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Mathematical Methods in the Applied Sciences

Received 27 February 2010 Published online in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.1361
MOS subject classification: 34 L 10; 34 L 20

Asymptotic behavior of a differential operator with discontinuities at two points

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In this paper, we study a Sturm–Liouville operator with eigenparameter-dependent boundary conditions and transmission conditions at two interior points. By establishing a new operator *A* associated with the problem, we prove that the operator *A* is self-adjoint in an appropriate space *H*, discuss completeness of its eigenfunctions in *H*, and obtain its Green function. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: Sturm–Liouville problem; transmission conditions; eigenparameter-dependent boundary conditions; eigenvalues; ~~eigenfunction;~~ Green function; completeness

1. Introduction

In recent years, more and more researchers are interested in the discontinuous Sturm–Liouville problem for its application in physics (see [1, 2]). Various physics applications of this kind of problem are found in many literatures, including some ~~boundary value~~ with transmission conditions that arise in the theory of heat and mass transfer (see [3–5]).

By using the techniques of [3, 6] and some new approaches, we define a new linear operator A associated with the problem in appropriate Hilbert space H such that the eigenvalues of the problem coincide with those of A. We discuss its eigenvalues and eigenfunctions, obtain asymptotic approximation formulas for eigenvalues, prove that the eigenfunctions of A are complete and construct its Green function, promote and deepen the previous conclusions.

In this study, we consider a discontinuous eigenvalue problem consisting of Sturm–Liouville equation

$$lu := -(a(x)u'(x))' + q(x)u(x) = \lambda u(x), \quad x \in J \tag{1}$$

where $J = [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$, $a(x) = a_1^2$ for $x \in [a, \xi_1)$, $a(x) = a_2^2$ for $x \in (\xi_1, \xi_2)$ and $a(x) = a_3^2$ for $x \in (\xi_2, b]$, $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$ are given real numbers; the real value function $q(x) \in L^1[J, \mathbb{R}]$; $\lambda \in \mathbb{C}$ is a complex eigenparameter; boundary conditions at the endpoints

$$l_1 u := \alpha_1 u(a) + \alpha_2 u'(a) = 0 \tag{2}$$

$$l_2 u := \lambda(\beta_1' u(b) - \beta_2' u'(b)) + (\beta_1 u(b) - \beta_2 u'(b)) = 0 \tag{3}$$

and the four transmission conditions at the points of discontinuities $x = \xi_1$ and $x = \xi_2$

$$l_3 u := u(\xi_1 + 0) - \alpha_3 u(\xi_1 - 0) - \beta_3 u'(\xi_1 - 0) = 0 \tag{4}$$

$$l_4 u := u'(\xi_1 + 0) - \alpha_4 u(\xi_1 - 0) - \beta_4 u'(\xi_1 - 0) = 0 \tag{5}$$

$$l_5 u := u(\xi_2 + 0) - \alpha_5 u(\xi_2 - 0) - \beta_5 u'(\xi_2 - 0) = 0 \tag{6}$$

$$l_6 u := u'(\xi_2 + 0) - \alpha_6 u(\xi_2 - 0) - \beta_6 u'(\xi_2 - 0) = 0 \tag{7}$$

where α_i, β_i and $\beta_j'(i = \overline{1,6}, j = 1, 2)$ are real numbers. Here we assume that $\alpha_1^2 + \alpha_2^2 \neq 0$ and

$$\theta_1 = \begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix} > 0, \quad \theta_2 = \begin{vmatrix} \alpha_5 & \beta_5 \\ \alpha_6 & \beta_6 \end{vmatrix} > 0, \quad \rho = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0$$

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2. Operator formulation

In this section, we introduce the special inner product in the Hilbert Space $H:=H_1\oplus\mathbb{C}$, where $H_1=(L^2(J),\langle\cdot,\cdot\rangle_1)$, $\mathbb{C}$ denotes the Hilbert space of complex numbers and a symmetric linear operator A defined on this Hilbert space such that (1)–(7) can be considered as the eigenvalue problem of this operator. Namely, we define an inner product in H by

$$\langle F,G\rangle=\frac{\theta_1\theta_2}{a_1^2}\int_a^{\xi_1} f\overline{g}\,\mathrm{d}x+\frac{\theta_2}{a_2^2}\int_{\xi_1}^{\xi_2} f\overline{g}\,\mathrm{d}x+\frac{1}{a_3^2}\int_{\xi_2}^{b} f\overline{g}\,\mathrm{d}x+\frac{1}{\rho}r\overline{s} \tag{8}$$

for

$$F:=(f,r),\quad G:=(g,s)\in H$$

In the Hilbert space H consider the operator A which is defined by

$$\begin{aligned}
&D(A)=\{(f(x),r)\in H|f_1,f_1'\in AC_{\mathrm{loc}}((a,\xi_1)),\ f_2,f_2'\in AC_{\mathrm{loc}}((\xi_1,\xi_2)),\ f_3,f_3'\in AC_{\mathrm{loc}}((\xi_2,b)),\\
&lf\in H_1,\ l_1f=l_3f=l_4f=l_5f=l_6f=0,\ r=\beta_1'f(b)-\beta_2'f'(b)\}\\
&AF=(lf,-(\beta_1f(b)-\beta_2f'(b)))\quad \text{for } F=(f,\beta_1'f(b)-\beta_2'f'(b))\in D(A)
\end{aligned}$$

For convenience, $\forall(f,r)\in D(A)$, let

$$N(f)=\beta_1f(b)-\beta_2f'(b),\quad N'(f)=\beta_1'f(b)-\beta_2'f'(b)$$

Now we can rewrite the considered problem (1)–(7) in the operator form $AF=\lambda F$.

Lemma 1
The eigenvalues and eigenfunctions of the problem (1)–(7) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator A respectively.

Lemma 2
The domain $D(A)$ is dense in H.

Proof
Let $F=(f(x),r)\in H$, $F\perp D(A)$ and $\widetilde{C}_0^\infty$ be a functional set such that

$$\varphi(x)=\begin{cases}\varphi_1(x), x\in[a,\xi_1)\\ \varphi_2(x), x\in(\xi_1,\xi_2)\\ \varphi_3(x), x\in(\xi_2,b]\end{cases}$$

for $\varphi_1(x)\in\widetilde{C}_0^\infty[a,\xi_1)$, $\varphi_2(x)\in\widetilde{C}_0^\infty(\xi_1,\xi_2)$ and $\varphi_3(x)\in\widetilde{C}_0^\infty(\xi_2,b]$. Since $\widetilde{C}_0^\infty\oplus 0\subset D(A)(0\in\mathbb{C})$, any $U=(u(x),0)\in\widetilde{C}_0^\infty\oplus 0$ is orthogonal to F, namely

$$\langle F,U\rangle=\frac{\theta_1\theta_2}{a_1^2}\int_a^{\xi_1} f_1\overline{u}\,\mathrm{d}x+\frac{\theta_2}{a_2^2}\int_{\xi_1}^{\xi_2} f_2\overline{u}\,\mathrm{d}x+\frac{1}{a_3^2}\int_{\xi_2}^{b} f_3\overline{u}\,\mathrm{d}x=\langle f,u\rangle_1$$

we can learn that $f(x)$ is orthogonal to $\widetilde{C}_0^\infty$ in H_1, this implies $f(x)=0$. So for all $G=(g(x),s)\in D(A)$, $\langle F,G\rangle=(1/\rho)r\overline{s}=0$. Thus $r=0$ since $s=N'(g)$ can be chosen arbitrarily. So $F=(0,0)$, which prove the assertion. □

Theorem 3
The operator A is self-adjoint in H.

