backslash\mathcal{C}(\{\widetilde{f}_k\})$, using the arguments in subsection 2.3.

It remains to check that each marked point y_i is on the image of f_∞ or one of the bubbles. If $x_{0,j}$ is not a blow up point of $\{\widetilde{f}_k\}$, it is obvious that $y_j \in f_\infty(U)$. Now assume $x_{0,j}$ is the only blow-up point in D. We take U_k, U_0, $\vartheta_k, \vartheta_0, \widehat{f}_k, V$ as in the proof of Theorem 2.8 for $z = x_{0,j}$. We will prove it by induction on the number of the levels of the bubble tree. We take z_k, r_k, ϕ_k and d_k as in subsection 2.3 for $\widehat{f}_k$. If $|z_k|/r_k < L$ for some fixed L, then we may assume $-z_k/r_k \to z_\infty$, by selecting a subsequence if necessary. Recalling that $\widehat{f}_k(0) \equiv y_j$, we get $y_j = \widehat{f}^F(z_\infty)$. Let (r, θ) be the polar coordinates centered at z_k, $T_k = -\ln r_k$ and $\phi_k : [0, T_k] \times S^1 \to \mathbb{R}^2$ be the conformal mapping given by $\phi_k(t, \theta) = (e^{-t}, \theta)$. We set $\phi_k^{-1}(0) = (t_k, \theta_k)$. Then $|z_k|/r_k \to +\infty$ means that $t_k \in [0, T_k]$ and $T_k - t_k \to +\infty$. Thus we may assume $t_k \in [d_k^i, d_k^{i+1}]$ for some i, where d_k^i are defined in Lemma 2.7. Then, if $t_k - d_k^i \to +\infty$ and $d_k^{i+1} - t_k \to +\infty$, we have $y_j = f_\infty^i(+\infty) = f_\infty^{i+1}(-\infty)$. If at least one of $t_k - d_k^i$ and $d_k^{i+1} - t_k$ is bounded above for all large k, then we repeat the above argument at the second level of the bubble tree, and proceed in this way for the finitely many levels of the bubble tree if necessary, and we conclude that y^j is on one of the bubbles of $\{\widetilde{f}_k^i\}$ or $\{\widetilde{f}_k^{i+1}\}$.

Finally, as $m \geq 2$ and all $y_i \in f_0(\Sigma_\infty)$, f_k cannot converge to a single point. $\square$

3. Branched conformal immersions and proof of Theorem 1

For a branched conformal immersion, we have the following result:

Theorem 3.1. [12] *Suppose that* $f \in W^{2,2}_{conf,loc}(D\backslash\{0\}, \mathbb{R}^n)$ *satisfies*

$$\int_D |A_f|^2 \, d\mu_g < \infty \quad \text{and} \quad \mu_g(D) < \infty,$$

where $g_{ij} = e^{2u}\delta_{ij}$ *is the induced metric. Then* $f \in W^{2,2}(D, \mathbb{R}^n)$ *and we have*

$$\begin{aligned} u(z) &= \lambda \log |z| + \omega(z) \quad \text{where } \lambda \geq 0, \quad \lambda \in \mathbb{Z}, \quad \omega \in C^0 \cap W^{1,2}(D), \\ -\Delta u &= -2\lambda\pi\delta_0 + K_g e^{2u} \quad \text{in } D. \end{aligned}$$

The density of $f(D_\sigma)$ *as varifolds at* $f(0)$ *is given by* $\lambda + 1$ *for any small* $\sigma > 0$.

The classical Gauss-Bonnet formula is generalized in [5] to smooth branched immersions and in [12] to $W^{2,2}$ conformal immersions. Following arguments in [5] and [12], we provide a Gauss-Bonnet formula for $W^{2,2}$ branched conformal immersions.

Lemma 3.2. *Let* (Σ, g) *be a closed Riemann surface. Then for any* $f \in W^{2,2}_{b,c}(\Sigma, g, \mathbb{R}^n)$, *there holds*

$$\int_\Sigma K_f d\mu_f = 2\pi\chi(\Sigma) + 2\pi b, \tag{3.1}$$

where b *is the number of branch points counted with multiplicities and at each branch point* p *the branching order is* $\lambda = \theta^2(p) - 1$.

Proof. Without loss of generality, we assume f has only one branch point p. Let $g_f = e^{2u}g$ be the metric induced by f and K_f be its Gauss curvature. It is shown in [12] that

$$-\Delta_g u = K_f e^{2u} - K_g$$

holds weakly on $\Sigma\backslash\{p\}$: for any smooth φ with support in $\Sigma\backslash\{p\}$, it holds

$$\int_\Sigma \nabla_g u \nabla_g \varphi \, d\mu_g = \int_\Sigma \varphi K_f e^{2u} d\mu_g - \int_\Sigma \varphi K_g du_g.$$

Take a complex coordinate chart $\{U; z\}$ around $p = 0$. For any small $\epsilon > 0$, we choose a function $\varphi_\epsilon(z) = \varphi_\epsilon(|z|)$ between 0 and 1 with $|\varphi'_\epsilon| < C/\epsilon$ and equals 1 outside D_ϵ and 0 in $D_{\epsilon/2}$. Then we have

$$\int_{D_\epsilon} \frac{\partial u}{\partial r} \varphi'_\epsilon dx = \int_\Sigma \varphi_\epsilon K_f e^{2u} d\mu_g - \int_\Sigma \varphi_\epsilon K_g du_g.$$

