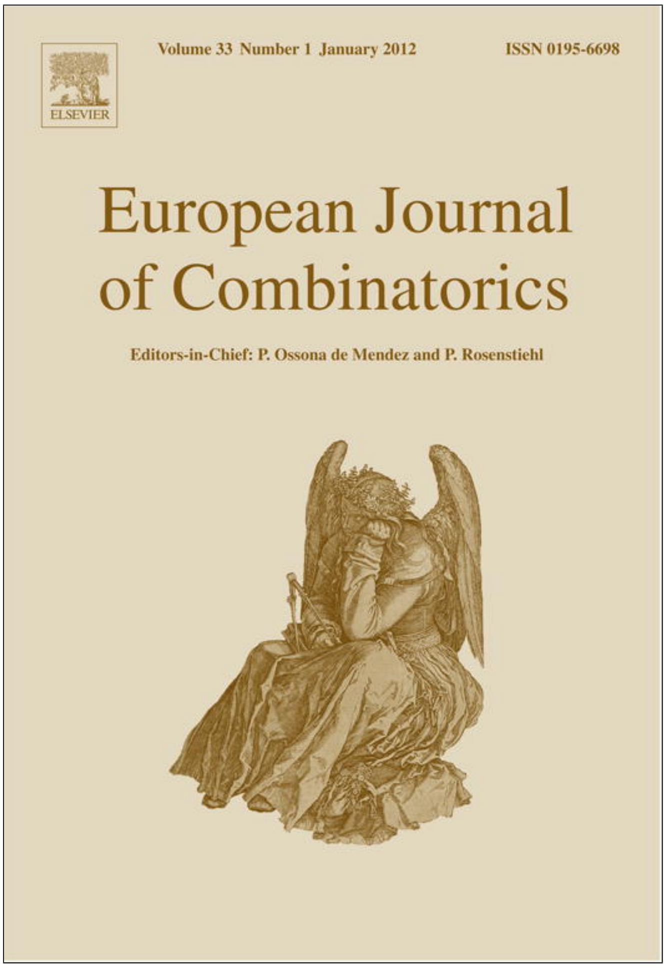
ELSEVIER
Volume 33 Number 1 January 2012
ISSN 0195-6698
European Journal of Combinatorics
Editors-in-Chief: P. Ossona de Mendez and P. Rosenstiehl

European Journal of Combinatorics 33 (2012) 1765–1776

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Spanning cycles in regular matroids without small cocircuits

Ping Li [a], Hong-Jian Lai [c,b,1], Yehong Shao [d], Mingquan Zhan [e]

[a] *Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China*
[b] *Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA*
[c] *College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China*
[d] *College of Arts and Sciences, Ohio University Southern, OH 45638, USA*
[e] *Department of Mathematics, Millersville University, Millersville, PA 17551, USA*

ARTICLE INFO

Article history:
Received 24 May 2011
Received in revised form
24 March 2012
Accepted 24 March 2012

ABSTRACT

A cycle of a matroid is a disjoint union of circuits. A cycle C of a matroid M is spanning if the rank of C equals the rank of M. Settling an open problem of Bauer in 1985, Catlin in [P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–44] showed that if G is a 2-connected graph on $n > 16$ vertices, and if $\delta(G) > \frac{n}{5} - 1$, then G has a spanning cycle. Catlin also showed that the lower bound of the minimum degree in this result is best possible. In this paper, we prove that for a connected simple regular matroid M, if for any cocircuit D, $|D| \geq \max\left\{\frac{r(M)-4}{5}, 6\right\}$, then M has a spanning cycle.

1. Introduction

Graphs and matroids in this note are finite and loopless. Undefined terms and notations can be found in [3] for graphs and in [16] for matroids. To be consistent with the matroid terminology, a nontrivial 2-regular connected graph will be called a *circuit*, and an edge disjoint union of circuits a *cycle*. A cycle C in a graph G is a *spanning cycle* if C contains a spanning tree of G. Graphs with a spanning cycle are also known as *supereulerian graphs*. The supereulerian graph problem, raised by Boesch et al. [2], seeks to characterize supereulerian graphs. Pulleyblank [17] showed that determining

E-mail address: hjlai@math.wvu.edu (H.-J. Lai).
[1] Tel.: +1 304 293 2011; fax: +1 304 293 3982.

0195-6698/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2012.03.033

if a graph is supereulerian, even when restricted to planar graphs, is NP-complete. For more on the literature on supereulerian graphs, see Catlin's survey [5] and its update by Chen and Lai [8].

For a matroid M on a set E, r_M, $\mathcal{B}(M)$ and $\mathcal{C}(M)$ denote the rank function of M, the collections of bases and circuits of M, respectively. As in [16], if $X \subseteq E$, then M/X and $M|X$ denote the matroid contractions and matroid restrictions, respectively. A *cycle* of M is a disjoint union of circuits in M, and $\mathcal{C}_0(M)$ denotes the set of all cycles of M. A cycle $C \in \mathcal{C}_0(M)$ is a *spanning cycle* if $r_M(C) = r_M(E)$.

Let G be a connected graph. For an edge subset $X \in E(G)$, we shall adopt the convention to use X to mean both the edge subset X as well as the subgraph induced by X. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of vertices that are adjacent to v in G, $N_G[v] = N_G(v) \cup \{v\}$, and $E_G(v)$ be the set of edges incident with v in G. As in [3], $d_G(v)$ denote the degree of v in G. If $X \subseteq V(G) \cup E(G)$, then $G - X$ is the subgraph obtained from G by deleting the elements in X from G. For $V_1, V_2 \subseteq V(G)$ with $V_1 \cap V_2 = \emptyset$, let $[V_1, V_2]_G = \{e = uv \in E(G) | u \in V_1, v \in V_2\}$. The subscript G will be omitted when it is understood from the context. For a matroid M, the *girth* of M is

$$g(M) = \begin{cases} \min\{k: M \text{ has a circuit } C \text{ with } |C| = k\}: & \text{if M has a circuit} \\ \infty: & \text{if } M \text{ has no circuits.} \end{cases}$$

The girth of a graph G is $g(G) = g(M(G))$. We also denote $g(M^*)$ by $g^*(M)$, called the *cogirth* of M.

Settling an open problem of Bauer [1], Catlin proved the following.

Theorem 1.1 (*Catlin, Theorem 9 of [4]*). *Let G be a 2-edge-connected simple graph G on $n > 16$ vertices. If $\delta(G) > \frac{n}{5} - 1$, then G has a spanning cycle.*

Catlin's result is best possible (see [4]) in the sense that there exists an infinite family of simple graphs G_n on n vertices, such that $\delta(G_n) = \frac{n}{5} - 1$ but each G_n does not has a spanning cycle. It is natural (as seen in [11]) to replace the minimum degree of a graph by the cogirth of a matroid when one tries to extend such a graphical result to its matroidal version. However, the cogirth of the cycle matroid $M(G)$ of a connected graph G equals to the edge-connectivity of G. Jaeger [12] and Catlin [4] independently proved the following theorem.