Proof
Let $F,G\in D(A)$. By two partial integrations we obtain

$$\begin{aligned}
\langle AF,G\rangle &= \langle F,AG\rangle+\theta_1\theta_2(W(f,\overline{g};\xi_1-0)-W(f,\overline{g};a))+\theta_2(W(f,\overline{g};\xi_2-0)-W(f,\overline{g};\xi_1+0))\\
&+W(f,\overline{g};b)-W(f,\overline{g};\xi_2+0)-\frac{1}{\rho}(N(f)\overline{N'(g)}-N'(f)\overline{N(g)})
\end{aligned}$$

where, as usual, by $W(f,g;x)$ we denote the Wronskians $f(x)g'(x)-f'(x)g(x)$. □

As f and g satisfy the boundary condition (2), it follows that $W(f,\overline{g};a)=0$. From the transmission conditions (4)–(7), we get

$$W(f,\overline{g};\xi_i-0)=\theta_iW(f,\overline{g};\xi_i+0)(i=1,2)$$

Further, it is easy to verify that

$$\rho W(f,\overline{g};b)=N(f)\overline{N'(g)}-N'(f)\overline{N(g)}$$

Then we have $\langle AF,G\rangle=\langle F,AG\rangle (F,G\in D(A))$. So A is symmetric.

It remains to show that if $\langle AF,W\rangle=\langle F,U\rangle$ for all $F=(f,N'(f))\in D(A)$, then $W\in D(A)$ and $AW=U$, where $W=(w(x),r)$ and $U=(u(x),s)$, i.e. (i) $w_1,w_1'\in AC_{\mathrm{loc}}((a,\xi_1))$, $w_2,w_2'\in AC_{\mathrm{loc}}((\xi_1,\xi_2))$, $w_3,w_3'\in AC_{\mathrm{loc}}((\xi_2,b))$ and $lw\in H_1$; (ii) $r=\beta_1'w(b)-\beta_2'w'(b)$; (iii) $l_1w=l_3w=l_4w=l_5w=l_6w=0$; (iv) $u=lw$; (v) $s=-\beta_1w(b)+\beta_2w'(b)$.

For an arbitrary point $F\in\widetilde{C}_0^{\infty}\oplus 0\subset D(A)$ such that

$$\frac{\theta_1\theta_2}{a_1^2}\int_a^{\xi_1}(lf)\overline{w}\,\mathrm{d}x+\frac{\theta_2}{a_2^2}\int_{\xi_1}^{\xi_2}(lf)\overline{w}\,\mathrm{d}x+\frac{1}{a_3^2}\int_{\xi_2}^{b}(lf)\overline{w}\,\mathrm{d}x=\frac{\theta_1\theta_2}{a_1^2}\int_a^{\xi_1}f\overline{u}\,\mathrm{d}x+\frac{\theta_2}{a_2^2}\int_{\xi_1}^{\xi_2}f\overline{u}\,\mathrm{d}x+\frac{1}{a_3^2}\int_{\xi_2}^{b}f\overline{u}\,\mathrm{d}x$$

that is $\langle lf,w\rangle_1=\langle f,u\rangle_1$. According to normal Sturm–Liouville theory, (i) and (iv) hold. By (iv), equation $\langle AF,W\rangle=\langle F,U\rangle$, $\forall F\in D(A)$, becomes

$$\langle lf,w\rangle_1=\langle f,lw\rangle_1+\frac{N(f)\overline{r}}{\rho}+\frac{N'(f)\overline{s}}{\rho}$$

However

$$\begin{aligned}\langle lf,w\rangle_1 &= \langle f,lw\rangle_1+\theta_1\theta_2(W(f,\overline{w};\xi_1-0)-W(f,\overline{w};a))\\ &\quad+\theta_2(W(f,\overline{w};\xi_2-0)-W(f,\overline{w};\xi_1+0))+W(f,\overline{w};b)-W(f,\overline{w};\xi_2+0)\end{aligned}$$

So

$$\frac{N(f)\overline{r}}{\rho}+\frac{N'(f)\overline{s}}{\rho}=\theta_1\theta_2(W(f,\overline{w};\xi_1-0)-W(f,\overline{w};a))+\theta_2(W(f,\overline{w};\xi_2-0)-W(f,\overline{w};\xi_1+0))+W(f,\overline{w};b)-W(f,\overline{w};\xi_2+0)\tag{9}$$

By Naimark's Patching Lemma [7], there is an $F\in D(A)$ such that $f(a)=f'(a)=f(\xi_1-0)=f'(\xi_1-0)=f(\xi_1+0)=f'(\xi_1+0)=f(\xi_2-0)=f'(\xi_2-0)=f(\xi_2+0)=f'(\xi_2+0)=0, f(b)=\beta_2'$ and $f'(b)=\beta_1'$. Thus $N'(f)=0$. Then from (9), equality (ii) be true. Similarly one proves (v).

Next choose $F\in D(A)$ so that $f(b)=f'(b)=f(\xi_1-0)=f'(\xi_1-0)=f(\xi_2-0)=f'(\xi_2-0)=0, f(a)=\alpha_2$ and $f'(a)=-\alpha_1$. Then $N'(f)=N(f)=0$. So from (9), we get $\alpha_1w(a)+\alpha_2w'(a)=0$. Let $F\in D(A)$ satisfies $f(b)=f'(b)=f(a)=f'(a)=f(\xi_1+0)=f(\xi_2-0)=f'(\xi_2-0)=f(\xi_2+0)=f'(\xi_2+0)=0$, $f(\xi_1-0)=-\beta_3$, $f'(\xi_1-0)=\alpha_3$ and $f'(\xi_1+0)=\theta_1$. Then $N'(f)=N(f)=0$. By (9), we have $w(\xi_1+0)=\alpha_3w(\xi_1-0)+\beta_3w'(\xi_1-0)$. Similarly one prove $l_4w=l_5w=l_6w=0$.

Corollary 4

All eigenvalues of the problem (1)–(7) are real, and if λ_1 and λ_2 are two different eigenvalues, then the corresponding eigenfunctions $f(x)$ and $g(x)$ of this problem are orthogonal in the sense of

$$\frac{\theta_1\theta_2}{a_1^2}\int_a^{\xi_1}f\overline{g}\,\mathrm{d}x+\frac{\theta_2}{a_2^2}\int_{\xi_1}^{\xi_2}f\overline{g}\,\mathrm{d}x+\frac{1}{a_3^2}\int_{\xi_2}^{b}f\overline{g}\,\mathrm{d}x+\frac{1}{\rho}(\beta_1'f(b)-\beta_2'f'(b))(\beta_1'\overline{g}(b)-\beta_2'\overline{g}'(b))=0$$

3. Simplicity of eigenvalues

Lemma 5

Let the real-valued function $q(x)\in C[a,b]$ be continuous on $[a,b]$ and $f(\lambda)$, $g(\lambda)$ are given entire functions. Then for $\forall\lambda\in\mathbb{C}$ Equation (1) has a unique solution $u=u(x,\lambda)$ satisfying the initial conditions

$$u(a)=f(\lambda),\quad u'(a)=g(\lambda)\quad (\text{or } u(b)=f(\lambda), u'(b)=g(\lambda))$$

For each fixed $x\in[a,b]$, $u(x,\lambda)$ is an entire function of λ.

