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Artificial boundary method for two-dimensional Burgers' equation

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Abstract

The numerical solution of the two-dimensional Burgers equation in unbounded domains is considered. By introducing a circular artificial boundary, we consider the initial-boundary problem on the disc enclosed by the artificial boundary. Based on the Cole–Hopf transformation and Fourier series expansion, we obtain the exact boundary condition and a series of approximating boundary conditions on the artificial boundary. Then the original problem is reduced to an equivalent problem on the bounded domain. Furthermore, the stability of the reduced problem is obtained. Finally, the finite difference method is applied to the reduced problem, and some numerical examples are given to demonstrate the feasibility and effectiveness of the approach.

Keywords: Two-dimensional Burgers equation; Artificial boundary conditions; Dirichlet to Neumann method; Unbounded domain; Stability analysis

1. Introduction

The Burgers equation is frequently used as testing ground for flows governed by Navier–Stokes equations [1], which also explains the different processes occurring in a wide range of physical phenomena. Therefore, mathematicians, engineers and physicists are attracted and devoted to the study of the Burgers equations. For analytic solution of the multidimensional cases, Steven Nerney et al. recapitulated the derivation of the solutions to the vector Burgers equation and showed that the generalized Cole–Hopf transformation went through for quite general coordinate systems (refer to [2] and the references therein), Frish et al. [3] extended the pole decomposition to multidimensional Burgers equation with no force term and derived its exact solution with a finite number of time-dependent parameters generated by the algebraic and trigonometric polynomial solutions of the heat equation in $\mathbb{R}^n$. For numerical solutions, Han et al. discussed the one-dimensional Burgers equation on unbounded domain by the nonlinear artificial boundary conditions [4], Nishinari et al. used the ultradiscrete method to derive the cellular automaton (CA) from the two-dimensional Burgers equation and studied some exact solutions of the CA [5]. In this paper, we consider the numerical solution of the Burgers equation with a source term on unbounded domains

$$\mathbf{u}_t = \nabla^2\mathbf{u} + 2(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{F}(x,t), \quad \text{in } \mathbb{R}^2, t > 0, \tag{1.1}$$

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$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \tag{1.2}$$

$$\mathbf{u} \to 0, \quad \text{when } |x| \to +\infty, \tag{1.3}$$

where $\mathbf{u} = (u_1, u_2)$, $\nabla \times \mathbf{u}$, the source term $\mathbf{F}(x, t) = (f_1, f_2)$ and initial data $\mathbf{u}_0(x)$ vanish outside a two-dimensional disc $B_0 = \{x : |x| \le R\}$, namely, the flow field is irrotational outside the disc B_0 and

$$\mathrm{supp}\{\mathbf{F}(x, t)\} \subset B_0 \times [0, T], \qquad \mathrm{supp}\{\mathbf{u}_0(x)\} \subset B_0. \tag{1.4}$$

For the numerical solution of problem (1.1)–(1.3), we need to introduce artificial boundaries to make the computational domain finite, find boundary conditions on the artificial boundaries, and reduce the original problem to an equivalent problem on a bounded domain. Using the Cole–Hopf transformation, we transform problem (1.1) into a heat equation outside the computational domain and find exact boundary conditions on the artificial boundaries by considering the heat equation. Then by inverting the above transformation, the exact boundary conditions on the artificial boundaries of the original problem are presented. This procedure is usually called artificial boundary method, which is the most important and efficient method for numerical solution of PDEs in unbounded domain and has been widely applied to elliptic and harmonic equations [6–10], wave equations [11–14], and the parabolic equation [15–19] etc. The artificial boundary method is also called the natural boundary integral method, or Dirichlet to Neumann mapping method. The exact boundary condition on the artificial boundary is just the natural integral equation, i.e. DtN mapping. One can refer to the related books [20,21] written by Yu. For nonlinear problems, generally speaking, it is difficult to use the artificial boundary condition directly. Recently, some nonlinear equations have been studied in detail in [22–24], where some equations can be linearized outside the artificial boundary, and in [4,25], where the transformation is used.

The brief organization of this article is as follows. In Section 2, we introduce the Cole–Hopf transformation and present exact and approximate artificial boundary conditions of the two-dimensional Burgers' equation. In Section 3, we are devoted to the stability analysis. In Section 4, the finite difference discretizations and some iteration tactics are given. Section 5 is to construct some numerical examples to show the effectiveness of our approach.