By Theorem 3.1, $u = \lambda \log |z| + \omega$,

$$\int_{D_\epsilon} \frac{\partial u}{\partial r} \varphi'_\epsilon dx = \int_{D_\epsilon} \frac{\partial \omega}{\partial r} \varphi'_\epsilon dx + 2\pi\lambda \left(\varphi_\epsilon(\epsilon) - \varphi_\epsilon(0)\right) = \int_{D_\epsilon} \frac{\partial \omega}{\partial r} \varphi'_\epsilon dx + 2\pi\lambda.$$

Since

$$\int_{D_\epsilon} \left|\frac{\partial \omega}{\partial r} \varphi'_\epsilon\right| \leq C \left(\int_{D_\epsilon \backslash D_{\epsilon/2}} \left|\frac{\partial \omega}{\partial r}\right|^2\right)^{1/2} \left(\int_{D_\epsilon \backslash D_{\epsilon/2}} \frac{1}{r^2}\right)^{1/2} \leq C\|\nabla\omega\|_{L^2(D_\epsilon)} \to 0, \quad \text{as} \quad \epsilon \to 0,$$

the classical Gauss-Bonnet theorem on (Σ, g) implies

$$\int_\Sigma K_f d\mu_f = \lim_{\epsilon \to 0} \int_\Sigma \varphi_\epsilon K_f d\mu_f = \lim_{\epsilon \to 0} \int_\Sigma \varphi_\epsilon K_g d\mu_g + \lim_{\epsilon \to 0} \int_\Sigma \frac{\partial u}{\partial r} \varphi'_\epsilon d\mu_g = 2\pi\chi(\Sigma) + 2\lambda\pi$$

and complete the proof. $\square$

Remark 3.3. *Since $\int_\Sigma K_f d\mu_f \leq W(f)$, it follows from Lemma 3.2*

$$b \leq \frac{1}{2\pi} W(f) - \chi(\Sigma).$$

Moreover,

$$\int_\Sigma |A_f|^2 d\mu_f = 4W(f) - 2\int K_f \leq 4W(f) - 2\pi\chi(\Sigma).$$

Then $\sup_k W(f_k) < +\infty$ *implies that* $\sup_k b_k < +\infty$ *and* $\sup_k \int_\Sigma |A_{f_k}|^2 d\mu_{f_k} < +\infty$.

To study convergence of conformal immersions, we recall an important result of Hélein.

Theorem 3.4. [8] *Let $f_k \in W^{2,2}_{conf}(D, \mathbb{R}^n)$ be a sequence of conformal immersions with induced metrics $(g_k)_{ij} = e^{2u_k}\delta_{ij}$, and assume*

$$\int_D |A_{f_k}|^2 \, d\mu_{g_k} \leq \gamma < \gamma_n = \begin{cases} 8\pi & \text{for } n = 3, \\ 4\pi & \text{for } n \geq 4. \end{cases}$$

Assume also that $\mu_{g_k}(D) \leq C$ and $f_k(0) = 0$. Then f_k is bounded in $W^{2,2}_{loc}(D, \mathbb{R}^n)$, and there is a subsequence such that one of the following two alternatives holds:

(a) *u_k is bounded in $L^\infty_{loc}(D)$ and f_k converges weakly in $W^{2,2}_{loc}(D, \mathbb{R}^n)$ to a conformal immersion $f \in W^{2,2}_{conf,loc}(D, \mathbb{R}^n)$.*

(b) *$u_k \to -\infty$ and $f_k \to 0$ locally uniformly on D.*

The above result is proved for $\gamma_n = 8\pi/3$ in [8, Theorem 5.1.1]. In [12] γ_n in Theorem 3.4 is shown to be optimal.

Before proving Theorem 1, we recall a monotonicity formula for proper $W^{2,2}$ conformal immersions $f : (\Sigma, h) \to \mathbb{R}^n$ (for more details, see [15, 31]). Since f is locally Lipschitz, the measure $\mu = f(\mu_h)$ is an integral varifold with multiplicity function $\theta^2(\mu, x) = \# f^{-1}\{x\}$ and approximate tangent space $T_x\mu = df_p T_p\Sigma$ a.e. for $x = f(p)$. The immersion f satisfies

$$\int_\Sigma \mathrm{div}_g X \, d\mu_g = -\int_\Sigma \langle X, H \rangle \, d\mu_g \quad \text{for any } X \in W^{1,1}_0(\Sigma, \mathbb{R}^n).$$

For the varifold μ this implies the first variation formula

$$\int \mathrm{div}_\mu \phi \, d\mu = -\int \langle \phi, H_\mu \rangle \, d\mu \quad \text{for } \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n),$$

where the weak mean curvature is given by

$$H_\mu(x) = \begin{cases} \frac{1}{\theta^2(\mu,x)} \sum_{p \in f^{-1}\{x\}} H(p) & \text{if } \theta^2(\mu, x) > 0, \\ 0 & \text{else.} \end{cases}$$