Theorem 1.2. *Let G be a 4-edge-connected graph.*

(i) *(Jaeger [12] and Catlin [4]) $M(G)$, the cycle matroid of G, has a spanning cycle.*
(ii) *(Catlin [4]) For any graph G' that contains G as a subgraph, $M(G')$ has a spanning cycle if and only if the contraction $M(G')/E(G)$ has a spanning cycle.*

It has been observed that Theorem 1.2 cannot be extended to regular matroids. In Section 2 of [14], using a result of Erdös in [10], an infinite family of cographic matroids has been found such that matroids in this family can have arbitrarily large cogirth yet none of these matroids will have a spanning cycle. This observation and Theorem 1.1 motivates the current research. The main result of this paper is the following:

Theorem 1.3. *Let M be a simple, connected regular matroid. If*

$$g^*(M) \geq \max\left\{\frac{r(M) - 4}{5}, 6\right\}, \tag{1}$$

then M has a spanning cycle.

We approach the problem by introducing the concept of contractible matroids. A matroid N is contractible if for any matroid M that contains N as a restriction, M has a spanning cycle if and only if the contraction M/N has a spanning cycle. The existence of nonempty contractible restrictions of M allows us to argue by induction. We shall first show that Theorem 1.3 holds if M is graphic or cographic. When M is a 2-sum or a 3-sum of its proper minors, we shall show that M will always have a contractible restriction, and so the proof will be done by induction.

This paper is arranged as follows. In Section 2, we formally define contractible matroids, review Catlin's reduction method to handle the graphic case, as well as Seymour's well known decomposition theorem of regular matroids. In Section 3, we show that Theorem 1.3 holds for cographic matroids. In Section 4, we shall show that when the girth is sufficiently high, cographic matroids will have a contractible restriction, which will serve as a useful step in our inductive argument to prove the main result in the last section.

2. Preliminaries

Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X, and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. Following [16], for a matroid M with a subset $X \subseteq E(M)$, M/X is the matroid obtained by contracting X.

Let $O(G)$ denote the set of all odd degree vertices in G. A graph H is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 (\mathrm{mod}\ 2)$, H has a connected subgraph Γ_R such that $O(\Gamma_R) = R$ and $V(\Gamma_R) = V(G)$. Catlin [4] showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The contraction $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$ is the *reduction* of G. A graph G that does not have a nontrivial collapsible subgraph is *reduced*. We summarize some of the former results below. Part (iv) of Theorem 2.1 below follows from the definition of reduced graphs and from Part (iii).

Theorem 2.1. *Let G be a connected graph, and let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with 2 edge-disjoint spanning trees. Each of the following holds.*

(i) *(Catlin, Theorem 3 of [4]). If H is a collapsible subgraph of G, then G has a spanning cycle if and only if G/H has a spanning cycle.*
(ii) *(Catlin, Theorem 5 of [4]). Any reduction is reduced.*
(iii) *(Catlin et al., Theorem of [7]). If $F(G) \le 2$, then the reduction of G is in $\{K_1, K_2, K_{2,t} : t \ge 1\}$.*
(iv) *If $G \notin \{K_1, K_2\}$ is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2 \ge 2$.*

As in [14], a binary matroid N with $|E(N)| \ge 1$ is *contractible* if for any binary matroid M that contains N as a restriction, it always holds that

$$M \text{ has a spanning cycle} \quad \text{if and only if} \quad M/N \text{ has a spanning cycle.} \tag{2}$$

Let $\tau(M)$ denote the maximum number of disjoint bases of M. If G is a connected graph, then $\tau(G) = \tau(M(G))$. Characterizations of matroids M with $\tau(M) \ge k$ have been obtained by Edmonds [9], extending the graphical results by Nash-Williams [15] and Tutte [20].

Lemma 2.2. *Let N be a binary matroid. Each of the following holds.*

(i) *(Theorem 5.4 of [14]). If $\tau(N) \ge 2$, then N is contractible. In particular, $U_{1,2}$ is contractible.*
(ii) *(Proposition 5.7 of [14]). $U_{2,3}$ is contractible.*

For sets X and Y, the *symmetric difference* of X and Y is defined by $X \Delta Y = (X \cup Y) - (X \cap Y)$.

Definition 2.3. Suppose that M_1, M_2 are binary matroids on E_1 and E_2, respectively. We follow Seymour [18,19] to define the *binary sum* $M_1 \triangle M_2$ to be the matroid on the set $E_1 \triangle E_2$ such that the set of cycles of $M_1 \triangle M_2$ equals $\{C_1 \triangle C_2 \subseteq E_1 \triangle E_2 : C_i \text{ is a cycle of } M_i, i = 1, 2\}$. Three special cases of this operation are introduced by Seymour [18,19] as follows.

(i) If $E_1 \cap E_2 = \emptyset$ and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a 1-*sum* of M_1 and M_2.
(ii) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{z\}$, say, and z is not a loop or coloop of M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a 2-*sum* of M_1 and M_2.
(iii) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, and Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a 3-*sum* of M_1 and M_2.

The following lemma follows from the definitions of matroid sums.

Lemma 2.4. *Suppose that for some $i \in \{2, 3\}$, M is Tutte i-connected and $M = M_1 \oplus_i M_2$. Then*

$$r(M) = r(M_1) + r(M_2) - (i - 1). \tag{3}$$

Proposition 2.5 (*Proposition 5.5 of [14]*). *Let M, M_1 and M_2 be binary matroids such that $M = M_1 \triangle M_2$ with $Z = E(M_1) \cap E(M_2)$ and such that one of the following holds.*

(i) $Z = \{e_0\}$ *and* $M = M_1 \oplus_2 M_2$ *is a 2-sum, or*
(ii) $Z = \{e_1, e_2, e_3\}$ *and* $M = M_1 \oplus_3 M_2$ *is a 3-sum, or*
(iii) $Z = \{e_1, e_2, e_3\}$ *and* $M^* = M_1^* \oplus_3 M_2^*$ *is a 3-sum.*
(iv) *Suppose that* $M_2 = M(G)$ *is graphic such that* $G - Z$ *contains a nontrivial collapsible subgraph* L. *If* $M/E(L)$ *has a spanning cycle, then* M *also has a spanning cycle.*

Let R_{10} denote the vector matroid of the following matrix over $GF(2)$:

$$R_{10} = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

We make the following observations.

Observation 2.6. *With* $E(R_{10}) = \{e_1, e_2, \ldots, e_{10}\}$ *as above, each of the following holds.*

(i) *(Seymour [18]).* R_{10} *has a doubly transitive automorphism group.*
(ii) R_{10}^* *is isomorphic to* R_{10}.
(iii) $E(R_{10})$ *is a disjoint union of a 4-circuit* $\{e_1, e_2, e_3, e_7\}$ *and a 6-circuit* $\{e_4, e_5, e_6, e_8, e_8, e_{10}\}$.