Let $\varphi_1(x,\lambda)$ be the solution of Equation (1) on the interval $[a,\xi_1)$, satisfying the initial conditions

$$\varphi_1(a,\lambda)=\alpha_2,\quad \varphi_1'(a,\lambda)=-\alpha_1\tag{10}$$

By virtue of Lemma 5, after defining this solution we can define the solution $\varphi_2(x,\lambda)$ of Equation (1) on the interval $[\xi_1,\xi_2)$ by the initial conditions

$$\varphi_2(\xi_1,\lambda)=\alpha_3\varphi_1(\xi_1,\lambda)+\beta_3\varphi_1'(\xi_1,\lambda),\quad \varphi_2'(\xi_1,\lambda)=\alpha_4\varphi_1(\xi_1,\lambda)+\beta_4\varphi_1'(\xi_1,\lambda)\tag{11}$$

After defining this solution we can define the solution $\varphi_3(x,\lambda)$ of Equation (1) on the interval $[\xi_2,b]$ by the initial conditions

$$\varphi_3(\xi_2,\lambda)=\alpha_5\varphi_2(\xi_2,\lambda)+\beta_5\varphi_2'(\xi_2,\lambda),\quad \varphi_3'(\xi_2,\lambda)=\alpha_6\varphi_2(\xi_2,\lambda)+\beta_6\varphi_2'(\xi_2,\lambda)\tag{12}$$

Analogously we shall define the solution $\chi_3(x,\lambda)$, $\chi_2(x,\lambda)$ and $\chi_1(x,\lambda)$ by initial conditions

$$\chi_3(b,\lambda)=\lambda\beta_2'+\beta_2,\quad \chi_3'(b,\lambda)=\lambda\beta_1'+\beta_1 \tag{13}$$

$$\chi_2(\xi_2,\lambda)=\frac{\beta_6\chi_3(\xi_2,\lambda)-\beta_5\chi_3'(\xi_2,\lambda)}{\theta_2},\quad \chi_2'(\xi_2,\lambda)=\frac{\alpha_6\chi_3(\xi_2,\lambda)-\alpha_5\chi_3'(\xi_2,\lambda)}{-\theta_2} \tag{14}$$

$$\chi_1(\xi_1,\lambda)=\frac{\beta_4\chi_2(\xi_1,\lambda)-\beta_3\chi_2'(\xi_1,\lambda)}{\theta_1},\quad \chi_1'(\xi_1,\lambda)=\frac{\alpha_4\chi_2(\xi_1,\lambda)-\alpha_3\chi_2'(\xi_1,\lambda)}{-\theta_1} \tag{15}$$

Let us consider the Wronskians

$$\omega_i(\lambda):=W_\lambda(\varphi_i,\chi_i;x):=\varphi_i\chi_i'-\varphi_i'\chi_i,\, x\in\Omega_i(i=\overline{1,3})$$

which are independent of $x\in\Omega_i(i=\overline{1,3})$ and are entire functions, where $\Omega_1=[a,\xi_1)$, $\Omega_2=(\xi_1,\xi_2)$ and $\Omega_3=(\xi_2,b]$. This sort of calculation gives $\omega_3(\lambda)=\theta_2\omega_2(\lambda)=\theta_1\theta_2\omega_1(\lambda)$. Now we may introduce in consideration the characteristic function $\omega(\lambda)$ as $\omega(\lambda):=\omega_1(\lambda)$.

Theorem 6
The eigenvalues of the problem (1)–(7) consist of the zeros of function $\omega(\lambda)$.

Proof
Let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0. Then the function $u_0(x)$ may be represented in the form

$$u_0(x)=\begin{cases} C_1\varphi_1(x,\lambda_0)+C_2\chi_1(x,\lambda_0), x\in[a,\xi_1)\\ C_3\varphi_2(x,\lambda_0)+C_4\chi_2(x,\lambda_0), x\in(\xi_1,\xi_2)\\ C_5\varphi_3(x,\lambda_0)+C_6\chi_3(x,\lambda_0), x\in(\xi_2,b]\end{cases} \tag{16}$$

where at least one of the constants c_i $(i=\overline{1,6})$ is not zero. □

Consider the true function

$$l_v(u_0(x))=0, v=\overline{1,6}$$

as the homogenous system of linear equations in the variables $c_i(i=\overline{1,6})$ and taking into account (10)–(15), it follows that the determinant of this system is

$$\begin{vmatrix} 0 & \omega_1(\lambda_0) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \omega_3(\lambda_0) & 0\\ -\varphi_2(\xi_1,\lambda_0) & -\chi_2(\xi_1,\lambda_0) & \varphi_2(\xi_1,\lambda_0) & \chi_2(\xi_1,\lambda_0) & 0 & 0\\ -\varphi_2'(\xi_1,\lambda_0) & -\chi_2'(\xi_1,\lambda_0) & \varphi_2'(\xi_1,\lambda_0) & \chi_2'(\xi_1,\lambda_0) & 0 & 0\\ 0 & 0 & -\varphi_3(\xi_2,\lambda_0) & -\chi_3(\xi_2,\lambda_0) & \varphi_3(\xi_2,\lambda_0) & \chi_3(\xi_2,\lambda_0)\\ 0 & 0 & -\varphi_3'(\xi_2,\lambda_0) & -\chi_3'(\xi_2,\lambda_0) & \varphi_3'(\xi_2,\lambda_0) & \chi_3'(\xi_2,\lambda_0)\end{vmatrix}=\theta_1^3\theta_2^2\omega(\lambda_0)^4=0$$

Definition 7
The analytic multiplicity of an eigenvalue λ of the problem (1)–(7) is its order as a root of the characteristic equation $\omega(\lambda)=0$.

Definition 8
The geometric multiplicity of an eigenvalue λ of the problem (1)–(7) is the dimension of its eigenspace, i.e. the number of its linearly independent eigenfunctions.

Theorem 9
The eigenvalues of the problem (1)–(7) are analytically simple.