Observing that $H_\mu(x)$ is μ a.e. perpendicular to $T_x\mu$, the monotonicity formula in [31] is valid for integral varifolds [15]: that for $B_\sigma(x_0) \subset B_\varrho(x_0)$ one has

$$g_{x_0}(\varrho) - g_{x_0}(\sigma) = \frac{1}{16\pi} \int_{B_\varrho(x_0) \setminus B_\sigma(x_0)} \left| H_\mu + 4\frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 d\mu,$$

where

$$g_{x_0}(r) = \frac{\mu(B_r(x_0))}{\pi r^2} + \frac{1}{4\pi} W(\mu, B_r(x_0)) + \frac{1}{2\pi r^2} \int_{B_r(x_0)} \langle x - x_0, H_\mu \rangle \, d\mu.$$

When Σ is compact and connected, if we let $\sigma \to +\infty$, and $\varrho \to 0$, then we get the Li-Yau inequality [19]

$$\theta^2(\mu, x_0) \leq \frac{1}{4\pi} W(f). \tag{3.2}$$

If we only let $\varrho \to 0$, then we get

$$\theta^2(\mu, x_0) \leq \frac{\mu(B_\sigma(x_0))}{\pi\sigma^2} + CW(f, B_\sigma(x_0)) + C\left(\frac{\mu(B_\sigma(x_0))}{\pi\sigma^2}\right)^{\frac{1}{2}} W(f, B_\sigma(x_0))^{\frac{1}{2}}. \tag{3.3}$$

Another useful consequence (cf. [31], [15]) is: for a compact surface Σ, it holds

$$\Big(\frac{\mu(f(\Sigma))}{W(f)}\Big)^{\frac{1}{2}} \leq \operatorname{diam} f(\Sigma) \leq C\Big(\mu(f(\Sigma))\, W(f)\Big)^{\frac{1}{2}}. \tag{3.4}$$

Proof of Theorem 1. Consider a branched conformal immersion $f_k \in W^{2,2}_{b,c}(\Sigma, h_k, \mathbb{R}^n)$, where h_k satisfies (1.1). The following equation clearly holds on Σ away from the finitely many branch points; the singularities at the branch points can be removed by using Theorem 3.1 in the isothermal coordinates, thus it holds on entire Σ:

$$\Delta_{h_k} f_k = \frac{1}{2} H_{f_k} |\nabla_{h_k} f_k|^2.$$

By Remark 3.3, the number of branch points and $\|A_{f_k}\|_{L^2}$ are both bounded from above.

By (3.4), $\operatorname{diam} f_k(\Sigma) \leq R$ for some $R > 0$. Then $f_k \in \mathcal{F}^2_{conf}(\Sigma, h_k, R + R_0)$. By Theorems 2.8 and 2.12, f_k converges in the bubble tree sense to a mapping f_0 which is a point or a conformal mapping from a stratified surface into $\mathbb{R}^n$. When f_0 is a constant map, we need to do nothing. Thus, we may assume f_0 is a conformal mapping from a stratified surface Σ_∞ into $\mathbb{R}^n$. On each component Σ^i_∞ of Σ_∞, f_0 is not a point and can be extended to a conformal mapping from $\overline{\Sigma^i_\infty}$, which is a smooth Riemann surface, into $\mathbb{R}^n$.

To finish the proof, we only need to prove that f_0 is also a branched $W^{2,2}$ conformal immersion of $\overline{\Sigma^i_\infty}$ in $\mathbb{R}^n$. Recalling that locally f_0 is the weakly $W^{2,p}$ limit of a sequence in $\mathcal{F}^2_{conf}(D, R + R_0)$ for some $p \in (1, \frac{4}{3})$, we only need to prove the following: For branched conformal immersions f'_k from D into $\mathbb{R}^n$ with uniform upper bounds on the number of branch points and $\mu(f_k) + \|A_{f'_k}\|_{L^2(D)}$, if the weak limit of f'_0 in $W^{2,p}_{loc}(D \setminus \mathcal{C}(\{f_k\}))$ is not a point, then f'_0 is a branched conformal immersion, and f'_k converges weakly in $W^{2,2}$ on D minus a finite set.

Let P be the limit set of the branch points of f'_k, and let

$$\mathcal{S}(\{f'_k\}) = \{z \in D : \lim_{r \to 0} \varliminf_{k \to +\infty} \int_{D_r(z)} |A_k|^2 \geq \hat{\epsilon}^2\},$$

where $0 < \hat{\epsilon} \leq \min\{\sqrt{4\pi}, 4\epsilon_0\}$. Using the arguments we get $\mathcal{C}(\{f_k\})$ is a finite set , we can prove that, after passing to a subsequence, $\mathcal{S}(\{f'_k\})$ is also a finite set. By Theorem 3.4, after passing a subsequence, f'_k will converge weakly in $W^{2,2}_{loc}(D \backslash (\mathcal{S} \cup P))$ to a conformal immersion of $D \setminus (\mathcal{S} \cup P)$ in $\mathbb{R}^n$. By Theorem 3.1, the limit can be extended across the finite set $\mathcal{S} \cup P$ to a branched conformal immersion of D in $\mathbb{R}^n$. $\square$

4. Willmore functional for surfaces in compact manifolds

Let N be a compact Riemannian manifold without boundary. We embed N into $\mathbb{R}^n$ isometrically so that any immersion of Σ in N can be regarded as an immersion in $\mathbb{R}^n$. Let $A_{\Sigma,N}, A_{\Sigma,\mathbb{R}^n}$ and $A_{N,\mathbb{R}^n}$ be the second fundamental forms of Σ in N, in $\mathbb{R}^n$ and N in $\mathbb{R}^n$ respectively. The L^2 integrals of these quantities can be related as in the following simple lemma, whose proof can be deduced from the compactness of N and Remark 3.3.