2. Artificial boundary conditions

Denote the artificial boundary by $\Gamma = \partial B_0$, the exterior domain by $\Omega_e = \mathbb{R}^2 \setminus B_0$, and the computational domain by $\Omega_i = B_0$. In order to obtain boundary conditions on the artificial boundary Γ, we consider firstly the restriction of the solution $\mathbf{u}$ on the exterior domain Ω_e. On the unbounded domain Ω_e the original problem (1.1)–(1.3) satisfies

$$\mathbf{u}_t = \nabla^2 \mathbf{u} + 2(\mathbf{u} \cdot \nabla)\mathbf{u}, \quad \text{in } \Omega_e \times (0, T] \tag{2.1}$$

$$\nabla \times \mathbf{u} = 0, \quad \text{in } \Omega_e \times (0, T] \tag{2.2}$$

$$\mathbf{u}(x, 0) = 0, \quad \text{in } \Omega_e \tag{2.3}$$

$$\mathbf{u} \to 0, \quad \text{when } |x| \to +\infty. \tag{2.4}$$

Since $\mathbf{u}(x, t)$ is an unknown function, problem (2.1)–(2.4) is an incompletely posed problem. It can not be solved independently. However, if we assume that the boundary values $\mathbf{u}(x, t)|_\Gamma$ are given, then (2.1)–(2.4) is a well-posed problem. Furthermore, by the Cole–Hopf transformation [2], we can reduce the nonlinear Burgers' equation to a linear parabolic equation. In fact, according to the irrotational condition $\nabla \times \mathbf{u} = 0$, we let

$$\omega(x_1, x_2, t) = -\int_{x_1}^{\infty} u_1(\xi, x_2, t)\mathrm{d}\xi = -\int_{x_2}^{\infty} u_2(x_1, \eta, t)\mathrm{d}\eta,$$

then,

$$\omega_t - (\nabla\omega)^2 - \triangle\omega = 0.$$

Let $v = \mathrm{e}^{w} - 1$, then v satisfies

$$v_t = \triangle v \quad \text{in } \Omega_e \times (0, T], \tag{2.5}$$

$$v(x, 0) = 0, \tag{2.6}$$

$$v \to 0, \quad \text{when } |x| \to +\infty, \tag{2.7}$$

$$v|_{x\in\Gamma} = v|_{\Gamma}. \tag{2.8}$$

In the polar coordinate, (2.5)–(2.8) can be written as

$$v_t = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} \quad \text{in } \Omega_e \times (0, T], \tag{2.9}$$

$$v(r, \theta, 0) = 0, \tag{2.10}$$

$$v \to 0, \quad \text{when } r \to +\infty, \tag{2.11}$$

$$v|_{\Gamma} = v(R, \theta, t). \tag{2.12}$$

We try to find the solution $v(r, \theta, t)$ in the form

$$v(r, \theta, t) = v_0(r, t) + \sum_{m=1}^{\infty} a_m(r, t)\cos m\theta + b_m(r, t)\sin m\theta, \tag{2.13}$$

with

$$a_m(t) = \frac{1}{\pi}\int_0^{2\pi} v(R, \theta, t)\cos(m\theta)\mathrm{d}\theta, \qquad b_m(t) = \frac{1}{\pi}\int_0^{2\pi} v(R, \theta, t)\sin(m\theta)\mathrm{d}\theta.$$

Substituting the above equality (2.13) into (2.9), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t} - \frac{\partial^2 v_0}{\partial r^2} - \frac{1}{r}\frac{\partial v_0}{\partial r} + \sum_{m=1}^{\infty}\left[\left(\frac{\partial a_m}{\partial t} - \frac{\partial^2 a_m}{\partial r^2} - \frac{1}{r}\frac{\partial a_m}{\partial r} + \frac{m^2}{r^2}a_m\right)\cos m\theta\right. \\ \left. + \left(\frac{\partial b_m}{\partial t} - \frac{\partial^2 b_m}{\partial r^2} - \frac{1}{r}\frac{\partial b_m}{\partial r} + \frac{m^2}{r^2}b_m\right)\sin m\theta\right] = 0.\end{aligned} \tag{2.14}$$

The Eq. (2.14) holds if and only if the coefficients of the series vanish, hence we have:

$$\frac{\partial v_0}{\partial t} - \frac{\partial^2 v_0}{\partial r^2} - \frac{1}{r}\frac{\partial v_0}{\partial r} = 0, \tag{2.15}$$

$$\frac{\partial a_m}{\partial t} - \frac{\partial^2 a_m}{\partial r^2} - \frac{1}{r}\frac{\partial a_m}{\partial r} + \frac{m^2}{r^2}a_m = 0, \quad m \geq 1, \tag{2.16}$$

$$\frac{\partial b_m}{\partial t} - \frac{\partial^2 b_m}{\partial r^2} - \frac{1}{r}\frac{\partial b_m}{\partial r} + \frac{m^2}{r^2}b_m = 0, \quad m \geq 1. \tag{2.17}$$