The next theorem follows immediately from Seymour's decomposition theorem of regular matroids. (For a verification, see the proof for Theorem 4.5 in [14].)

Theorem 2.7 (*Seymour [18]*). *For a connected regular matroid* M, *one of the following must hold.*

(i) M *is graphic, cographic, or* $M \cong R_{10}$.
(ii) M *is 2-connected and* $M = M_1 \oplus_2 M_2$ *is a 2-sum of* M_1 *and* M_2, *such that each of* M_1 *and* M_2 *is isomorphic to a proper minor of* M, *and such that either* M_2 *is isomorphic to* R_{10}, *or* M_2 *is graphic or* M_2 *is cographic.*
(iii) M *is 3-connected and* $M = M_1 \oplus_3 M_2$ *is a nontrivial 3-sum of* M_1 *and* M_2, *such that each of* M_1 *and* M_2 *is isomorphic to a proper minor of* M, *and such that either* M_2 *is graphic or* M_2 *is cographic.*

Lemma 2.8. *Suppose that* M *is Tutte* i*-connected and* M *is an* i*-sum for some* $i \in \{2, 3\}$ *with one of the summand being isomorphic to* R_{10}, *or graphic or cographic. Then we can choose* M_1 *and* M_2 *with* $M = M_1 \oplus_i M_2$ *such that* M_2 *is isomorphic to* R_{10}, *or graphic or cographic and such that*

$$r(M_2) \le (r(M) - i + 1)/2, \quad \text{or} \quad \text{equivalently}, r(M) \ge 2r(M_2) + i - 1. \tag{4}$$

Proof. We assume that this lemma holds for matroids M with smaller value of $|E(M)|$. By Theorem 2.7, we can choose M_1 and M_2 with $M = M_1 \oplus_i M_2$ such that M_2 is isomorphic to R_{10}, or is graphic or cographic and such that subject to being isomorphic to R_{10}, or being graphic or cographic, $r(M_2)$ is minimized. Suppose that $r(M_1) < r(M_2)$. If M_1 is isomorphic to R_{10}, or is graphic or cographic, then the choice of M_2 is violated. Hence M_1 is also an i-sum of its proper minors, and so by induction, $M_1 = M_{11} \oplus_i M_{12}$ such that M_{12} is R_{10}, or graphic or cographic, and such that $r(M_{12}) \le r(M_{11}) < r(M_1) < r(M_2)$, contrary to the choice of M_2. Hence we may assume that $r(M_2) \le r(M_1)$, and so (4) follows from (3). $\square$

3. Spanning cycles in cographic matroids with large cogirths

In this section, we shall show that Theorem 1.3 holds for cographic matroids. We need a few more notations and former results. The *vertex arboricity* of a graph G, denoted by $a(G)$, is the minimum number of sets in a partition of $V(G)$ such that each set induces an acyclic graph. The theorem below will be useful.

Theorem 3.1 (*Kronk and Mitchem, [13]*). *If G is connected, not complete and $a(G) = k \geq 3$, then $\Delta(G) \geq 2k-1$.*

Lemma 3.2. *Let G be a graph and $M = M(G)$. The following are equivalent.*

(i) *G has a cocycle X such that $r^*(X) = r^*(M)$.*
(ii) *$V(G)$ has a partition $\{V_1, V_2\}$ such that both $G[V_1]$ and $G[V_2]$ are forests.*
(iii) *$a(G) \leq 2$.*

Proof. (i) $\Longrightarrow$ (ii). Let $X = [V_1, V_2]_G$ denote a cocycle of G with $r^*(X) = r^*(M)$. Since X is a cospanning, $E(G) - X$ is independent. Thus $G[E(G) - X]$ is a forest in G.

(ii) $\Longrightarrow$ (iii). This follows by the definition of arboricity.

(iii) $\Longrightarrow$ (i). Let V_1, V_2 be the two sets in a partition of $V(G)$ such that $G[V_i]$ is acyclic, for $i \in \{1, 2\}$, and let $X = [V_1, V_2]_G$. Then X is a cocycle. As $E(G) - X$ is independent in $M = M(G)$, X is cospanning in M, and so the cocycle X satisfies $r^*(X) = r^*(M)$. $\square$

Lemma 3.3. *Let G be a connected graph with $\delta(G) \geq 4$. Let $m = |E(G)|$, $n = |V(G)|$ and $d = g(G)$. If $d \geq \max\left\{\frac{m-n-3}{5}, 6\right\}$, then one of the following must hold.*

(i) *G is 4-regular.*
(ii) *$d = 6$ and $a(G) = 2$.*

Proof. Let $t = \Delta(G)$, and let $R = \bigcup_{i\geq 5} D_i(G)$ and $r = |R|$. Counting degrees we have $2m \geq 4n + r$. Since $g(G) = d \geq \frac{m-n-3}{5}$, we have

$$2n \leq 10d - r + 6. \tag{5}$$

In the rest of the proof, we always assume that v is a vertex of G of degree t. We have the following claims.

Claim 1. *If d is odd, then G is 4-regular.*

If not, then $t \geq 5$. Since d is odd, for some integer $s \geq 3$, $d = 2s + 1$. Since v has degree $t \geq 5$, and since $\delta(G) \geq 4$, G has at least $1 + t + 3t + \cdots + 3^{s-1}t = 1 + \frac{t}{2}(3^s - 1)$ vertices of distance at most s from v. Thus by (5), $20s + 14 - r \geq t(3^s - 1)$. As $s \geq 3$, we have $3^s \geq 9s$, and so by $t \geq 5$, $t(3^s - 1) \geq 5(9s - 1) > 20s + 14$, contrary to the fact that $20s + 14 - r \geq t(3^s - 1)$. This proves Claim 1.

Claim 2. *If $d \geq 8$ is even, then G is 4-regular.*

If not, then $t \geq 5$. Since $d \geq 8$ is even, for some $s \geq 4$, $d = 2s$. Let $e = uv$ be an edge incident with v. Since v has degree t, and since $\delta(G) \geq 4$, G has at least $2 + (t + 2) + 3(t + 2) + \cdots + 3^{s-2}(t + 2) = 2 + \frac{t+2}{2}(3^{s-1} - 1)$ vertices of distance at most $s - 1$ from e. Hence by (5) $20s + 2 - r \geq (t + 2)(3^{s-1} - 1)$. As $s \geq 4$, $3^{s-1} \geq 6s$, and so by $t \geq 5$, $20s + 2 - r \geq (t + 2)(3^{s-1} - 1) \geq 7(6s - 1) = 42s - 7$, contrary to the fact that $s \geq 4$. This proves Claim 2.