Lemma 4.1. *For any $f \in W^{2,2}_{b,c}(\Sigma, h, N)$, and $i : \Sigma \to \mathbb{R}^n$, which is isometric embedding, we have*

$$\int_\Sigma |H_{i\circ f,\Sigma,\mathbb{R}^n}|^2 d\mu_{i\circ f} \leq C\mu(f) + \int_\Sigma |H_{f,\Sigma,N}|^2 d\mu_f, \tag{4.1}$$

and

$$\int_\Sigma |A_{i\circ f,\Sigma,\mathbb{R}^n}|^2 d\mu_f \leq C \int_\Sigma (1 + |H_{f,\Sigma,N}|^2) d\mu_f + C', \tag{4.2}$$

where C only depends on N and C' only depends on the Euler characteristic of Σ.

4.1. Willmore sphere passing through fixed points.

In this subsection, we let

$$W_n(f) = \int_{S^2} \left(1 + \frac{1}{4}|H_f|^2\right) d\mu_f$$

where f is a $W^{2,2}$ conformal immersion of S^2 in the round unit sphere $\mathbb{S}^n$ for some $n > 2$. It is known that $W_n(f)$ corresponds to the Willmore functional in $\mathbb{R}^n$ under the stereographic projection.

We consider the existence of minimizers of

$$\beta^n_0(y_1, \ldots, y_m) = \inf\{W_n(f) : f \in W^{2,2}_{conf}(S^2, \mathbb{S}^n), \quad y_1, \ldots, y_m \in f(S^2)\}$$

where $y_1, \ldots, y_m$ are fixed distinct points in $\mathbb{S}^n$. When $m \geq 2$, $\beta^n_0(y_1, \ldots, y_m)$ is positive by the conformality of the functions f.

Proposition 4.2. *Let $m \geq 2$. If $\beta^n_0(y_1, \ldots, y_m)$ is less than 8π, then $\beta^n_0(y_1, \ldots, y_m)$ is attained by a $W^{2,2}$-conformally embedded S^2 in $\mathbb{S}^n$.*

Proof. Let $\{f_k\}$ be a minimizing sequence of $\beta^n_0(y_1, \ldots, y_m)$. We can consider f_k as conformal map from S^2 into $\mathbb{R}^{n+1}$. By Theorem 1, f_k (pass to a subsequence if necessary) will converge to a mapping f_0 which is a $W^{2,2}$ branched conformal immersion from a stratified sphere Σ_∞ into $\mathbb{S}^n$ with

$$y_1, \ldots, y_m \in f_0(\Sigma_\infty), \quad W_n(f_0) \leq \beta^n_0(y_1, \ldots, y_m) < 8\pi.$$

Composing with a stereographic projection Π from $\mathbb{S}^n$ minus a point not on $f_0(\Sigma_\infty)$ into $\mathbb{R}^n$, we see $W_n(f_0) = W(\Pi \circ f_0)$ and $\theta^2_{f_0(p)} = \theta^2_{\Pi\circ f_0(p)}$. Now, by (3.2) we have

$$\theta^2_{f_0(p)} \leq \frac{1}{4\pi} W_n(f_0).$$

By Theorem 3.1

$$\lambda(p) + 1 = \theta^2_{f_0(p)} \leq \frac{1}{4\pi} W_n(f_0) < 2$$

thus $\lambda(p) = 0$ which means f_0 has no branched points. Moreover, that the area density of Σ_∞ is one everywhere implies that Σ_∞ has only 1 component and f_0 has no intersection points. Thus $\Sigma_\infty = S^2$, and f_0 is an (Lipschitz) embedding. $\square$

Corollary 4.3. *For any $\epsilon > 0$, there is a Willmore sphere $f : S^2 \to \mathbb{S}^n$ with $W_n(f) < 4\pi + \epsilon$, which has at least 2 nonremovable singular points.*

Proof. Take five distinct points $y_1, \dots, y_5 \in \mathbb{S}^n$, such that there is no round 2-sphere passing through all of them. Recall the Willmore functional W_n of a round 2-sphere is 4π. We can choose the five points to be very closed to a round 2-sphere, such that there is a 2-sphere Σ which is not round and contains $y_1, \dots, y_5$ with

$$W_n(\Sigma) < 4\pi + \epsilon.$$

Then we can find a $W^{2,2}$ conformal embedding $f : S^2 \to \mathbb{S}^n$, such that $f(S^2)$ passes through $y_1, \dots, y_5$, and attains $\beta_0^n(y_1, \dots, y_5)$, by Proposition 4.2.