Solving the above equations with the corresponding initial and boundary conditions, respectively, we have the following exact boundary conditions on the artificial boundary Γ (refer to [17] and the references therein):

$$\begin{aligned}\frac{\partial v(R, \theta, t)}{\partial r} &= -\frac{1}{2R\sqrt{\pi^3}}\int_0^t\int_0^{2\pi}\frac{\partial v(R, \phi, \tau)}{\partial \tau}\mathrm{d}\phi\frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\ &\quad - \frac{1}{R\sqrt{\pi^3}}\int_0^t\sum_{m=1}^{\infty}\int_0^{2\pi}\frac{\partial v(R, \phi, \tau)}{\partial \tau}\cos m(\phi-\theta)\mathrm{d}\phi\frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\ &\quad - \frac{1}{R\pi}\sum_{m=1}^{\infty} m\int_0^{2\pi} v(R, \phi, t)\cos m(\phi-\theta)\mathrm{d}\phi\end{aligned} \tag{2.18}$$

and

$$\frac{\partial v(R, \theta, t)}{\partial \theta} = \frac{1}{\pi}\sum_{m=1}^{\infty} m\int_0^{2\pi} v(R, \phi, t)\sin m(\phi-\theta)\mathrm{d}\phi, \tag{2.19}$$

with

$$H_m(t) = \frac{4\sqrt{t}}{\sqrt{\pi^3}} \int_0^\infty \frac{\mathrm{e}^{-\mu^2 t}}{J_m^2(\mu R) + Y_m^2(\mu R)} \frac{\mathrm{d}\mu}{\mu}, \tag{2.20}$$

where $J_m(\cdot)$ and $Y_m(\cdot)$ are two independent solutions of Bessel's equation of order m, respectively, $Y_m(\cdot)$ is called Weber's Bessel function of order m. To transform the boundary conditions (2.18) and (2.19) back into the original variables $\mathbf{u}$, we consider firstly

$$\mathrm{e}^{\omega} u_1 = v_{x_1} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x_1}, \qquad v_t = \mathrm{e}^{\omega}\omega_t = \mathrm{e}^{\omega}(u_{1x_1} + u_{2x_2} + u_1^2 + u_2^2), \tag{2.21}$$

in fact, the first term of (2.21) is trivial, for the second term, by the irrotational condition $\nabla \times \mathbf{u} = 0$, i.e. $u_{1x_2} = u_{2x_1}$, we have

$$\begin{aligned}
\omega_t &= -\int_{x_1}^{\infty} u_{1t}(x_1, x_2, t)\mathrm{d}x_1 \\
&= -\int_{x_1}^{\infty} (u_{1x_1x_1} + u_{1x_2x_2} + 2u_1u_{1x_1} + 2u_2u_{1x_2})\mathrm{d}x_1 \\
&= -\int_{x_1}^{\infty} (u_{1x_1x_1} + u_{2x_1x_2} + 2u_1u_{1x_1} + 2u_2u_{2x_1})\mathrm{d}x_1 \\
&= (u_{1x_1} + u_{2x_2} + u_1^2 + u_2^2) \triangleq G(x_1, x_2, t).
\end{aligned} \tag{2.22}$$

Hence on the artificial boundary we have

$$\begin{aligned}
\mathrm{e}^{\omega(R,\theta,t)} u_1(R,\theta,t) &= \Bigg[-\frac{1}{2R\sqrt{\pi^3}} \int_0^t \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\mathrm{d}\phi \frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\
&\quad - \frac{1}{R\sqrt{\pi^3}} \int_0^t \sum_{m=1}^{\infty} \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\cos m(\phi-\theta)\mathrm{d}\phi \frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\
&\quad - \frac{1}{R\pi} \sum_{m=1}^{\infty} m \int_0^{2\pi} (\mathrm{e}^{\omega(R,\phi,t)} - 1)\cos m(\phi-\theta)\mathrm{d}\phi \Bigg] \cos\theta \\
&\quad - \frac{1}{R\pi} \sum_{m=1}^{\infty} m \int_0^{2\pi} (\mathrm{e}^{\omega(R,\phi,t)} - 1)\sin m(\phi-\theta)\mathrm{d}\phi \sin\theta \\
&= \Bigg[-\frac{1}{2R\sqrt{\pi^3}} \int_0^t \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\mathrm{d}\phi \frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\
&\quad - \frac{1}{R\sqrt{\pi^3}} \int_0^t \sum_{m=1}^{\infty} \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\cos m(\phi-\theta)\mathrm{d}\phi \frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\
&\quad - \frac{1}{R\pi} \sum_{m=1}^{\infty} m \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \cos m(\phi-\theta)\mathrm{d}\phi \Bigg] \cos\theta \\
&\quad - \frac{1}{R\pi} \sum_{m=1}^{\infty} m \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \sin m(\phi-\theta)\mathrm{d}\phi \sin\theta
\end{aligned} \tag{2.23}$$

noting that

$$\int_0^{2\pi} \sin m(\phi-\theta)\mathrm{d}\phi = 0 \quad \text{and} \quad \int_0^{2\pi} \cos m(\phi-\theta)\mathrm{d}\phi = 0.$$