Proof
Let $\lambda=u+iv$. For convenience, set $\varphi=\varphi(x,\lambda)$, $\varphi_{1\lambda}=\frac{\partial\varphi_1}{\partial\lambda}$, $\varphi_{1\lambda}'=\frac{\partial\varphi_1'}{\partial\lambda}$, etc. We differentiate equation $l\chi=\lambda\chi$ with respect to λ and have

$$l\chi_\lambda=\lambda\chi_\lambda+\chi \tag{17}$$

By integration by parts, we get

$$\langle l\chi_\lambda,\varphi\rangle_1-\langle\chi_\lambda,l\varphi\rangle_1=\theta_1\theta_2(\chi_{1\lambda}\overline{\varphi}_1'-\chi_{1\lambda}'\overline{\varphi}_1)|_a^{\xi_1}+\theta_2(\chi_{2\lambda}\overline{\varphi}_2'-\chi_{2\lambda}'\overline{\varphi}_2)|_{\xi_1}^{\xi_2}+(\chi_{3\lambda}\overline{\varphi}_3'-\chi_{3\lambda}'\overline{\varphi}_3)|_{\xi_2}^{b} \tag{18}$$

Substituting (17) and $l\varphi=\lambda\varphi$ into the left side of (18), we have

$$\lambda\langle\chi_\lambda,\varphi\rangle_1+\langle\chi,\varphi\rangle_1-\langle\chi_\lambda,\lambda\varphi\rangle_1=\langle\chi,\varphi\rangle_1+2iv\langle\chi_\lambda,\varphi\rangle_1$$

Moreover

$$\theta_1\theta_2(\chi_{1\lambda}\overline{\varphi}_1'-\chi_{1\lambda}'\overline{\varphi}_1)|_a^{\xi_1}+\theta_2(\chi_{2\lambda}\overline{\varphi}_2'-\chi_{2\lambda}'\overline{\varphi}_2)|_{\xi_1}^{\xi_2}+(\chi_{3\lambda}\overline{\varphi}_3'-\chi_{3\lambda}'\overline{\varphi}_3)\Big|_{\xi_2}^{b}=\theta_1\theta_2(\alpha_1\chi_{1\lambda}(a,\lambda)+\alpha_2\chi_{1\lambda}'(a,\lambda))+(\beta_2'\overline{\varphi}_3'(b,\lambda)-\beta_1'\overline{\varphi}_3(b,\lambda))$$

Note that

$$\omega'(\lambda)=\alpha_1\chi_{1\lambda}(a,\lambda)+\alpha_2\chi_{1\lambda}'(a,\lambda)$$

So, (18) becomes

$$\theta_1\theta_2\omega'(\lambda)=\langle\chi,\varphi\rangle_1+2iv\langle\chi_\lambda,\varphi\rangle_1-(\beta_2'\overline{\varphi}_3'(b,\lambda)-\beta_1'\overline{\varphi}_3(b,\lambda)) \tag{19}$$

Next, let μ be arbitrary zero of $\omega(\lambda)$. By Corollary 4, μ is real. Since

$$\omega(\mu)=\begin{vmatrix}\varphi_1(x,\mu) & \chi_1(x,\mu)\\ \varphi_1'(x,\mu) & \chi_1'(x,\mu)\end{vmatrix}=0$$

We have $\varphi_1(x,\mu)=c_1\chi_1(x,\mu)(c_1\neq 0)$, $\varphi_2(x,\mu)=c_2\chi_2(x,\mu)(c_2\neq 0)$ and $\varphi_3(x,\mu)=c_3\chi_3(x,\mu)(c_3\neq 0)$, where $c_1,c_2,c_3\in\mathbb{C}$. From

$$\varphi_2(\xi_1,\mu)=c_1(\alpha_3\chi_1(\xi_1,\mu)+\beta_3\chi_1'(\xi_1,\mu))=c_1\chi_2(\xi_1,\mu)$$

and

$$\varphi_3(\xi_2,\mu)=c_2(\alpha_5\chi_2(\xi_2,\mu)+\beta_5\chi_2'(\xi_2,\mu))=c_2\chi_3(\xi_2,\mu)$$

we get $c_1=c_2=c_3\neq 0$. Thus, a short calculation, (19) becomes

$$\theta_1\theta_2\omega'(\mu)=\overline{c}_1\left(\frac{\theta_1\theta_2}{a_1^2}\int_a^{\xi_1}|\chi_1(x,\mu)|^2\mathrm{d}x+\frac{\theta_2}{a_2^2}\int_{\xi_1}^{\xi_2}|\chi_2(x,\mu)|^2\mathrm{d}x+\frac{1}{a_3^2}\int_{\xi_2}^{b}|\chi_3(x,\mu)|^2\mathrm{d}x+\rho\right)$$

Here $\theta_1>0$, $\theta_2>0$, $\rho>0$ and $\overline{c}_1>0$, so $\omega'(\mu)\neq 0$. Hence the analytic multiplicity of μ is one. By Definition 7, the proof is completed. □

Theorem 10
All eigenvalues of the problem (1)–(7) are also geometrically simple.

Proof
If f and g are two eigenfunctions for an eigenvalue λ_0 of (1)–(7), then (2) implies that $f(a)=cg(a)$ and $f'(a)=cg'(a)$ for some constant $c\in\mathbb{R}$. By the uniqueness theorem for solutions of ordinary differential equation and the transmission conditions (4)–(7), we have that $f=cg$ on $[a,\xi_1)$, (ξ_1,ξ_2) and $(\xi_2,b]$. Thus, the geometric multiplicity of λ_0 is one. □

Lemma 11
Let $\lambda=s^2$, $s=\sigma+it$. Then the following integral equations hold for $k=0,1$:

$$\frac{d^k}{dx^k}\varphi_1(x,\lambda)=\alpha_2\frac{d^k}{dx^k}\cos\frac{s(x-a)}{a_1}-\frac{a_1\alpha_1}{s}\frac{d^k}{dx^k}\sin\frac{s(x-a)}{a_1}+\frac{1}{a_1 s}\int_a^x\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_1}q(y)\varphi_1(y,\lambda)\mathrm{d}y \tag{20}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_2(x,\lambda)&=(\alpha_3\varphi_1(\xi_1,\lambda)+\beta_3\varphi_1'(\xi_1,\lambda))\frac{d^k}{dx^k}\cos\frac{s(x-\xi_1)}{a_2}+\frac{a_2}{s}(\alpha_4\varphi_1(\xi_1,\lambda)+\beta_4\varphi_1'(\xi_1,\lambda))\frac{d^k}{dx^k}\sin\frac{s(x-\xi_1)}{a_2}\\&\quad+\frac{1}{a_2 s}\int_{\xi_1}^x\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_2}q(y)\varphi_2(y,\lambda)\mathrm{d}y\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_3(x,\lambda)&=(\alpha_5\varphi_2(\xi_2,\lambda)+\beta_5\varphi_2'(\xi_2,\lambda))\frac{d^k}{dx^k}\cos\frac{s(x-\xi_2)}{a_3}+\frac{a_3}{s}(\alpha_6\varphi_2(\xi_2,\lambda)+\beta_6\varphi_2'(\xi_2,\lambda))\frac{d^k}{dx^k}\sin\frac{s(x-\xi_2)}{a_3}\\&\quad+\frac{1}{a_3 s}\int_{\xi_2}^x\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_3}q(y)\varphi_3(y,\lambda)\mathrm{d}y\end{aligned} \tag{22}$$