Choose a point $P \in \mathbb{S}^n \backslash \Sigma$ as the north pole. Let Π be the stereographic projection from $\mathbb{S}^n \backslash \{P\}$ to $\mathbb{R}^n$, and denote $\widetilde{y}_i = \Pi(y_i)$ and $\widetilde{f} = \Pi(f)$. By the conformal invariance of the Willmore functional, we have

$$W_n(f) = \frac{1}{4}\int_{S^2} |H_{\widetilde{f}}|^2 d\mu_{\widetilde{f}}.$$

Then $\widetilde{f}$ attains

$$\inf\left\{\frac{1}{4}\int_{S^2} |H_\varphi|^2 d\mu_\varphi : \varphi \in W^{2,2}_{conf}(S^2, \mathbb{R}^n), \quad \widetilde{y}_1, \dots, \widetilde{y}_5 \in \varphi(S^2)\right\}.$$

Then by results in [27], $\widetilde{f}(S^2)$ is smooth on $\widetilde{f}(S^2)\backslash\{\widetilde{y}_1, \dots, \widetilde{y}_5\}$. However, the Gap Lemma in [14, Theorem 2.7] tells us that there is an $\epsilon > 0$, such that any closed smooth Willmore sphere with Willmore functional $< 4\pi + \epsilon$ is a round sphere. Therefore, at least one of $\widetilde{y}_1, \dots, \widetilde{y}_5$ is a nonremovable singular point. However, a Willmore sphere cannot have only one singular point, by Lemma 4.2 in [15] (which is true in $\mathbb{R}^n$), therefore $\widetilde{f}$ has at least 2 singular points. $\square$

4.2. **Minimizing Willmore functional subject to area constraint.** In this subsection, N stands for a compact submanifold of $\mathbb{R}^n$ with induced metric. We say $f \in W^{2,2}_{conf}(\Sigma, h, N)$ if $f \in W^{2,2}_{conf}(\Sigma, h, \mathbb{R}^n)$ and $f(\Sigma) \subset N$. For $f \in W^{2,2}_{conf}(\Sigma, h, N)$, we define

$$W(f) = W(f, \Sigma, N) = \frac{1}{4}\int_\Sigma |H_{f,\Sigma,N}|^2 d\mu_f.$$

First, we consider the case of genus zero. Set

$$\beta_0(N, a) = \inf\{W(f) : \mu(f) = a, \quad f \in W^{2,2}_{conf}(S^2, N)\}.$$

Proposition 4.4. *We have*

$$\lim_{a\to 0} \beta_0(N, a) = 4\pi.$$

Moreover, when a is sufficiently small, there is an embedding $f \in W^{2,2}_{conf}(S^2, N)$, such that

$$\mu(f) = a, \quad \textit{and} \quad W(f) = \beta_0(N, a).$$

Proof. First, we show that

$$\limsup_{a\to 0} \beta_0(N,a) \le 4\pi. \tag{4.3}$$

Take a point $p \in N$ and a normal coordinate neighborhood U around p. Let

$$S_r = \{(x^1, x^2, x^3, 0, \dots, 0) \in T_pN : (x^1)^2 + (x^2)^2 + (x^3)^2 = r^2\}.$$

It is easy to check that

$$\lim_{r\to 0} W(\exp_p(S_r), N) = 4\pi.$$

For any a which is sufficiently small, we can find $r = r(a)$ such that $\mu(\exp_p(S_r)) = a$ and $r \to 0$ as $a \to 0$. Then (4.3) follows from $\beta_0(N,a) \le W(\exp_p(S_r))$.

Next, we prove that $\beta_0(N,a)$ can be attained by an embedded 2-sphere. Let $f_k \in W^{2,2}_{conf}(S^2, N)$ be a minimizing sequence of $\beta_0(N,a)$. By Lemma 4.1 and (4.3), when a is sufficiently small and k is sufficiently large

$$W(f_k, S^2, \mathbb{R}^n) \le W(f_k, S^2, N) + C\mu(f_k) < 4\pi + \epsilon(a,k) + Ca$$

where $\epsilon(a,k) \to 0$ as $a \to 0$ and $k \to \infty$. By Theorem 1, $\{f_k\}$ has a limit f_0, which is a branched conformal immersion from a stratified sphere S into N with

$$\mu(f_0) = a \quad \text{and} \quad W(f_0) \le \beta_0(N,a).$$

Then by (3.2), for any $p \in S$ it holds

$$\theta^2(f_0(p)) < 2.$$

Thus S is a 2-sphere and f_0 has no branch points and no self-intersection points. Hence f_0 is an embedding. Therefore f_0 is a minimizer for $\beta_0(N,a)$:

$$W(f_0) = \beta_0(N,a).$$

Finally, we prove

$$\varliminf_{a\to 0} \beta_0(N,a) \ge 4\pi.$$

By Lemma 4.1,

$$W(f_0, S^2, \mathbb{R}^n) \le W(f_0, S^2, N) + Ca.$$

It is well-known that $W(f_0, S^2, \mathbb{R}^n) \ge 4\pi$, which completes the proof. □

We now consider the case of genus larger than 0. Recall a result of Schoen-Yau [30] and Sacks-Uhlenbeck [29]: If $\varphi : \Sigma \to N$ induces an injection from the fundamental group of Σ to that of N, then there is a branched minimal immersion $f : \Sigma \to N$ so that f induces the same action on the fundamental groups as φ and f has least area among all such maps. If $\pi_2(N) = 0$ then f is minimizing in its homotopy class. We denote the area of the branched minimal immersion f_φ by a_φ.

Let $g > 0$ be the genus of a compact Riemann surface Σ and let $\phi : \Sigma \to N$ be a continuous map. Define

$$\beta_g(N,a,\phi) = \inf\left\{W(f) : f \in \widetilde{W}^{2,2}(\Sigma, N),\ \mu(f) = a,\ f \sim \phi\right\},$$

where $f \sim \phi$ means that f is homotopic to ϕ.