Since the function $\omega(R,\theta,t)$ contains an infinite integral over Ω_e in the above boundary condition (2.23), it is rather difficult to be used directly for computation. $\omega(R,\theta,t)$ can be greatly simplified as long as we integrate the both sides of (2.22) term-by-term with respect to time t, namely,

$$\omega(R,\theta,t) = \int_0^t G(R,\theta,\tau)\mathrm{d}\tau$$
$$= \int_0^t (u_{1r}(R,\theta,\tau)\cos\theta - u_{1\theta}(R,\theta,\tau)\frac{\sin\theta}{R} + u_{2r}\sin\theta$$
$$+ u_{2\theta}\frac{\cos\theta}{R} + u_1(R,\theta,\tau)^2 + u_2(R,\theta,\tau)^2)\mathrm{d}\tau.$$

Thus we have the first artificial condition

$$\begin{aligned} u_1(R,\theta,t) = \mathrm{e}^{-\omega(R,\theta,t)} \Bigg[& -\frac{1}{2R\sqrt{\pi^3}} \int_0^t \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\mathrm{d}\phi \frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\ & - \frac{1}{R\sqrt{\pi^3}} \int_0^t \sum_{m=1}^{\infty} \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\cos m(\phi-\theta)\mathrm{d}\phi \frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\ & - \frac{1}{R\pi} \sum_{m=1}^{\infty} m \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \cos m(\phi-\theta)\mathrm{d}\phi \Bigg] \cos\theta \\ & - \frac{1}{R\pi} \mathrm{e}^{-\omega(R,\theta,t)} \sum_{m=1}^{\infty} m \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \sin m(\phi-\theta)\mathrm{d}\phi \sin\theta \\ \triangleq\ & \kappa_{1\infty}(u_1(R,\cdot,\cdot), u_2(R,\cdot,\cdot), u_{1r}(R,\cdot,\cdot), u_{1\theta}(R,\cdot,\cdot), u_{2r}(R,\cdot,\cdot), u_{2\theta}(R,\cdot,\cdot))(\theta,t). \end{aligned} \tag{2.24}$$

Then we consider

$$\mathrm{e}^{\omega} u_2 = v_{x_2} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x_2}, \qquad v_t = \mathrm{e}^{\omega}\omega_t = \mathrm{e}^{\omega}(u_{1x_1} + u_{2x_2} + u_1^2 + u_2^2), \tag{2.25}$$

by the same argument as obtaining $u_1(R,\theta,t)$, we have the second artificial condition

$$\begin{aligned} u_2(R,\theta,t) = \mathrm{e}^{-\omega(R,\theta,t)} \Bigg[& -\frac{1}{2R\sqrt{\pi^3}} \int_0^t \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\mathrm{d}\phi \frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\ & - \frac{1}{R\sqrt{\pi^3}} \int_0^t \sum_{m=1}^{\infty} \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R,\phi,\tau)\cos m(\phi-\theta)\mathrm{d}\phi \frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \\ & - \frac{1}{R\pi} \sum_{m=1}^{\infty} m \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \cos m(\phi-\theta)\mathrm{d}\phi \Bigg] \sin\theta \\ & + \frac{1}{R\pi} \mathrm{e}^{-\omega(R,\theta,t)} \sum_{m=1}^{\infty} m \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \sin m(\phi-\theta)\mathrm{d}\phi \cos\theta \\ \triangleq\ & \kappa_{2\infty}(u_1(R,\cdot,\cdot), u_2(R,\cdot,\cdot), u_{1r}(R,\cdot,\cdot), u_{1\theta}(R,\cdot,\cdot), u_{2r}(R,\cdot,\cdot), u_{2\theta}(R,\cdot,\cdot))(\theta,t). \end{aligned} \tag{2.26}$$

On the other hand, By using the relationships between the partial derivatives of v with respect to the pole coordinates r,θ and the Descartes coordination x_1, x_2, we have

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial v}{\partial x_2}\frac{\partial x_2}{\partial r}, \qquad \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial v}{\partial x_2}\frac{\partial x_2}{\partial \theta}. \tag{2.27}$$