Proof
Consider $\varphi_1(x,\lambda)$ as the solution of the following non-homogeneous Cauchy problem:

$$\begin{cases}a_1^2u''(x)+s^2u(x)=q(x)\varphi_1(x,\lambda)\\ \varphi_1(a,\lambda)=\alpha_2,\ \varphi_1'(a,\lambda)=-\alpha_1\end{cases}$$

Using the method of constant changing, $\varphi_1(x,\lambda)$ satisfies

$$\varphi_1(x,\lambda)=\alpha_2\cos\frac{s(x-a)}{a_1}-\frac{a_1\alpha_1}{s}\sin\frac{s(x-a)}{a_1}+\frac{1}{a_1 s}\int_a^x \sin\frac{s(x-y)}{a_1}(y)\varphi_1(x,\lambda)dy$$

Then differentiating it with respect to x, we have (20). The proof for (21) and (22) are similar. □

Lemma 12
Let $\lambda=s^2$, $\mathrm{Im}s=t$. Then ~~for $\alpha_2\neq0$~~

$$\frac{d^k}{dx^k}\varphi_1(x,\lambda)=\alpha_2\frac{d^k}{dx^k}\cos\frac{s(x-a)}{a_1}+O(|s|^{k-1}e^{|t|\frac{x-a}{a_1}})$$

$$\frac{d^k}{dx^k}\varphi_2(x,\lambda)=-\frac{\alpha_2\beta_3 s}{a_1}\sin\frac{s(\xi_1-a)}{a_1}\frac{d^k}{dx^k}\cos\frac{s(x-\xi_1)}{a_2}+O\left(|s|^k e^{|t|\left(\frac{\xi_1-a}{a_1}+\frac{x-\xi_1}{a_2}\right)}\right)$$

$$\frac{d^k}{dx^k}\varphi_3(x,\lambda)=\frac{\alpha_2\beta_3\beta_5 s^2}{a_1a_2}\sin\frac{s(\xi_1-a)}{a_1}\sin\frac{s(\xi_2-\xi_1)}{a_2}\frac{d^k}{dx^k}\cos\frac{s(x-\xi_2)}{a_3}+O(|s|^{k+1}e^{|t|(\frac{\xi_1-a}{a_1}+\frac{\xi_2-\xi_1}{a_2}+\frac{x-\xi_2}{a_3})})$$

~~while if $\alpha_2=0$~~

$$\cancel{\frac{d^k}{dx^k}\varphi_1(x,\lambda)=-\frac{a_1\alpha_1}{s}\sin\frac{s(x-a)}{a_1}+O(|s|^{k-2}e^{|t|\frac{x-a}{a_1}})}$$

$$\cancel{\frac{d^k}{dx^k}\varphi_2(x,\lambda)=-\alpha_1\beta_3\cos\frac{s(\xi_1-a)}{a_1}\frac{d^k}{dx^k}\cos\frac{s(x-\xi_1)}{a_2}+O(|s|^{k-1}e^{|t|(\frac{\xi_1-a}{a_1}+\frac{x-\xi_1}{a_2})})}$$

$$\cancel{\frac{d^k}{dx^k}\varphi_3(x,\lambda)=\frac{\alpha_1\beta_3\beta_5}{a_2}s\cos\frac{s(\xi_1-a)}{a_1}\sin\frac{s(\xi_2-\xi_1)}{a_2}\frac{d^k}{dx^k}\cos\frac{s(x-\xi_2)}{a_3}+O(|s|^k e^{|t|(\frac{\xi_1-a}{a_1}+\frac{\xi_2-\xi_1}{a_2}+\frac{x-\xi_2}{a_3})})}$$

$k=0,1$. Each of this asymptotic equalities hold uniformly for x as $|\lambda|\to\infty$.

Theorem 13
Let $\lambda=s^2$, $\mathrm{Im}s=t$. Then for $\alpha_2\neq0$ and $\beta_2'\neq0$, the characteristic function $\omega(\lambda)$ has the following asymptotic representation:

$$\omega(\lambda)=\frac{\alpha_2\beta_2'\beta_3\beta_5 s^5}{a_1a_2a_3\theta_1\theta_2}\sin\frac{s(\xi_1-a)}{a_1}\sin\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(b-\xi_2)}{a_3}+O(|s|^4 e^{|t|(\frac{\xi_1-a}{a_1}+\frac{\xi_2-\xi_1}{a_2}+\frac{b-\xi_2}{a_3})})$$

Proof
The proof is obtained by substituting the asymptotic equalities $(d^k/dx^k)\varphi_3(x,\lambda)$ into the representation

$$\theta_1\theta_2\omega(\lambda)=(\beta_1+\lambda\beta_1')\varphi_3(b,\lambda)-(\beta_2+\lambda\beta_2')\varphi_3'(b,\lambda)$$

□

Theorem 14
Let $\alpha_2\neq0$, $\beta_2'\neq0$, the following asymptotic formulas hold for the eigenvalues and eigenfunctions of the boundary value transmission problem (1)–(7):

$$\sqrt{\lambda_n'}=\frac{a_1(n-1)\pi}{\xi_1-a}+O\left(\frac{1}{n}\right),\quad \sqrt{\lambda_n''}=\frac{a_2(n-1)\pi}{\xi_2-\xi_1}+O\left(\frac{1}{n}\right),\quad \sqrt{\lambda_n'''}=\frac{a_3(n-1)\pi}{b-\xi_2}+O\left(\frac{1}{n}\right)$$

All these asymptotic formulas hold uniformly for x.

Proof
By applying the known Rouche theorem, we can obtain these conclusions (cf. [8, Theorem 2.3]).
~~The conclusions of the other cases $\alpha_2\beta_2'=0$ are omitted, and the proof are similar.~~ □

4. Completeness of eigenfunctions

Theorem 15
The operator A has only point spectrum, i.e. $\sigma(A)=\sigma_p(A)$.

Proof
It suffices to prove that if γ is not an eigenvalue of A, then $\gamma\in\rho(A)$. Here we investigate the equation $(A-\gamma)Y=F\in H$, where $\gamma\in\mathbb{R}$, $F=(f,r)$. □

Consider the initial-value problem

$$\begin{cases} lu-\gamma u=f,\ x\in J \\ \alpha_1 u(a)+\alpha_2 u'(a)=0 \\ u(\xi_1+0)=\alpha_3 u(\xi_1-0)+\beta_3 u'(\xi_1-0) \\ u'(\xi_1+0)=\alpha_4 u(\xi_1-0)+\beta_4 u'(\xi_1-0) \\ u(\xi_2+0)=\alpha_5 u(\xi_2-0)+\beta_5 u'(\xi_2-0) \\ u'(\xi_2+0)=\alpha_6 u(\xi_2-0)+\beta_6 u'(\xi_2-0) \end{cases} \tag{23}$$

Let $u(x)$ be the solution of the equation $lu-\gamma u=0$ satisfying the transmission conditions (4)–(7). In fact

$$u(x)=\begin{cases} u_1(x), & x\in[a,\xi_1) \\ u_2(x), & x\in(\xi_1,\xi_2) \\ u_3(x), & x\in(\xi_2,b] \end{cases}$$

where $u_1(x)$ is the unique solution of the initial-value problem

$$\begin{cases} -a_1^2 u''+q(x)u=\gamma u,\ x\in[a,\xi_1) \\ u(a)=\alpha_2,\ u'(a)=-\alpha_1 \end{cases}$$