Proposition 4.5. *Let Σ be a closed Riemann surface with genus $g > 0$ and let N be a compact Riemannian manifold with $\pi_2(N) = 0$. Let $\varphi : \Sigma \to N$ be a map which induces an injective $\varphi_\# : \pi_1(\Sigma) \to \pi_1(N)$. Then we can find an $\delta > 0$, such that for any $a \in [a_\varphi, a_\varphi + \delta)$, there is a branched conformal immersion f_0 of a smooth Riemann surface (Σ, h) of genus g in $\mathbb{R}^n$, such that $\mu(f_0) = a$ and $W(f_0) = \beta_g(N, a, \varphi)$ and f_0 is homotopic to φ. Moreover, when $\dim N = 3$, we can choose δ to be small such that f_0 is an immersion.*

Proof. The proof will be divided into several steps.

Step 1. We prove that $\lim_{a\to a_0} \beta_g(N, a, \varphi) = 0$.

Let $F \in C^\infty(\Sigma \times [0, 1], \mathbb{R}^n)$, such that $F(\cdot, t)$ is an immersion for each t and

$$F(\cdot, 0) = f_\varphi, \quad \mu(F(\cdot, 1)) \geq a_\varphi.$$

As $F(\cdot, t) \sim \varphi$ and f_φ is a minimal surface,

$$\lim_{a\to a_\varphi} \beta_p(N, a, \varphi) \leq \lim_{t\to 0} W(F(\cdot, t)) = W(f_\varphi) = 0.$$

Step 2. Smooth convergence of conformal structures.

We take a minimizing sequence $\{f_k\}$ of $\beta_g(N, a, f)$. Recall that f_k are $W^{2,2}$ branched conformal immersions from (Σ, h_k) into $\mathbb{R}^n$, where h_k are the smooth metrics with curvature 0 or -1. Because $\pi_2(N) = 0$ and $f_k \sim \varphi$ for each k, f_k induces the same injective action on the fundamental groups as φ does; hence the conformal structures of h_k stay in a compact set of the moduli space for both the cases $g > 1$ and $g = 1$, therefore, after passing to a subsequence if necessary, $\Sigma_k = (\Sigma, h_k)$ converges to a Riemann surface (Σ, h_0) in $\mathcal{M}_g$ (cf. [30]). The results in [30] applies as f_k belong to $W^{1,2} \cap C^0$.

Step 3. We prove that $\{f_k\}$ has no bubbles, i.e. the limit f_0 is a map defined on Σ.

By Remark 2.11, f_0 is defined on $\Sigma_\infty = \Sigma_0 \cup S_1 \cup S_2 \cdots \cup S_m$, where S_i are all 2-spheres and Σ_0 is a smooth surface of genus g. We prove $m = 0$. Assume $m \geq 1$. By Theorem 1, $\mu(f_0) = a$ and $W(f_0) \leq \beta_p(N, a, \varphi)$. Further, $f_k(\Sigma)$ converges to $f_0(\Sigma_\infty)$ in Hausdorff distance and $f_0|_{S_j}$ is homotopic to a constant map for each $j = 1, \ldots, m$ as $\pi_2(N) = 0$. We conclude that $f_0|_{\Sigma_0}$ is homotopic to φ. Consequently, $\mu(f_0(\Sigma_0)) \geq a_\varphi$. Then we get

$$\mu(f_0, S_i) \leq \mu(\Sigma) - \mu(\Sigma_0) \leq a - a_\varphi \quad \text{and} \quad W(f_0, S_i, N) \leq \beta_g(N, a, \varphi).$$

By Lemma 4.1 and Step 1,

$$W(f_0, S_i, \mathbb{R}^n) \leq C(a - a_\varphi) + \beta_g(N, a, \varphi) \to 0 \quad \text{as} \quad a \to a_\varphi.$$

This, however, contradicts Proposition 2.2 when the Willmore functional of S_i goes below the gap constant.

Step 4. We consider the case of $\dim N = 3$.

We will use the result that there are no branch points for minimal surfaces [6, 23] to prove that f_0 has no branch points when δ is sufficiently small.

If the claimed result is not true, then there is a sequence of numbers $a_k > a_\varphi$ with $a_k \to a_\varphi$ and a sequence of $W^{2,2}$ branched conformal immersions $f_{0,k}$ of (Σ, h_k) in N with $\mu(f_{0,k}) = a_k$, $W(f_{0,k}, \Sigma, N) = \beta_g(N, a_k, \varphi)$ by the first part of the proposition, and each $f_{0,k}$ has at least a branch point p_k. By Step 1, $W(f_{0,k}, \Sigma, N) \to 0$.

As in Step 2, (Σ, h_k) converges to a smooth surface (Σ, h_0) in $\mathcal{M}_g$. For simplicity, we will still denote $f_{0,k} \circ \psi_k$ (see Remark 2.11) by $f_{0,k}$ which is a branched conformal immersion from $(\Sigma, \psi_k^*(h_k))$ into $\mathbb{R}^n$. By Theorem 2.8, we may set $f_{0,0}$ to be the limit of

$f_{0,k}$ with $\mu(f_{0,0}) = a$ and $W(f_{0,0}) = 0$. Arguing as in Step 3, $\{f_{0,k}\}$ has no bubbles, and $f_{0,0} \in W^{2,2}_{b,c}(\Sigma, h_0, \mathbb{R}^n)$ for some smooth h_0. Moreover, $f_{0,0}$ is a minimal surface in N. By the results of Gulliver and Osserman, $f_{0,0}$ is a smooth immersion of Σ in N.