Substituting (2.18), (2.19), (2.21) and (2.25) into (2.27), we obtain the equivalent conditions

$$
\begin{aligned}
&u_1(R,\theta,t)\cos\theta+u_2(R,\theta,t)\sin\theta=\left[-\frac{1}{2R\sqrt{\pi^3}}\int_0^t\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,\tau)}G(R,\phi,\tau)\mathrm{d}\phi\frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau\right.\\
&\quad-\frac{1}{R\sqrt{\pi^3}}\int_0^t\sum_{m=1}^{\infty}\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,\tau)}G(R,\phi,\tau)\cos m(\phi-\theta)\mathrm{d}\phi\frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau\\
&\quad\left.-\frac{1}{R\pi}\sum_{m=1}^{\infty}m\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,t)}\cos m(\phi-\theta)\mathrm{d}\phi\right]\mathrm{e}^{-\omega(R,\theta,t)}\\
&\quad+\frac{1}{\pi}\sum_{m=1}^{\infty}m\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,t)-\omega(R,\theta,t)}\sin m(\phi-\theta)\mathrm{d}\phi\\
&\triangleq\kappa'_{1\infty}(u_1(R,\cdot,\cdot),u_2(R,\cdot,\cdot),u_{1r}(R,\cdot,\cdot),u_{1\theta}(R,\cdot,\cdot),u_{2r}(R,\cdot,\cdot),u_{2\theta}(R,\cdot,\cdot))(\theta,t).
\end{aligned}\tag{2.28}
$$

$$
\begin{aligned}
&-Ru_1(R,\theta,t)\sin\theta+Ru_2(R,\theta,t)\cos\theta=\left[-\frac{1}{2R\sqrt{\pi^3}}\int_0^t\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,\tau)}G(R,\phi,\tau)\mathrm{d}\phi\frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau\right.\\
&\quad-\frac{1}{R\sqrt{\pi^3}}\int_0^t\sum_{m=1}^{\infty}\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,\tau)}G(R,\phi,\tau)\cos m(\phi-\theta)\mathrm{d}\phi\frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau\\
&\quad\left.-\frac{1}{R\pi}\sum_{m=1}^{\infty}m\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,t)}\cos m(\phi-\theta)\mathrm{d}\phi\right]\mathrm{e}^{-\omega(R,\theta,t)}\\
&\quad+\frac{1}{\pi}\sum_{m=1}^{\infty}m\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,t)-\omega(R,\theta,t)}\sin m(\phi-\theta)\mathrm{d}\phi\\
&\triangleq\kappa'_{2\infty}(u_1(R,\cdot,\cdot),u_2(R,\cdot,\cdot),u_{1r}(R,\cdot,\cdot),u_{1\theta}(R,\cdot,\cdot),u_{2r}(R,\cdot,\cdot),u_{2\theta}(R,\cdot,\cdot))(\theta,t).
\end{aligned}\tag{2.29}
$$

Furthermore, taking the first few terms of the above summations, we obtain a series of approximating artificial boundary conditions on Γ:

$$
\begin{aligned}
u_1(R,\theta,t)&=\mathrm{e}^{-\omega(R,\theta,t)}\left[-\frac{1}{2R\sqrt{\pi^3}}\int_0^t\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,\tau)}G(R,\phi,\tau)\mathrm{d}\phi\frac{H_0(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau\right.\\
&\quad-\frac{1}{R\sqrt{\pi^3}}\int_0^t\sum_{m=1}^{M}\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,\tau)}G(R,\phi,\tau)\cos m(\phi-\theta)\mathrm{d}\phi\frac{H_m(t-\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau\\
&\quad\left.-\frac{1}{R\pi}\sum_{m=1}^{M}m\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,t)}\cos m(\phi-\theta)\mathrm{d}\phi\right]\cos\theta\\
&\quad-\frac{1}{R\pi}\mathrm{e}^{-\omega(R,\theta,t)}\sum_{m=1}^{M}m\int_0^{2\pi}\mathrm{e}^{\omega(R,\phi,t)}\sin m(\phi-\theta)\mathrm{d}\phi\sin\theta\\
&\triangleq\kappa_{1M}(u_1(R,\cdot,\cdot),u_2(R,\cdot,\cdot),u_{1r}(R,\cdot,\cdot),u_{1\theta}(R,\cdot,\cdot),u_{2r}(R,\cdot,\cdot),u_{2\theta}(R,\cdot,\cdot))(\theta,t).
\end{aligned}\tag{2.30}
$$