By Claims 1 and 2, in the rest of the proof, we assume that $s = 3$. Let $e = uv$ be an edge incident with v and define

$$A_1 = N(u) - \{v\}, \qquad B_1 = N(v) - \{u\}, \qquad A_2 = N(A_1) - \{u\}, \quad \text{and} \quad B_2 = N(B_1) - \{v\}.$$

It follows by $g(G) \geq 6$ that

$$\begin{aligned} &\{v\} \cup A_1 \cup B_2, \qquad \{u\} \cup A_2 \cup B_1 \text{ are independent sets} \\ &\text{for } i = 1, 2, A_i \cap (B_1 \cup B_2) = \emptyset \quad \text{and} \quad B_i \cap (A_1 \cup A_2) = \emptyset. \end{aligned} \tag{6}$$

If for some $x \in N(v)$, $d(x) = 5$, then by $g(G) \geq 6$,

$$\begin{aligned} n &= |V(G)| \\ &\geq |\{x, v\}| + |N(x) - \{v\}| + |N(v) - \{x\}| + |N(N(x) - \{v\}) - \{x\}| + |N(N(v) - \{x\}) - \{v\}| \\ &\geq 2 + 8 + 3(8) = 34, \end{aligned}$$

and so by (5), $68 \leq 2n \leq 66 - r$, a contradiction. Hence

$$\forall x \in N(v), \quad d(x) = 4. \tag{7}$$

By (7), $d(u) = 4$, $|A_2| = 12$ and $|B_1| = 3$. Thus $|\{u, v\} \cup A_1 \cup A_2 \cup B_1| = 21$. Let $B_2' = V(G) - (\{u, v\} \cup A_1 \cup A_2 \cup B_1)$. By (5) and by $\delta(G) \geq 4$, $|B_2'| \in \{9, 10, 11\}$. As $|A_2| = 12$ and $\delta(G) \geq 4$, $|N(A_2) \cap B_2'| \geq 12 \cdot 3 = 36$. We have $|N(B_2') \cap A_2| = |N(A_2) \cap B_2'| \geq 36$.

If $|B_2'| = 9$, then $B_2' = B_2$, and so $36 \leq |N(B_2) \cap A_2| \leq 3|B_2| + (r-1) \leq 27 + 5 = 32$, a contradiction. If $|B_2'| = 10$, then $B_2 \subseteq B_2'$, and $|N(B_2') \cap A_2| \leq 3|B_2| + (r - 1) + 4|B_2' - B_2| \leq 27 + 5 + 4 = 36$. Thus $r = 6$ and $n = 31$. By (5), $62 = 2n \leq 66 - r = 60$, a contradiction. It follows that $|B_2'| = 11$ and $n = 21 + 11 = 32$. By (5), $r = 2$. Thus $36 \leq |N(A_2) \cap B_2'| = |N(B_2') \cap A_2| \leq 3|B_2| + (r-1) + 4|B_2' - B_2| = 27 + 1 + 8 = 36$, forcing $|B_2| = 9$.

Let $B_2' - B_2 = \{w_1, w_2\}$. Then $d(w_1) = d(w_2) = 4$, $N(w_1) \subseteq A_2$ and $N(w_2) \subseteq A_2$, and $w_1 w_2 \notin E(G)$. Thus $\{w_1, w_2\} \cup B_2$ is an independent set in G, and B_2 contains exactly one degree five vertex and the other vertices in B_2 have degree four.

Let $V_1 = \{v, w_1\} \cup A_1 \cup B_2$ and $V_2 = V(G) - V_1 = B_1 \cup A_2 \cup \{u, w_2\}$. By $g(G) \geq 6$, any circuit of $G[V_1]$ must use w_1. By (6), $\{v\} \cup A_1 \cup B_2$ is an independent set, and so by $g(G) \geq 6$ again, no circuit in $G[V_1]$ contains w_1. Thus $G[V_1]$ is acyclic. Similarly, $G[V_2]$ is also acyclic. Hence by Lemma 3.2, $a(G) \leq 2$. □

Lemma 3.4. *Let G be a connected graph with $m = |E(G)|$ and $n = |V(G)|$. Each of the following holds.*

(i) *Suppose that G has a vertex v with $d_G(v) = i \leq 3$. If $a(G - v) \geq 2$, then $a(G) = a(G - v)$.*
(ii) *Suppose that G has a vertex v with $d_G(v) = i \leq 3$, and let $G' = G - v$. If $a(G) = k \geq 3$, then $a(G') = k$, Furthermore, if $g(G) \geq \max\{\frac{m-n-3}{5}, 6\}$, then $g(G') \geq \max\{\frac{m'-n'-3}{5}, 6\}$, where $m' = |E(G')|$, $n' = |V(G')|$.*
(iii) *If $a(G) \geq 3$, then G has a subgraph H with $\delta(H) \geq 4$.*

Proof. Let v be a vertex of G of degree at most 3 in G, and let $G' = G - v$.

(i) As $a(G) \geq a(G - v)$, it suffices to show that $a(G) \leq a(G - v)$. Assume that $a(G') = k' \leq 2$, and that $(V_1', V_2', \ldots, V_{k'}')$ is a partition of $V(G')$ such that $G[V_i']$ $(i = 1, 2, \ldots, k')$ is acyclic. Since $d_G(v) \leq 3$ and $k' \geq 2$, there must be a V_i', say V_1', such that $|N_G(v) \cap V_1'| \leq 1$. Thus $(V_1, V_2, \ldots, V_{k'}) = (V_1' \cup \{v\}, V_2', \ldots, V_{k'}')$ is a partition of $V(G)$ such that $G[V_i]$ is a forest, and so $a(G) \leq k'$.
(ii) The conclusion $a(G') = k$ follows from (i). Now assume that $g(G) \geq \max\{\frac{m-n-3}{5}, 6\}$. Since deleting a vertex will not decrease the girth, $g(G') \geq 6$ and $g(G') \geq g(G) \geq \frac{m-n-3}{5}$. Since $d_G(v) = i$, we have $\frac{m-n-3}{5} = \frac{(m-i)-(n-1)-3+i-1}{5} \geq \frac{(m-i)-(n-1)-3}{5} = \frac{m'-n'-3}{5}$. Thus $g(G') \geq \max\{\frac{m'-n'-3}{5}, 6\}$. This proves (ii).
(iii) If $\max\{\delta(H) : H \text{ is a subgraph of } G\} \leq 3$, then by (i), we can argue by induction to show that $a(G) \leq 2$, contrary to the assumption that $a(G) \geq 3$. □

Theorem 3.5. *Suppose that G is a connected graph, $m = |E(G)|$ and $n = |V(G)|$. Let $M = M(G)$. If $g(G) = d \geq \max\{\frac{m-n-3}{5}, 6\}$, then G has a cocycle X such that $r^*(X) = r^*(M)$.*

Proof. By Lemma 3.2, it suffices to prove that $a(G) \leq 2$. By Lemma 3.3, we may assume that either $\delta(G) \leq 3$ or G is 4-regular. Arguing by contradiction, we assume that $a(G) = k \geq 3$.