$u_2(x)$ is the unique solution of the problem

$$\begin{cases} -a_2^2 u''+q(x)u=\gamma u,\ x\in(\xi_1,\xi_2) \\ u(\xi_1+0)=\alpha_3 u(\xi_1-0)+\beta_3 u'(\xi_1-0) \\ u'(\xi_1+0)=\alpha_4 u(\xi_1-0)+\beta_4 u'(\xi_1-0) \end{cases}$$

and $u_3(x)$ is the unique solution of the problem

$$\begin{cases} -a_3^2 u''+q(x)u=\gamma u,\ x\in(\xi_2,b] \\ u(\xi_2+0)=\alpha_5 u(\xi_2-0)+\beta_5 u'(\xi_2-0) \\ u'(\xi_2+0)=\alpha_6 u(\xi_2-0)+\beta_6 u'(\xi_2-0) \end{cases}$$

Let

$$w(x)=\begin{cases} w_1(x), & x\in[a,\xi_1) \\ w_2(x), & x\in(\xi_1,\xi_2) \\ w_3(x), & x\in(\xi_2,b] \end{cases}$$

be a solution of $lw-\gamma w=f$ satisfying

$$\alpha_1 w(a)-\alpha_2 w'(a)=0$$

$$w(\xi_1+0)=\alpha_3 w(\xi_1-0)+\beta_3 w'(\xi_1-0),\quad w'(\xi_1+0)=\alpha_4 w(\xi_1-0)+\beta_4 w'(\xi_1-0)$$
$$w(\xi_2+0)=\alpha_5 w(\xi_2-0)+\beta_5 w'(\xi_2-0),\quad w'(\xi_2+0)=\alpha_6 w(\xi_2-0)+\beta_6 w'(\xi_2-0)$$

Then, (23) has the general solution

$$y(x)=\begin{cases} du_1+w_1, & x\in[a,\xi_1) \\ du_2+w_2, & x\in(\xi_1,\xi_2) \\ du_3+w_3, & x\in(\xi_2,b] \end{cases} \tag{24}$$

where $d\in\mathbb{C}$.

As γ is not an eigenvalue of (1)–(7), we have

$$\beta_1 u_3(b)-\beta_2 u_3'(b)+\gamma(\beta_1' u_3(b)-\beta_2' u_3'(b))\neq 0 \tag{25}$$

The second component of $(A-\gamma)Y=F$ means the equation

$$\beta_2 y'(b)-\beta_1 y(b)-\gamma(\beta_1' y(b)-\beta_2' y'(b))=r \tag{26}$$

substituting (24) into (26), we get

$$(\beta_2 u_3'(b)-\beta_1 u_3(b)+\gamma\beta_2' u_3'(b)-\gamma\beta_1' u_3(b))d=r+\beta_1 w_3(b)-\beta_2 w_3'(b)+\gamma\beta_1' w_3(b)-\gamma\beta_2' w_3'(b)$$

In view of (25), we know that d is uniquely solvable. So y is uniquely determined.

The above equation show that $(A-\gamma I)^{-1}$ is defined on all of H. We get that $(A-\gamma I)^{-1}$ is bounded by Theorem 3 and the closed Graph Theorem. Thus $\gamma\in\rho(A)$. Hence, $\sigma(A)=\sigma_p(A)$.

Lemma 16
The eigenvalues of the boundary value problem (1)–(7) are bounded below, and they are countably infinite and can cluster only at ∞.

Lemma 17
The operator A has compact resolvents, i.e. for each $\delta\in\mathbb{R}/\sigma_p(A), (A-\delta I)^{-1}$ is compact on H (cf. [4, Theorem 6.3.3]).

By the above Lemmas and the spectral theorem for compact operator, we obtain the following theorem:

Theorem 18
The eigenfunctions of the problem (1)–(7), augmented to become eigenfunctions of A, are complete in H, i.e. if we let $\{\Phi_n=(\varphi_n(x),N'(\varphi_n));n\in\mathbb{N}\}$ be a maximum set of orthonormal eigenfunctions of A, where $\{\varphi_n(x);n\in\mathbb{N}\}$ are eigenfunctions of the problem (1)–(7), then for all $F\in H$, $F=\sum_{n=1}^{\infty}\langle F,\,\Phi_n\rangle\Phi_n$.

5. Green function

Let us consider the following differential equation:

$$-(a(x)u'(x))'+q(x)u(x)-\lambda u(x)=-f(x),\quad x\in J \tag{27}$$

where $J=[a,\xi_1)\cup(\xi_1,\xi_2)\cup(\xi_2,b]$, $a(x)=a_1^2$ for $x\in[a,\xi_1)$, $a(x)=a_2^2$ for $x\in(\xi_1,\xi_2)$ and $a(x)=a_3^2$ for $x\in(\xi_2,b]$, $a_1>0$, $a_2>0$ and $a_3>0$ are given real numbers; together with the eigenparameter-dependent boundary and transmission conditions (2)–(7).

We can represent the general solution (16) of homogeneous differential Equation (1), appropriate to Equation (27). By applying the standard method of variation of constants, we shall search the general solution of the non-homogeneous differential Equation (27) in the form

$$U(x)=\begin{cases} C_1(x,\lambda)\varphi_1(x,\lambda)+C_2(x,\lambda)\chi_1(x,\lambda), & x\in[a,\xi_1)\\ C_3(x,\lambda)\varphi_2(x,\lambda)+C_4(x,\lambda)\chi_2(x,\lambda), & x\in(\xi_1,\xi_2)\\ C_5(x,\lambda)\varphi_3(x,\lambda)+C_6(x,\lambda)\chi_3(x,\lambda), & x\in(\xi_2,b]\end{cases} \tag{28}$$

where the functions $C_i(x,\lambda)$ $(i=\overline{1,6})$ satisfy the linear system of equation

$$\begin{cases} C_1'(x,\lambda)\varphi_1(x,\lambda)+C_2'(x,\lambda)\chi_1(x,\lambda)=0\\ C_1'(x,\lambda)\varphi_1'(x,\lambda)+C_2'(x,\lambda)\chi_1'(x,\lambda)=f(x)\end{cases}$$

for $x\in[a,\xi_1)$,

$$\begin{cases} C_3'(x,\lambda)\varphi_2(x,\lambda)+C_4'(x,\lambda)\chi_2(x,\lambda)=0\\ C_3'(x,\lambda)\varphi_2'(x,\lambda)+C_4'(x,\lambda)\chi_2'(x,\lambda)=f(x)\end{cases}$$

for $x\in(\xi_1,\xi_2)$, and

$$\begin{cases} C_5'(x,\lambda)\varphi_3(x,\lambda)+C_6'(x,\lambda)\chi_3(x,\lambda)=0\\ C_5'(x,\lambda)\varphi_3'(x,\lambda)+C_6'(x,\lambda)\chi_3'(x,\lambda)=f(x)\end{cases}$$

for $x\in(\xi_2,b]$. Because the characteristic function $\omega(\lambda)\neq 0$, the following relations can be easily obtained:

$$C_1(x,\lambda)=\frac{1}{\omega(\lambda)}\int_x^{\xi_1} f\chi_1\,dy+C_1,\quad C_2(x,\lambda)=\frac{1}{\omega(\lambda)}\int_a^x f\varphi_1\,dy+C_2,\quad x\in[a,\xi_1)$$

$$C_3(x,\lambda)=\frac{1}{\omega_2(\lambda)}\int_x^{\xi_2} f\chi_2\,dy+C_3,\quad C_4(x,\lambda)=\frac{1}{\omega_2(\lambda)}\int_{\xi_1}^x f\varphi_2\,dy+C_4,\quad x\in(\xi_1,\xi_2)$$

$$C_5(x,\lambda)=\frac{1}{\omega_3(\lambda)}\int_x^{b} f\chi_3\,dy+C_5,\quad C_6(x,\lambda)=\frac{1}{\omega_3(\lambda)}\int_{\xi_2}^x f\varphi_3\,dy+C_6,\quad x\in(\xi_2,b]$$