For brevity, the approximating conditions are written as

$$
\begin{aligned}
&u_1(R,\theta,t)=\kappa_{1M}(R,\cdot,\cdot)(\theta,t),\\
&u_2(R,\theta,t)=\kappa_{2M}(R,\cdot,\cdot)(\theta,t),\\
&u_1(R,\theta,t)\cos\theta+u_2(R,\theta,t)\sin\theta=\kappa'_{1M}(R,\cdot,\cdot)(\theta,t),\\
&-Ru_1(R,\theta,t)\sin\theta+Ru_2(R,\theta,t)\cos\theta=\kappa'_{2M}(R,\cdot,\cdot)(\theta,t).
\end{aligned}
$$

Let $\mathbf{n}=(\cos\theta,\sin\theta)$ and $\mathbf{s}=(-\sin\theta,\cos\theta)$ denote the normal and tangential vectors along the artificial boundary Γ, respectively, we have

$$
\mathbf{u}|_{\Gamma}=(\kappa_{1M}(R,\cdot,\cdot)(\theta,t),\kappa_{2M}(R,\cdot,\cdot)(\theta,t))\tag{2.31}
$$

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma} = \kappa'_{1M}(R, \cdot, \cdot)(\theta, t), \tag{2.32}$$

$$R\mathbf{u} \cdot \mathbf{s}|_{\Gamma} = \kappa'_{2M}(R, \cdot, \cdot)(\theta, t). \tag{2.33}$$

It is obvious that when $M \longrightarrow +\infty$, the approximating boundary conditions are exact. By the truncated boundary conditions (2.31)–(2.33), the original problem (1.1)–(1.3) is reduced to the following approximation problem:

$$\mathbf{u}_t = \nabla^2 \mathbf{u} + 2(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{F}(x, t), \quad \text{in } \Omega_i, t > 0, \tag{2.34}$$

$$\mathbf{u}(r, \theta, 0) = \mathbf{u}_0(r, \theta) \quad \text{in } \Omega_i, \tag{2.35}$$

$$\mathbf{u}|_{\Gamma} = (\kappa_{1M}(R, \cdot, \cdot)(\theta, t), \kappa_{2M}(R, \cdot, \cdot)(\theta, t)) \quad \text{or} \tag{2.36}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma} = \kappa'_{1M}(R, \cdot, \cdot)(\theta, t), \qquad R\mathbf{u} \cdot \mathbf{s}|_{\Gamma} = \kappa'_{2M}(R, \cdot, \cdot)(\theta, t). \tag{2.37}$$

3. Stability analysis

For $0 \le t \le T$ we consider the stability analysis of Burgers equation with a source term on computational domains

$$\mathbf{u}_t = \nabla^2 \mathbf{u} + 2(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{F}(x, t), \quad \text{in } \Omega_i \times (0, T], \tag{3.1}$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \tag{3.2}$$

$$u_1|_{\Gamma} = g_1, \qquad u_2|_{\Gamma} = g_2. \tag{3.3}$$

Multiply the system (3.1) by (u_1, u_2), integrate over the domain $\Omega_i \times [0, t]$, respectively, then simplify by using Green's theorem we get

$$\begin{aligned} \frac{1}{2}\|u_1\|^2_{\Omega_i} &= \frac{1}{2}\|u_{10}\|^2_{\Omega_i} + \int_0^t \int_{\Gamma} \frac{\partial u_1}{\partial n} g_1 - |u_1|^2_{\Omega_i \times [0,t]} \\ &\quad + \frac{2}{3}\int_0^t \int_{\Gamma} u_1^3 \mathrm{d}x_2 + 2\int_0^t \int_{\Omega_i} u_1 u_2 u_{1x_2} + \int_0^t \int_{\Omega_i} f_1 u_1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{1}{2}\|u_2\|^2_{\Omega_i} &= \frac{1}{2}\|u_{20}\|^2_{\Omega_i} + \int_0^t \int_{\Gamma} \frac{\partial u_2}{\partial n} g_2 - |u_2|^2_{\Omega_i \times [0,t]} \\ &\quad + \frac{2}{3}\int_0^t \int_{\Gamma} u_2^3 \mathrm{d}x_1 + 2\int_0^t \int_{\Omega_i} u_1 u_2 u_{2x_1} + \int_0^t \int_{\Omega_i} f_2 u_2, \end{aligned} \tag{3.5}$$

where $\mathbf{u}_0(x) = (u_{10}, u_{20})$, here and below, $\|\cdot\|_D$ is the standard L^2-norm and

$$\|v\|^2_{\Omega_i \times [0,T]} = \int_0^T \int_{\Omega_i} v^2(x, y, t)\mathrm{d}x\mathrm{d}y\mathrm{d}t = \int_0^T \int_{\Omega_i} v^2.$$

$$|v|^2_{\Omega_i \times [0,T]} = \int_0^T \int_{\Omega_i} \left(v_x^2(x, y, t) + v_y^2(x, y, t)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t = \int_0^T \int_{\Omega_i} \left(v_x^2 + v_y^2\right).$$