Let $G_0 = G$, $G_{i+1} = G_i - v_i (i = 0, 1, 2, \ldots)$, where $v \in V(G_i)$ and $d_{G_i}(v_i) \leq 3$. Since $a(G) = k \geq 3$ and by Lemma 3.4, there is a G_n such that $\delta(G_n) \geq 4$, $a(G_n) = k \geq 3$ and $g(G_n) \geq \max\{\frac{|E(G_n)|-|V(G_n)|-3}{5}, 6\}$. Since $a(G_n) = k \geq 3 > 2$ and by Lemma 3.3, G_n must be 4-regular. On the other hand, as $k \geq 3$, by Theorem 3.1, $\Delta(G_n) \geq 5$, contrary to the fact that G_n is 4-regular. This contradiction establishes the corollary. □

4. Contractible cographic restrictions

The main result (Theorem 4.8 of this section) will show that if a cographic matroid has sufficiently large cogirth, then it must have a either a $U_{2,3}$ or a restriction N such that $\tau(N) \geq 2$. By Lemma 2.2, such a cographic matroid must have a contractible restriction. This result allow us to argue by induction in the next section to prove Theorem 1.3.

Throughout this section, for a graph G, we always denote $n = |V(G)|$ and $m = |E(G)|$. For any two vertices $u, v \in V(G)$, let $\text{dist}_G(u, v)$ denote the distance between u and v in G. As our approach here needs some of the former results in [6], we start with some terminology and notations from [6]. Let M be a matroid with $r(M) > 0$. For any $X \subseteq E(M)$ with $r(X) > 0$, define

$$d_M(X) = \frac{|X|}{r(M)}.$$

When M is understood from the context, we often use $d(X)$ for $d_M(X)$. Following the notation in [6], define the strength and the fractional arboricity of a matroid M by

$$\eta(M) = \min_{X\subseteq E(M), r(X)<r(M)} d(G/X) \quad \text{and} \quad \gamma(M) = \max_{X\neq\emptyset} d(X),$$

respectively. We list some of facts related to $\eta(M)$ and $\gamma(M)$ in the next theorem.

Theorem 4.1. *Let $q > 0$ be a fractional number and let M be a matroid with $r(M) > 0$. Each of the following holds.*

(i) *([9], Corollary 5 of [6]) $\frac{|E|}{r(M)} \geq \eta(M)$, and $\tau(M) = \lfloor \eta(M) \rfloor$.*
(ii) *(Corollary 5 of [6]) $E(M)$ has a nonempty subset X with $\eta(M|X) \geq q$ if and only if $\gamma(M) \geq q$.*
(iii) *(Theorem 1 of [6])*

$$\eta(M^*) = \frac{\gamma(M)}{\gamma(M)-1} \quad \textit{and} \quad \gamma(M^*) = \frac{\eta(M)}{\eta(M)-1}.$$

(iv) *(Lemma 9 of [6]) For any closed set $X \subseteq E(M)$ with $r(X) < r(M)$, $\eta(M) \leq \eta(M/X)$.*

To show that a cographic matroid M contains a restriction N with $\tau(N) \geq 2$, by Theorem 4.1(ii) (with $q = 2$) and (iii), it suffices to show that $\eta(M^*) \leq 2$, or for some $Z \subseteq E$, $\eta(M^*/Z) \leq 2$. In the rest of this section, we shall show that this can be done when the girth of M^* is sufficiently large.

Definition 4.2. Let $s > 0$ be an integer and $\mathcal{F}(s)$ be the collection of all 2-connected graph G that has a distinguished edge subset $Z \subseteq E(G)$ with $|Z| = 3$ such that Z is a cocircuit of $M(G)$ that do not contain any pair of parallel edges, and such that

(i) G does not have an edge cut X of size at most 3 such that $X \cap Z = \emptyset$,
(ii) For any circuit C of $G - Z$, $|C| \geq s$.

Lemma 4.3. *Suppose $G \in \mathcal{F}(s)$. If for some $u_0 \in V(G)$, $Z = E_G(u_0) = \{e_1, e_2, e_3\}$. Then either $s \leq 3$ and G is spanned by a K_4, or there exists a vertex $v_0 \in V(G)$ such that the distance from u_0 to v_0 in G is at least $s/2$.*

Proof. Assume that G is not spanned by a K_4. We shall show that there exists a vertex $v_0 \in V(G)$ such that the distance from u_0 to v_0 in G is at least $\frac{s}{2}$. By contradiction, we assume that

$$\forall v \in V(G - u_0), \quad \text{dist}_G(u_0, v) < \frac{s}{2}. \tag{8}$$

Let $e_j = u_0u_i$, $1 \leq i \leq 3$. Choose a depth-first-search tree T of G rooted at u_0. Then for any $v \in V(G)$, $\text{dist}_G(u_0, v) = \text{dist}_T(u_0, v)$, and $Z \subseteq E(T)$. Thus $T - u_0$ has three components: T_1, T_2, T_3 with $u_i \in V(T_i)$ $(1 \leq i \leq 3)$. Fix an $i \in \{1, 2, 3\}$. If $G[V(T_i)]$ has an edge $e = v'v'' \in E(G[V(T_i)] - E(T_i))$, then $T_i + e$ has a circuit C. Let P' and P'' denote the (v', u_0)-path and the (v'', u_0)-path in T, respectively. It follows that $|C| \le |E(P' - u_0) \cup E(P'' - u_0) \cup \{e\}| \le \mathrm{dist}_T(u_0, v') - 1 + \mathrm{dist}(u_0, v'') - 1 + 1 < s - 1$, contrary to Definition 4.2(ii). Hence

$$G[V(T_i)] \text{ is a tree}, \quad \text{for any } i \in \{1, 2, 3\}. \tag{9}$$

Let $i, j \in \{1, 2, 3\}$ with $i \neq j$. By (9), both T_i and T_j are trees. Suppose that for some $v \in V(T_i)$, v is adjacent to two distinct vertices $v', v'' \in V(T_j)$. Let P denote the unique (v', v'')-path in T_j, and let $C = G[E(P) \cup \{vv', vv''\}]$. Let P' and P'' denote the (v', u_0)-path and the (v'', u_0)-path in T, respectively. Then $C \subseteq E(P' - u_0) \cup E(P'' - u_0) \cup \{vv', vv''\}$, and so $|C| \le \mathrm{dist}_T(u_0, v') - 1 + \mathrm{dist}(u_0, v'') - 1 + 2 < s$, contrary to Definition 4.2(ii). Thus we have

$$\text{If } v \in V(T_i), \quad \text{then } \forall j \neq i, \left|\left[v, V(T_j)\right]\right| \le 1. \tag{10}$$

For each i with $1 \le i \le 3$, let z_i be a vertex in T_i such that $\mathrm{dist}_T(u_0, z_i)$ is maximized. If for all i, $\mathrm{dist}_T(u_0, z_i) = 1$, then each $V(T_i) = \{z_i\}$ and G is spanned by a K_4 (and so by Definition 4.2(ii), $s \le 3$). Otherwise, we may assume that $\mathrm{dist}_T(z_1, u_0) > 1$. Then $E_G(z_1) \cap Z = \emptyset$. By Definition 4.2(i), $|E_G(z_1)| \ge 4$. By the choice of z_1, z_1 has degree one in T_1. Therefore, $|[z_1, V(T_2) \cup V(T_3)]| \ge 3$, contrary to (10), and so the lemma holds. □

Let Z be a cocircuit of G with $|Z| = 3$. By Theorem 4.1, if for some $X \subseteq E(G)$, $\eta(G/(X \cup Z)) \le 2$, then $\gamma(M^*(G) - (X \cup Z)) \ge 2$. It follows by Theorem 4.1 again and by Lemma 2.2 that $M^*(G) - (X \cup Z)$ has a contractible restriction. Therefore, we shall investigate graphs G in $\mathcal{F}(s)$ such that for some $X \subseteq E(G)$, $\eta(G/(X \cup Z)) \le 2$.