Here, $C_i(i=\overline{1,6})$ are arbitrary constants. Substituting the above equations in (28), the general solution of the non-homogeneous differential Equation (27) are obtained as

$$U(x,\lambda)=\begin{cases}\dfrac{\varphi_1(x,\lambda)}{\omega(\lambda)}\displaystyle\int_x^{\xi_1} f\chi_1\,dy+\dfrac{\chi_1(x,\lambda)}{\omega(\lambda)}\displaystyle\int_a^x f\varphi_1\,dy+C_1\varphi_1(x,\lambda)+C_2\chi_1(x,\lambda), & x\in[a,\xi_1)\\ \dfrac{\varphi_2(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_x^{\xi_2} f\chi_2\,dy+\dfrac{\chi_2(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1}^x f\varphi_2\,dy+C_3\varphi_2(x,\lambda)+C_4\chi_2(x,\lambda), & x\in(\xi_1,\xi_2)\\ \dfrac{\varphi_3(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_x^{b} f\chi_3\,dy+\dfrac{\chi_3(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_{\xi_2}^x f\varphi_3\,dy+C_5\varphi_3(x,\lambda)+C_6\chi_3(x,\lambda), & x\in(\xi_2,b]\end{cases}\tag{29}$$

where $C_i(i=\overline{1,6})$ are arbitrary constants. By differentiating (29) we have

$$U'(x,\lambda)=\begin{cases}\dfrac{\varphi_1'(x,\lambda)}{\omega(\lambda)}\displaystyle\int_x^{\xi_1} f\chi_1\,dy+\dfrac{\chi_1'(x,\lambda)}{\omega(\lambda)}\displaystyle\int_a^x f\varphi_1\,dy+C_1\varphi_1'(x,\lambda)+C_2\chi_1'(x,\lambda), & x\in[a,\xi_1)\\ \dfrac{\varphi_2'(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_x^{\xi_2} f\chi_2\,dy+\dfrac{\chi_2'(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1}^x f\varphi_2\,dy+C_3\varphi_2'(x,\lambda)+C_4\chi_2'(x,\lambda), & x\in(\xi_1,\xi_2)\\ \dfrac{\varphi_3'(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_x^{b} f\chi_3\,dy+\dfrac{\chi_3'(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_{\xi_2}^x f\varphi_3\,dy+C_5\varphi_3'(x,\lambda)+C_6\chi_3'(x,\lambda), & x\in(\xi_2,b]\end{cases}\tag{30}$$

By using the system of Equation (29) and the proof process of Theorem 6, the following equalities are obtained for $l_i(U)$, $i=\overline{1,6}$:

$$l_1(U)=C_2\omega(\lambda)\tag{31}$$

$$l_2(U)=C_5\omega_2(\lambda)\tag{32}$$

$$\begin{aligned}l_3(U)=&\frac{\varphi_2(\xi_1+0,\lambda)}{\omega_2(\lambda)}\int_{\xi_1+0}^{\xi_2-0} f\chi_2\,dy-\frac{\chi_2(\xi_1+0,\lambda)}{\omega(\lambda)}\int_a^{\xi_1-0} f\varphi_1\,dy\\&-C_1\varphi_2(\xi_1+0,\lambda)-C_2\chi_2(\xi_1+0,\lambda)+C_3\varphi_2(\xi_1+0,\lambda)+C_4\chi_2(\xi_1+0,\lambda)\end{aligned}\tag{33}$$

$$\begin{aligned}l_4(U)=&\frac{\varphi_2'(\xi_1+0,\lambda)}{\omega_2(\lambda)}\int_{\xi_1+0}^{\xi_2-0} f\chi_2\,dy-\frac{\chi_2'(\xi_1+0,\lambda)}{\omega(\lambda)}\int_a^{\xi_1-0} f\varphi_1\,dy\\&-C_1\varphi_2'(\xi_1+0,\lambda)-C_2\chi_2'(\xi_1+0,\lambda)+C_3\varphi_2'(\xi_1+0,\lambda)+C_4\chi_2'(\xi_1+0,\lambda)\end{aligned}\tag{34}$$

$$\begin{aligned}l_5(U)=&\frac{\varphi_3(\xi_2+0,\lambda)}{\omega_3(\lambda)}\int_{\xi_2+0}^{b} f\chi_3\,dy-\frac{\chi_3(\xi_2+0,\lambda)}{\omega_2(\lambda)}\int_{\xi_1+0}^{\xi_2-0} f\varphi_2\,dy\\&-C_3\varphi_3(\xi_2+0,\lambda)-C_4\chi_3(\xi_2+0,\lambda)+C_5\varphi_3(\xi_2+0,\lambda)+C_6\chi_3(\xi_2+0,\lambda)\end{aligned}\tag{35}$$

$$\begin{aligned}l_6(U)=&\frac{\varphi_3'(\xi_2+0,\lambda)}{\omega_3(\lambda)}\int_{\xi_2+0}^{b} f\chi_3\,dy-\frac{\chi_3'(\xi_2+0,\lambda)}{\omega_2(\lambda)}\int_{\xi_1+0}^{\xi_2-0} f\varphi_2\,dy\\&-C_3\varphi_3'(\xi_2+0,\lambda)-C_4\chi_3'(\xi_2+0,\lambda)+C_5\varphi_3'(\xi_2+0,\lambda)+C_6\chi_3'(\xi_2+0,\lambda)\end{aligned}\tag{36}$$

Because $U(x,\lambda)$ is a solution and $\omega(\lambda)\neq 0$, from boundary condition (2) and equality (31) we have $C_2=0$. Similarly from equality (32) and boundary condition (3) we have $C_5=0$.