In order to simplify (3.4) and (3.5), we introduce an auxiliary problem on unbounded domain Ω_e:

$$\mathbf{u}_t = \nabla^2 \mathbf{u} + 2(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{F}(x, t), \quad \text{in } \Omega_e \times (0, T], \tag{3.6}$$

$$\mathbf{u}(x, 0) = (u_{10}, u_{20}) = 0, \tag{3.7}$$

$$u_1|_{\Gamma} = g_1, \qquad u_2|_{\Gamma} = g_2. \tag{3.8}$$

Multiply Eq. (3.6) by (u_1, u_2), respectively, integrate over $\Omega_e \times [0, t]$, and integrate by parts, we obtain

$$\frac{1}{2}\|u_1\|^2_{\Omega_e} = -\int_0^t \int_{\Gamma} \frac{\partial u_1}{\partial n} g_1 - |u_1|^2_{\Omega_e \times [0,t]} - \frac{2}{3}\int_0^t \int_{\Gamma} u_1^3 \mathrm{d}x_2 + 2\int_0^t \int_{\Omega_e} u_1 u_2 u_{1x_2}, \tag{3.9}$$

$$\frac{1}{2}\|u_2\|^2_{\Omega_e} = -\int_0^t \int_{\Gamma} \frac{\partial u_2}{\partial n} g_2 - |u_2|^2_{\Omega_e \times [0,t]} - \frac{2}{3}\int_0^t \int_{\Gamma} u_2^3 \mathrm{d}x_1 + 2\int_0^t \int_{\Omega_e} u_1 u_2 u_{2x_1}. \tag{3.10}$$

Combine (3.4), (3.5), (3.9) and (3.10) together, we have

$$\begin{aligned}\frac{1}{2}\|u_1\|^2_{\mathbb{R}^2} &= \frac{1}{2}\|u_{10}\|^2_{\Omega_i} - |u_1|^2_{\mathbb{R}^2\times[0,t]} + 2\int_0^t\int_{\mathbb{R}^2} u_1u_2u_{1x_2} + \int_0^t\int_{\Omega_i} f_1u_1\\ &\le \frac{1}{2}\|u_{10}\|^2_{\Omega_i} + \int_0^t\int_{\mathbb{R}^2}(u_1u_2)^2 + \int_0^t\int_{\Omega_i}\frac{1}{2}(f_1^2+u_1^2),\end{aligned}\tag{3.11}$$

$$\begin{aligned}\frac{1}{2}\|u_2\|^2_{\mathbb{R}^2} &= \frac{1}{2}\|u_{20}\|^2_{\Omega_i} - |u_2|^2_{\mathbb{R}^2\times[0,t]} + 2\int_0^t\int_{\mathbb{R}^2} u_1u_2u_{2x_1} + \int_0^t\int_{\Omega_i} f_2u_2\\ &\le \frac{1}{2}\|u_{20}\|^2_{\Omega_i} + \int_0^t\int_{\mathbb{R}^2}(u_1u_2)^2 + \int_0^t\int_{\Omega_i}\frac{1}{2}(f_2^2+u_2^2).\end{aligned}\tag{3.12}$$

If the value $|u_1u_2|$ is bounded in the domain $\mathbb{R}^2$, i.e., there exists a constant C such that $|u_1u_2| \le C$. From (3.11) and (3.12) we derive

$$\frac{1}{2}\|\mathbf{u}\|^2_{\mathbb{R}^2} \le \frac{1}{2}\|\mathbf{u}_0\|^2_{\Omega_i} + \left(C+\frac{3}{2}\right)\|\mathbf{u}\|^2_{\mathbb{R}^2\times[0,t]} + \frac{1}{2}\int_0^t\int_{\Omega_i}(f_1^2+f_2^2).\tag{3.13}$$

Using the abbreviations

$$y(t) = \int_0^t\int_{\mathbb{R}^2}(u_1^2+u_2^2) = \|\mathbf{u}\|^2_{\mathbb{R}^2\times[0,t]}, \qquad \phi(t) = \|\mathbf{u}_0\|_{\Omega_i} + \int_0^t\int_{\Omega_i}(f_1^2+f_2^2),$$

we have the differential inequality

$$y'(t) \le (2C+3)y(t) + \phi(t), \quad 0 \le t \le T,$$

Now **Gronwall's Lemma** [26], formulated next, allows us to estimate the above $y(t)$.