Lemma 4.4. *Let $G \in \mathcal{F}(s)$ with $s = \lceil \frac{2(m-n)}{5} \rceil$, and with $m - n \ge 9$ (or with $n \ge 13$). If G has a vertex $u_0 \in V(G)$ such that $Z = E_G(u_0) = \{e_1, e_2, e_3\}$, then either G/Z has a cocircuit D with $|D| \le 3$, or $\eta(G/Z) \le 2$.*

Proof. By contradiction, assume that

$$G/Z \text{ does not have a cocircuit } D \quad \text{with } |D| \le 3, \tag{11}$$

and that $\eta(G/Z) > 2$. By Lemma 4.3, G has a vertex v with $\mathrm{dist}_G(u_0, v) \ge h = \lceil \frac{s}{2} \rceil$. Let V_i be the set of vertices in G that has distance i to v. By (11), every vertex in G/Z has degree at least 4. As $G \in \mathcal{F}(s)$, $|V_1| \ge 4$, $|V_2| \ge 3|V_1|, \ldots |V_{i+1}| \ge 3|V_i|$, for $i \in \{1, 2, \ldots, h\}$. Hence, with $V_0 = \{v\}$,

$$n = |V(G)| \ge \sum_{i=0}^{h} |V_i| \ge 1 + 4\left(1 + 3 + 3^2 + \cdots + 3^{h-1}\right) = 1 + 2(3^h - 1).$$

Let $t = |E(G[N_G[u_0]])| \ge |Z| = 3$. Since $\eta(G/Z) > 2$, by Theorem 4.1(i),

$$2 < \eta(G/Z) \le \frac{|E(G/Z)|}{|V(G/Z)| - 1} = \frac{m - t}{(n - 3) - 1}, \quad \text{or} \quad m - n \ge n - (7 - t) \ge n - 4.$$

It follows that

$$m - n \ge n - 4 \ge 2\left(3^{\frac{2(m-n)}{10}} - 1\right) - 3.$$

Hence $m - n \le 8$ and $n \le 12$, contrary to the assumptions that $m - n \ge 9$ or $n \ge 13$. This completes the proof. □

Lemma 4.5. *Let $G \in \mathcal{F}(s)$ with $s = \lceil \frac{2(m-n)}{5} \rceil \ge 6$ an $Z = \{e_1, e_2, e_3\}$ denote the distinguished cocircuit of $M(G)$. If $m - n \ge 9$ or $n \ge 13$, then either G/Z has a cocircuit D with $|D| \le 3$, or for some $X \subseteq E(G)$, $\eta(G/(X \cup Z)) \le 2$.*

Proof. Again we assume (11) holds. If for some vertex $u_0 \in V(G)$, $Z = E_G(u_0)$, then the conclusion follows from Lemma 4.4. Suppose that $G - Z$ has two nontrivial components G_1, G_2. Let $m_i =$

$|E(G/G_{3-i})|$ and $n_i = |V(G/G_{3-i})|$. Thus $m_1 + m_2 = m + 3$ and $n_1 + n_2 = n + 2$. We may assume that $m_1 - n_1 \le m_2 - n_2$. Then

$$2(m_1 - n_1) \le (m_1 - n_1) + (m_2 - n_2) = (m - n) + 1.$$

Since Z is a cocircuit, any circuit C of G not intersecting Z must be a circuit of G_1 or of G_2. It follows by $G \in \mathcal{F}(s)$ that for any circuit C of G_1 not intersecting Z,

$$|C| \ge \frac{2(m-n)}{5} \ge \frac{4(m_1 - n_1) - 2}{5}. \tag{12}$$

Let $s' = \lceil \frac{4(m_1-n_1)-2}{5} \rceil$, and $G' = G/G_2$ with u_0' denote the vertex onto which G_2 is contracted. Then $G' \in \mathcal{F}(s')$. If G' is spanned by a K_4, then $|C| \le 4$, contrary to (12) and the assumption of $s = \lceil \frac{2(m-n)}{5} \rceil \ge 6$. Hence G' cannot be spanned by a K_4. By Lemma 4.3, G' has a vertex v with distance $s'/2 \ge 2$ from u_0' in G'. It follows that

$$n_1 \ge 1 + 4\left(1 + 3 + 3^3 + \cdots + 3^{h-1}\right) = 1 + 2(3^{\lceil s'/2 \rceil} - 1).$$

By contradiction, we assume that $\eta(G'/Z) > 2$. Then

$$m_1 - n_1 \ge n_1 - 4 \ge 2\left(3^{\frac{4(m_1-n_1)-2}{10}} - 1\right) - 3,$$

implying $m_1 - n_1 \le 2$. Since $n_1 \ge 5$ and since every vertex in G' no incident with edges in Z must have degree at last 4, $m_1 - n_1 \ge 3$. This contradiction establishes the lemma. $\square$

Lemma 4.6. *Let $s \ge 2$ be an integer and G be a 2-connected graph with $m = |E(G)|$, $n = |V(G)|$ and with a distinguished edge $e_0 = u_0v_0$ such that for any circuit C of $G - e_0$, $|C| \ge s$. Then each of the following holds.*

(i) *Either G itself is a circuit, or G has a vertex $z_0 \in V(G)$ such that the distance from z_0 to each of u_0 and v_0 in G is at least $s/2$.*
(ii) *Suppose that $s \ge \frac{2(m-n)-1}{5}$. If $m - n \ge 10$ or $n \ge 12$, either G/e_0 has a cocircuit D with $|D| \le 3$, or $\eta(G/e_0) \le 2$.*

Proof. (i) Suppose that G is not a circuit, and that for every vertex $z \in V(G)$, the distance from z to u_0 in G is less than $\frac{s}{2}$.