Since p_k is a branch point, by Theorem 3.1, the area density

$$\theta^2_{f_{0,k}(p_k)}(f_{0,k}(U)) \geq 2,$$

where U is a neighborhood of $p_k \to p$ in Σ for sufficiently large k. As $f_{0,0}$ is immersive, we can take U small so that $f_{0,0}$ is an embedding on U and $\mu(f_{0,0}(U)) < \epsilon'$. Further, by the monotonicity formula for minimal surfaces, for small r and geodesic balls $B^N_r(f_{0,0}(p))$ in N, it holds

$$\mu(f_{0,0}(U) \cap B^N_r(f_{0,0}(p))) \leq (1+\epsilon')\pi r^2.$$

From the expansion of metric in normal coordinates, for small r and the Euclidean ball $B_r(f_{0,0}(p))$ in $\mathbb{R}^n$ we have

$$\mu(f_{0,0}(U) \cap B_r(f_{0,0}(p))) \leq \mu(f_{0,0}(U) \cap B^N_{r+cr^2}(f_{0,0}(p))) \leq (1+\epsilon')\pi r^2 + O(r^3)$$

where c depends on N.

In light of Lemma 4.1, $W(f_{0,k}, U, \mathbb{R}^n) < \epsilon_0^2$ if we choose ϵ' to be very small and k large enough. Then $\{f_{0,k}\}$ has no blow-up points in U by the ϵ-regularity. Then we have

$$\mu(f_{0,k}(U) \cap B_r(f_{0,k}(p_k))) \to \mu(f_{0,0}(U) \cap B_r(f_{0,0}(p))) \quad \text{as } k \to \infty.$$

By Lemma 4.1,

$$W(f_{0,k}, U, \mathbb{R}^n) \leq C\epsilon' + W(f_{0,k}, U, N).$$

Then by (3.3),

$$\theta^2_{f_{0,k}(p)}(f_{0,k}(U)) \quad \leq \quad \frac{\mu(f_{0,k}(U) \cap B_r(f_{0,k}(p_k)))}{\pi r^2} + W(f_{0,k}, U, \mathbb{R}^n) + CW(f_{0,k}, U, \mathbb{R}^n)^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned} 2 \quad &\leq \quad \lim_{U \to p} \lim_{k \to \infty} \left(\frac{\mu(f_{0,k}(U) \cap B_r(f_{0,k}(p_k)))}{\pi r^2} + W(f_{0,k}, U, \mathbb{R}^n) + CW(f_{0,k}, U, \mathbb{R}^n)^{\frac{1}{2}} \right) \\ &\leq \quad 1 + \epsilon'. \end{aligned}$$

This is impossible for ϵ' small. □

4.3. **Minimizing Willmore functional of surfaces with a Douglas type condition.** In this subsection, we consider a sufficient condition of Douglas type as in the minimal surface theory for existence of minimizers of the Willmore functional.

First, we assume N to be a compact Riemannian manifold with negative sectional curvatures. In negatively curved N, surface area is bounded by the Willmore functional and the genus of the surface.

Lemma 4.6. *Let N be a compact Riemannian manifold with $K \leq -c < 0$. Then for any $f \in \widetilde{W}^{2,2}(\Sigma, N)$,*

$$\mu(f) \leq c^{-1}\left(W(f, \Sigma, N) - 2\pi\chi(\Sigma)\right).$$

Especially, when $g(\Sigma) = 0$ or 1,

$$\mu(f) \leq c^{-1} W(f, \Sigma, N).$$

Proof. From the Gauss equation:

$$R^{\Sigma}(X,Y,X,Y) = R^N(X,Y,X,Y) + \langle A(X,X), A(Y,Y)\rangle - \langle A(X,Y), A(X,Y)\rangle.$$

we have

$$K_\Sigma \leq K_{f_*(T\Sigma)} + \frac{1}{4}|H_{f,\Sigma,N}|^2.$$

Then from the generalized Gauss-Bonnet formula - Lemma 3.2, we have

$$2\pi\chi(\Sigma) + 2\pi b \leq -c\mu_f(\Sigma) + W(f,\Sigma,N)$$

where b is the number of branch points, in turn

$$c\mu_f(\Sigma) \leq W(f,\Sigma,N) - 2\pi\chi(\Sigma).$$

When $g(\Sigma) \leq 1$ the Euler number $\chi(\Sigma)$ is nonnegative, in this case

$$c\mu_f(\Sigma) \leq W(f,\Sigma,N).$$

Dividing by c yields the desired area bounds. □

Recall that any connected stratified surface Σ can be written as union of finitely many connected 2-dimensional components: $\Sigma = \bigcup_i \Sigma_i$. Denote the genus of Σ and Σ_i by $g(\Sigma)$ and $g(\Sigma_i)$, accordingly. We introduce a subset $S(g)$ of all stratified surfaces as follows.