On the other hand, by taking into account Equations (35), (36), and transmission conditions, the following linear equation system according to the variables C_1, C_3, C_4 and C_6 are obtained as:

$$\begin{cases}C_1\varphi_2(\xi_1+0,\lambda)-C_3\varphi_2(\xi_1+0,\lambda)-C_4\chi_2(\xi_1+0,\lambda)=\dfrac{\varphi_2(\xi_1+0,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1+0}^{\xi_2-0} f\chi_2\,dy-\dfrac{\chi_2(\xi_1+0,\lambda)}{\omega(\lambda)}\displaystyle\int_a^{\xi_1-0} f\varphi_1\,dy\\ C_1\varphi_2'(\xi_1+0,\lambda)-C_3\varphi_2'(\xi_1+0,\lambda)-C_4\chi_2'(\xi_1+0,\lambda)=\dfrac{\varphi_2'(\xi_1+0,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1+0}^{\xi_2-0} f\chi_2\,dy-\dfrac{\chi_2'(\xi_1+0,\lambda)}{\omega(\lambda)}\displaystyle\int_a^{\xi_1-0} f\varphi_1\,dy\\ C_3\varphi_3(\xi_2+0,\lambda)+C_4\chi_3(\xi_2+0,\lambda)-C_6\chi_3(\xi_2+0,\lambda)=\dfrac{\varphi_3(\xi_2+0,\lambda)}{\omega_3(\lambda)}\displaystyle\int_{\xi_2+0}^{b} f\chi_3\,dy-\dfrac{\chi_3(\xi_2+0,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1+0}^{\xi_2-0} f\varphi_2\,dy\\ C_3\varphi_3'(\xi_2+0,\lambda)+C_4\chi_3'(\xi_2+0,\lambda)-C_6\chi_3'(\xi_2+0,\lambda)=\dfrac{\varphi_3'(\xi_2+0,\lambda)}{\omega_3(\lambda)}\displaystyle\int_{\xi_2+0}^{b} f\chi_3\,dy-\dfrac{\chi_3'(\xi_2+0,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1+0}^{\xi_2-0} f\varphi_2\,dy\end{cases}\tag{37}$$

By using the definitions of solutions $\varphi_i(x,\lambda)$ and $\chi_i(x,\lambda)$ $(i=1,2)$, the following relation is obtained for the determinant of this linear equation system:

$$\begin{vmatrix} \varphi_2(\xi_1+0,\lambda) & -\varphi_2(\xi_1+0,\lambda) & -\chi_2(\xi_1+0,\lambda) & 0 \\ \varphi_2'(\xi_1+0,\lambda) & -\varphi_2'(\xi_1+0,\lambda) & -\chi_2'(\xi_1+0,\lambda) & 0 \\ 0 & \varphi_3(\xi_2+0,\lambda) & \chi_3(\xi_2+0,\lambda) & -\chi_3(\xi_2+0,\lambda) \\ 0 & \varphi_3'(\xi_2+0,\lambda) & \chi_3'(\xi_2+0,\lambda) & -\chi_3'(\xi_2+0,\lambda) \end{vmatrix} = -\omega_2(\lambda)\omega_3(\lambda)$$

As this determinant is different from zero, the solution of (37) is unique. If we solve system (37), we get the following equality for the coefficients C_1, C_3, C_4 and C_6:

$$C_1 = \frac{1}{\omega_2(\lambda)}\int_{\xi_1}^{\xi_2} f\chi_2\,dy + \frac{1}{\omega_3(\lambda)}\int_{\xi_2}^{b} f\chi_3\,dy, \quad C_3 = \frac{1}{\omega_3(\lambda)}\int_{\xi_2}^{b} f\chi_3\,dy$$

$$C_6 = \frac{1}{\omega(\lambda)}\int_{a}^{\xi_1} f\varphi_1\,dy + \frac{1}{\omega_2(\lambda)}\int_{\xi_1}^{\xi_2} f\varphi_2\,dy, \quad C_4 = \frac{1}{\omega(\lambda)}\int_{a}^{\xi_1} f\varphi_1\,dy$$

Finally, by substituting the coefficients $C_i(i=\overline{1,6})$ in (29), the following formulae are obtained for the resolvent $U(x,\lambda)$:

$$\begin{cases} \dfrac{\varphi_1(x,\lambda)}{\omega(\lambda)}\displaystyle\int_x^{\xi_1} f\chi_1\,dy + \dfrac{\chi_1(x,\lambda)}{\omega(\lambda)}\displaystyle\int_a^x f\varphi_1\,dy + \dfrac{\varphi_1(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1}^{\xi_2} f\chi_2\,dy + \dfrac{\varphi_1(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_{\xi_2}^{b} f\chi_3\,dy, & x\in[a,\xi_1) \\ \dfrac{\varphi_2(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_x^{\xi_2} f\chi_2\,dy + \dfrac{\chi_2(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1}^x f\varphi_2\,dy + \dfrac{\varphi_2(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_{\xi_2}^{b} f\chi_3\,dy + \dfrac{\chi_2(x,\lambda)}{\omega(\lambda)}\displaystyle\int_{a}^{\xi_1} f\varphi_1\,dy, & x\in(\xi_1,\xi_2) \\ \dfrac{\varphi_3(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_x^{b} f\chi_3\,dy + \dfrac{\chi_3(x,\lambda)}{\omega_3(\lambda)}\displaystyle\int_{\xi_2}^x f\varphi_3\,dy + \dfrac{\chi_3(x,\lambda)}{\omega(\lambda)}\displaystyle\int_{a}^{\xi_1} f\varphi_1\,dy + \dfrac{\chi_3(x,\lambda)}{\omega_2(\lambda)}\displaystyle\int_{\xi_1}^{\xi_2} f\varphi_2\,dy, & x\in(\xi_2,b] \end{cases} \tag{38}$$

Let

$$\varphi(x) = \begin{cases} \varphi_1(x), & x\in[a,\xi_1) \\ \varphi_2(x), & x\in(\xi_1,\xi_2), \\ \varphi_3(x), & x\in(\xi_2,b] \end{cases} \quad \chi(x) = \begin{cases} \chi_1(x), & x\in[a,\xi_1) \\ \chi_2(x), & x\in(\xi_1,\xi_2) \\ \chi_3(x), & x\in(\xi_2,b] \end{cases}$$

Then

$$U(x,\lambda) = \frac{\varphi(x)}{\omega_i(\lambda)}\int_x^b f\chi\,dy + \frac{\chi(x)}{\omega_i(\lambda)}\int_a^x f\varphi\,dy, \quad i=\overline{1,3} \tag{39}$$

Thus, the resolvent of the boundary-value transmission problem is obtained. We can find the Green function from the resolvent (39). Namely, denoting

$$G(x,y;\lambda) = \begin{cases} \dfrac{\varphi(y,\lambda)\chi(x,\lambda)}{\omega_i(\lambda)}, & a\leqslant y\leqslant x\leqslant b, x\neq\xi_1,\xi_2, \quad y\neq\xi_1,\xi_2 \\ \dfrac{\varphi(x,\lambda)\chi(y,\lambda)}{\omega_i(\lambda)}, & a\leqslant x\leqslant y\leqslant b, x\neq\xi_1,\xi_2, \quad y\neq\xi_1,\xi_2 \end{cases} \quad i=\overline{1,3}$$

We can rewrite the resolvent (39) in the next form

$$U(x,\lambda) = \int_a^b G(x,y;\lambda)f(y)dy$$

Acknowledgements

This work was supported by supported by the National Nature Science Foundation of China (10661008) and the National Nature Science Foundation of Neimongo (200711020102).

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