Subdivide e_0 by inserting a new vertex w_0 and replace e_0 by two edges w_0u_0 and w_0v_0. Let G' denote the resulting graph. Choose a depth-first-search tree T of G' rooted at w_0. Then for any $v \in V(G')$, $\mathrm{dist}_{G'}(w_0, v) = \mathrm{dist}_T(u_0, v)$, and $\{w_0u_0, w_0v_0\} \subseteq E(T)$. Thus $T - w_0$ has two components: T_1, T_2. Let $e_1 = w_0u_0$ and $e_2 = w_0v_0$ and assume that $e_i \in V(T_i)$. If every vertex in $G' - w_0$ has distance to w_0 at most $s/2$, then with the same arguments used to prove (9) and (10), we conclude that each $G'[V(T_i)]$ is a tree, and that there is at most one edge in $G - e_0$ joining a vertex in T_1 to a vertex in T_2. Since G is 2-connected, this forces that both T_1 and T_2 are paths, and so G must be a circuit itself, contrary to the assumption that G is not a circuit. Hence $G' - w_0$ has a vertex z_0 whose distance to w_0 in G' is at least $s/2 + 1$, and so (i) follows.

(ii) Assume that G/Z does not have a cocircuit D with $|D| \le 3$. If G is a circuit with at least 3 vertices, then G/e_0 has a cocircuit of size 2. Hence we assume that G is not a circuit, and so by (i), G has a vertex $z_0 \in V(G)$ such that the distance from z_0 to u_0 in G is at least $s/2$. By assumption, every vertex in $G - \{u_0, v_0\}$ has degree at least 4 in G, and so with $h = \lfloor s/2 \rfloor$,

$$n = |V(G)| \ge 1 + 4(1 + 3 + 3^3 + \cdots + 3^{h-1}) = 1 + 2(3^h - 1).$$

Since $\eta(G/e_0) > 2$,

$$2 < \frac{|E(G/e_0)|}{|V(G/e_0)| - 1} = \frac{m-1}{(n-1)-1}, \quad \text{or} \quad m - n \ge n - 2.$$

It follows that

$$m - n \geq n - 2 \geq 2\left(3^{\frac{2(m-n)-1}{10}} - 1\right) - 3.$$

Hence $m - n \leq 9$, or $n \leq 11$, contrary to the assumptions that $m - n \geq 10$ or $n \geq 12$. This completes the proof. □

Lemma 4.7. *Let G be a 2-connected loopless graph on $n = |V(G)| \geq 4$, $m = |E(G)|$, and with a distinguished edge subset Z such that*

$$\text{if } C \text{ is a circuit of } G \text{ with } C \cap Z = \emptyset, \quad \text{then } |C| \geq 6. \tag{13}$$

Then each of the following holds.

(i) *If $Z = \{e_0\}$ and $n \leq 12$, then G/Z has a cocircuit D with $|D| \leq 3$.*
(ii) *If $Z = \{e_0\}$ and $m - n \leq 9$, then either $\eta(G/e_0) \leq 2$ or G/Z has a cocircuit D with $|D| \leq 3$.*
(iii) *If Z is a cocircuit with $|Z| = 3$, and if $n \leq 12$, then G/Z has a cocircuit D with $|D| \leq 3$.*
(iv) *If Z is a cocircuit with $|Z| = 3$, and if $m - n \leq 8$, then either $\eta(G/Z) \leq 2$ or G/Z has a cocircuit D with $|D| \leq 3$.*

Proof. We assume that

$$G/Z \text{ has no cocircuit } D \quad \text{with } |D| \leq 3, \tag{14}$$

and we will find contradictions in (i) and (iii), and prove $\eta(G/Z) \leq 2$ in (ii) and (iv).

(i) Let $e_0 = u_0v_0$. If $V(G) = N_G(u_0) \cup N_G(v_0)$, then by (14), every vertex of $G - \{u_0, v_0\}$ has degree at least 4, which implies that G has a 4-circuit containing at most one vertex in $\{u_0, v_0\}$, contrary to (13). Hence G must have a vertex z with distance at least 2 to both u_0 and v_0. Let $V_i = \{v \in V(G)\colon \mathrm{dist}_G(z, v) = i\}$. It follows by (13) that $n = |V(G)| \geq \sum_{i=0}^{2} |V_i| \geq 1 + 4 + 4(3) = 17$, contrary to the assumption that $n \leq 12$.

(ii) If not, then by definition of η,

$$2 < \eta(G/e_0) \leq \frac{|E(G/e_0)|}{|V(G/e_0)| - 1} = \frac{m-1}{n-2} \leq \frac{n+9-1}{n-2}.$$

Thus $n \leq 12$, and so (ii) follows from (i).

(iii) Suppose first that for some $v_0 \in V(G)$, $Z = E_G(v_0)$. Let $e_i = v_0v_i$, $(1 \leq i \leq 3)$. If $V(G) = \cup_{i=0}^{3} N_G(v_i)$, then by (14), every vertex of $G - \{v_0, v_1, v_2, v_3\}$ has degree at least 4. It follows that a vertex in $N_G(v_1) - \{v_0\}$ must be adjacent to either a vertex in $N_G(v_1) - \{v_0\}$, or two vertices in $N_G(v_i) - \{v_0\}$, for some $i \in \{2, 3\}$, and so $G - v_0$ has a circuit of length at most 4, contrary to (13). Hence G must have a vertex z with distance at least 3 to v_0. Let $V_i = \{v \in V(G)\colon \mathrm{dist}_G(z, v) = i\}$. It follows by (13) that $n = |V(G)| \geq \sum_{i=0}^{2} |V_i| \geq 1 + 4 + 4(3) = 17$, contrary to the assumption that $n \leq 12$. The proof for the case when $G - Z$ has two nontrivial components is similar, and will be omitted. This proves (iii).

(iv) If not, then by definition of η,

$$2 < \eta(G/e_0) \leq \frac{|E(G/Z)|}{|V(G/Z)| - 1} = \frac{m-3}{(n-3)-1} \leq \frac{n+8-3}{n-4}.$$

Thus $n \leq 12$, and so (iv) follows from (iii). □

Theorem 4.8. *Let G be a 2-connected loopless graph on n vertices and m edges with a distinguished edge subset Z, and let $M = M(G)$. If one of the following holds:*

(i) *$Z = \{e_0\}$, and for any circuit $C \subseteq E(G) - Z$, $|C| \geq \max\{\frac{2(m-n)-1}{5}, 6\}$,*
(ii) *$n \geq 5$, $Z = \{e_1, e_2, e_3\}$ is a circuit of M that does not contain any cocircuits of M, and for any circuit $C \subseteq E(G) - Z$, $|C| \geq \max\{\frac{2(m-n)}{5}, 6\}$,*

then $M^* - Z$ contains a nonempty set D such that either $\tau(M^*|D) \geq 2$ or $M^*|D \cong U_{2,3}$.

Proof. Since $\tau(U_{1,2}) = 2$, by Theorem 4.1(iii), it suffices to show that if G/Z does not have a cocircuit with 3-elements, then $G - Z$ has an edge subset X such that $\eta(G/(X \cup Z)) \le 2$.