(1) If $g > 0$, $S(g) = \{\Sigma : \Sigma = \bigcup_i \Sigma_i \text{ with } g(\Sigma_i) < g \text{ for all } i\}$.
(2) If $g = 0$, $S(0) = \{\Sigma : \Sigma = \bigcup_i \Sigma_i \text{ with } g(\Sigma) = 0 \text{ and } i \geq 2\}$.

Note that any $\Sigma \in S(g)$ with $g(\Sigma) = g$ must be singular, in the sense that it has more than one components. Especially, $S(g) \cap \mathcal{M}_g = \emptyset$. However, when $g \geq 1$, $S(g)$ contains smooth surfaces of genus $\leq g-1$.

Define

$$\begin{aligned}\alpha^*(g) &= \inf\{W(f,\Sigma,\mathbb{R}^n) : f \in W^{2,2}_{b,c}(\Sigma,\mathbb{R}^n), f(\Sigma) \subset N, \Sigma \in S(g)\}\\ \alpha(g) &= \inf\{W(f,\Sigma,\mathbb{R}^n) : f \in W^{2,2}_{b,c}(\Sigma,\mathbb{R}^n), f(\Sigma) \subset N, \Sigma \in \mathcal{M}_g\}.\end{aligned}$$

We now state a sufficient condition, similar to the Douglas condition for minimal surfaces, for existence of minimizers for the Willmore functional.

Proposition 4.7. *Let N be a compact Riemannian manifold with negative sectional curvatures. If $0 < \alpha(g) < \alpha^*(g)$, then there is a $W^{2,2}$ branched conformal immersion f from a smooth surface of genus g into N which minimizes the Willmore functional among all such maps.*

Proof. Let $f_k : (\Sigma, h_k) \to N \hookrightarrow \mathbb{R}^n$ be a minimizing sequence of $\alpha(g)$. By Lemma 4.6, the areas $\mu(f_k(\Sigma))$ are uniformly bounded. Since $S^2 \in S(g)$ for any $g \geq 0$, $\alpha^*(g) \leq 4\pi$ by Proposition 4.4, hence by assumption $\alpha(g) < \alpha^*(g) \leq 4\pi$. The sequence $\{f_k\}$ cannot converge to a point since otherwise the images $f_k(\Sigma)$ would lie in a coordinate chart of the point in N and then for any $\epsilon > 0$, $W(f_k) \geq 4\pi - \epsilon$ for large k, in turn $\alpha(g) \geq 4\pi$ as $\{f_k\}$ is a minimizing sequence of $\alpha(g)$. Then from Theorem 1, there exists a subsequence of $\{f_k\}$, still denoted by $\{f_k\}$, a limit map $f_0 \in W^{2,2}_{b,c}(\Sigma_\infty, \mathbb{R}^n)$ from a stratified Riemann surface Σ_∞ with $g(\Sigma_\infty) \leq g$ into $N \hookrightarrow \mathbb{R}^n$, and

$$W(f_0, \Sigma_\infty, \mathbb{R}^n) \leq \lim_{k\to\infty} W(f_k, \Sigma, \mathbb{R}^n) = \alpha(g).$$

We write $\Sigma_\infty = \bigcup_{i=1}^m \Sigma_i$. If $g(\Sigma_\infty) = g$, we consider two cases. Case 1: $g(\Sigma_i) = g$ for some $i = 1, ..., m$. In this case,

$$W(f_0|_{\Sigma_1}, \Sigma_1, \mathbb{R}^n) \leq W(f_0, \Sigma_\infty, \mathbb{R}^n) = \alpha(g).$$

So $f_0(\Sigma_i)$ is a smooth genus g surface attains $\alpha(g)$. Case 2: $g(\Sigma_i) < g$ for all $i = 1, ..., m$. Thus $\Sigma_\infty \in S(g)$, and in turn

$$\alpha^*(g) \leq W(f_0, \Sigma_\infty, \mathbb{R}^n) \leq \alpha(g) < \alpha^*(g).$$

This contradiction rules out Case 2. If $g(\Sigma_\infty) < g$ then $\Sigma_g \in S(g)$. Therefore

$$\alpha^*(g) \leq W(f_0, \Sigma_\infty, \mathbb{R}^n) \leq \alpha(g) < \alpha^*(g)$$

and this is impossible. □

Instead of the curvature assumption on N, we set, for $0 < a < \infty$,

$$\begin{aligned} \gamma^*(g,a) &= \inf\{W(f,\Sigma,\mathbb{R}^n) : f \in W^{2,2}_{b,c}(\Sigma,\mathbb{R}^n), f(\Sigma) \subset N, \Sigma \in S(g), \mu(f(\Sigma)) \leq a\} \\ \gamma(g,a) &= \inf\{W(f,\Sigma,\mathbb{R}^n) : f \in W^{2,2}_{b,c}(\Sigma,\mathbb{R}^n), f(\Sigma) \subset N, \Sigma \in \mathcal{M}_g, \mu(f(\Sigma)) \leq a\}. \end{aligned}$$

Since there is no loss of measure in the limit process, as asserted in Theorem 1, the same proof above allows us to conclude

Proposition 4.8. *Let N be a compact Riemannian manifold. If $0 < \gamma(g,a) < \gamma^*(g,a)$, then there is a $W^{2,2}$ branched conformal immersion f from a smooth surface of genus g into N which minimizes the Willmore functional among all such maps.*

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