(i) If $m - n \ge 10$ or $n \ge 14$, then by Lemma 4.6, $\eta(G/e_0) \le 2$. If $m - n \le 9$ or $n \le 13$, then by Lemma 4.7(i) and (ii), $\eta(G/e_0) \le 2$.
(ii) If $m - n \ge 9$ or $n \ge 13$, then by Lemma 4.5, for some edge subset $X \subseteq E(G) - Z$, $\eta(G/(X \cup Z)) \le 2$. If $m - n \le 8$ or $n \le 12$, then by Lemma 4.7(iii) and (iv), $\eta(G/Z) \le 2$ as well. □

5. Proofs of the main results

We start with an auxiliary lemma for our arguments.

Lemma 5.1. *Let H be a simple graph on $n \ge 4$ vertices. Let $Z \subseteq E(H)$ and let $V(Z)$ denote the set of vertices in H that is incident with an edge in Z. Suppose that for any $v \in V(H) - V(Z)$,*

$$d_H(v) \ge \max\left\{\frac{2n}{5} - 1, 4\right\}. \tag{15}$$

Each of the following holds.

(i) *If $Z = \{e\}$, then $H - e$ contains a nontrivial collapsible subgraph.*
(ii) *If $Z = \{e_1, e_2, e_3\}$ is a circuit of H, then $H - Z$ contains a nontrivial collapsible subgraph.*

Proof. We shall only prove (ii) as the proof for (i) is similar. For integers $i \ge 1$, let

$$D_i(H) = \{v \in V(H) : d_H(v) = i\}, \quad \text{and} \quad d_i = |D_i(H)|.$$

By contradiction, we assume that $H - Z$ has no nontrivial collapsible subgraphs. By Theorem 2.1 (iv), $F(H) \ge 2$. If $F(H) = 2$, then by Theorem 2.1(iii), $d_1 + d_2 + d_3 \ge 4$. If $F(H) \ge 3$, then by Theorem 2.1(iv),

$$4 \le 2F(H) = 4\sum_{i\ge1} d_i - \sum_{i\ge1} i d_i - 6.$$

It follows that

$$3d_1 + 2d_2 + d_3 \ge \sum_{i\ge5}(i-4)d_i + 10.$$

Hence $d_1 + d_2 + d_3 \ge 4$ also. Since $|V(Z)| = 3$, there exists a vertex $v \in D_1(H - Z) \cup D_2(H - Z) \cup D_3(H - Z) - V(Z)$. As $d_H(v) \ge 4$, this contradicts that $v \in D_1(H - Z) \cup D_2(H - Z) \cup D_3(H - Z)$, and so H must have a nontrivial collapsible subgraph. □

Proof of Theorem 1.3. Let M be a connected simple regular matroid such that (1) holds. The theorem holds trivially if $|E| \le 3$. We argue by contradiction and assume that

$$M \text{ is a counterexample to Theorem 1.3 with } |E(M)| \text{ minimized.} \tag{16}$$

If $M = M(G)$ is the cycle matroid of a 2-connected simple graph, then Theorem 1.3 follows from Theorem 1.2, contrary to (16). If $M = M^*(G)$ is a cocycle matroid of a connected graph G, then by Theorem 3.5, M has a spanning cycle, contrary to (16) also. If $M \cong R_{10}$, then by Observation 2.6, R_{10} itself is also a cycle, again contrary to (16). Therefore, by Lemma 2.8, we can express $M = M_1 \oplus_i M_2$, for some $i \in \{2, 3\}$ such that M_2 is either R_{10}, graphic or cographic and such that (4) holds. By Observation 2.6(i) and (iii), for any $e \in E(R_{10})$, R_{10} has a spanning circuit (a 6-circuit) that contains e, and a spanning cycle (a 6-circuit) that does not contain e. Using symmetric difference, if M_1 has a spanning cycle and if $M_2 \cong R_{10}$, then $M_1 \oplus_2 M_2$ also has a spanning cycle. Hence we only have these two cases.

Case 1. M_2 is graphic.

Thus for some 2-connected simple graph H, $M_2 = M(H)$. By (1) and by (4) and (15) must hold. It follows from Lemma 5.1 that $H - E(M_1)$ must have a nontrivial collapsible subgraph L. By (16), M/L has a spanning cycle. By Proposition 2.5(iv), M also has a spanning cycle, contrary to (16).

Case 2. M_2 is cographic.

Then for some connected graph G, $M_2 = M^*(G)$. As $M = M_1 \oplus_i M_2$ for some $i \in \{2, 3\}$, by (4) and by (1), if C is a circuit of $G - Z$, then $|C| \geq \frac{2(m-n)+i-3}{5}$. It follows by Theorem 4.8 that $M_2 - Z$ has a subset D such that either $\tau(M_2|D) \geq 2$, or $M_2|D \cong U_{2,3}$. By (16), M/D has a spanning cycle. By Lemma 2.2, M also has a spanning cycle, contrary to (16). $\square$

References

[1] D. Bauer, A note on degree conditions for Hamiltonian cycles in line graphs, Congr. Numer. 49 (1985) 11–18.
[2] F.T. Boesch, C. Suffel, R. Tindell, The spanning subgraphs of Eulerian graphs, J. Graph Theory 1 (1977) 79–84.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[4] P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–44.
[5] P.A. Catlin, Supereulerian graphs, a survey, J. Graph Theory 16 (1992) 177–196.
[6] P.A. Catlin, J.W. Grossman, A.M. Hobbs, H.-J. Lai, Fractional arboricity, strength and principal partitions in graphs and matroids, Discrete Appl. Math. 40 (1992) 285–302.
[7] P.A. Catlin, Z.Y. Han, H.-J. Lai, Graphs without spanning closed trails, J. Discrete Math. 160 (1996) 81–91.
[8] Z.H. Chen, H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics-a survey, in: Combinatorics and Graph Theory'95, vol. 1 (Hefei), World Sci. Publishing, River Edge, NJ, 1995, pp. 53–69.
[9] J. Edmonds, Lehman's switching game and a theorem of Tutte and Nash-Williams, J. Res. Nat. Bur. Stand Sect. B 69B (1965) 73–77.
[10] P. Erdös, Graph theory and probability, Can. J. Math. 11 (1959) 34–38.
[11] Winfried Hochstättler, Bill Jackson, Large circuits in binary matroids of large cogirth I, J. Combin. Theory, Ser. B 74 (1998) 35–52.
[12] F. Jaeger, A note on subeulerian graphs, J. Graph Theory 3 (1979) 91–93.
[13] H.V. Kronk, J. Mitchem, Critical point arboritic graphs, J. Lond. Math. Soc. 9 (1974/75) 459–466.
[14] H.-J. Lai, B. Liu, Y. Liu, Y. Shao, Spanning cycles in regular matroids without $M^*(K_5)$ minors, European J. Combin. 29 (1) (2008) 298–310.
[15] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[16] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.
[17] W.R. Pulleyblank, A note on graphs spanned by Eulerian graphs, J. Graph Theory 3 (1979) 309–310.
[18] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory, Ser. B 28 (1980) 305–359.
[19] P.D. Seymour, Matroids and multicommodity flows, European J. Combin. Theory Ser. B 2 (1981) 257–290.
[20] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. Lond. Math. Soc. 36 (1961) 221–230.