Lemma 1. *Suppose that* $y \in C^1[0,T]$, $\psi \in C[0,T]$ *satisfy*

$$y'(t) \le cy(t) + \psi(t), \quad 0 \le t \le T,$$

for some $c \ge 0$. *Then*

$$y(t) \le \mathrm{e}^{ct}\left\{y(0) + \int_0^t|\psi(\tau)|\mathrm{d}\tau\right\}, \quad 0 \le t \le T.$$

Using Lemma 1 and noting that $\|\mathbf{u}\|^2_{\Omega_i\times[0,t]} \le \|\mathbf{u}\|^2_{\mathbb{R}^2\times[0,t]}$ we obtain

Theorem 1. *Assume that* $\mathbf{u}$ *solves the problem* (3.1)–(3.3)*,* $|u_1u_2| \le C$*, and* $\mathbf{u} \in L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$*, then we have*

$$\|\mathbf{u}\|^2_{\Omega_i\times[0,t]} \le \mathrm{e}^{(2C+3)t}\left(\|\mathbf{u}_0\|^2_{\Omega_i} + \int_0^t\phi(\tau)\mathrm{d}\tau\right).\tag{3.14}$$

4. Numerical approximation

In this section we discuss the numerical approximation of the reduced problem (2.34)–(2.37) by Crank–Nicholson scheme, which is unconditionally stable. In the computational domain $[a,R]\times[0,2\pi]$, let $\Delta r = (R-a)/I$ and $\Delta\theta = 2\pi/J$ be the spatial mesh sizes in r and θ, respectively, and let $\Delta t = T/N$ be the time step, where I, J and N are positive integers. Let the grid points and temporal mesh points be

$$r_i = a + i\Delta r, \qquad \theta_j = j\Delta\theta, \quad t_n = n\Delta t,$$

where $i = 0,1,\ldots,I$, $j = 0,1,\ldots,J$, $n = 0,1,\ldots,N$.

For brevity, denote $\mathbf{u} = (u, v)$ and denote the approximation of $u(r_i, \theta_j, t_n)$, $v(r_i, \theta_j, t_n)$ by u_{ij}^n, v_{ij}^n. By the second-order implicit Crank–Nicolson scheme, we have the following approximation of the two-dimensional Burgers equation:

$$\begin{aligned}
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} &= \frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta r^2} + \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{2r_i \Delta r} + \frac{u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}}{r_i^2 \Delta\theta^2} \\
&\quad + 2\left[u_{i,j}^{n+\frac{1}{2}} \left(\cos\theta_j \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r} - \frac{\sin\theta_j}{r_i} \frac{u_{i,j+1}^{n+\frac{1}{2}} - u_{i,j-1}^{n+\frac{1}{2}}}{2\Delta\theta} \right) \right. \\
&\quad \left. + \ v_{ij}^{n+\frac{1}{2}} \left(\sin\theta_j \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r} + \frac{\cos\theta_j}{r_i} \frac{u_{i,j+1}^{n+\frac{1}{2}} - u_{i,j-1}^{n+\frac{1}{2}}}{2\Delta\theta} \right) \right] + f_1(r_i, \theta_j, t_{n+\frac{1}{2}}), \\
\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta t} &= \frac{v_{i+1,j}^{n+\frac{1}{2}} - 2v_{i,j}^{n+\frac{1}{2}} + v_{i-1,j}^{n+\frac{1}{2}}}{\Delta r^2} + \frac{v_{i+1,j}^{n+\frac{1}{2}} - v_{i-1,j}^{n+\frac{1}{2}}}{2r_i \Delta r} + \frac{v_{i,j+1}^{n+\frac{1}{2}} - 2v_{i,j}^{n+\frac{1}{2}} + v_{i,j-1}^{n+\frac{1}{2}}}{r_i^2 \Delta\theta^2} \\
&\quad + 2\left[u_{i,j}^{n+\frac{1}{2}} \left(\cos\theta_j \frac{v_{i+1,j}^{n+\frac{1}{2}} - v_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r} - \frac{\sin\theta_j}{r_i} \frac{v_{i,j+1}^{n+\frac{1}{2}} - v_{i,j-1}^{n+\frac{1}{2}}}{2\Delta\theta} \right) \right. \\
&\quad \left. + \ v_{ij}^{n+\frac{1}{2}} \left(\sin\theta_j \frac{v_{i+1,j}^{n+\frac{1}{2}} - v_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r} + \frac{\cos\theta_j}{r_i} \frac{v_{i,j+1}^{n+\frac{1}{2}} - v_{i,j-1}^{n+\frac{1}{2}}}{2\Delta\theta} \right) \right] + f_2(r_i, \theta_j, t_{n+\frac{1}{2}}),
\end{aligned}$$

with $i = 1, \ldots, I$, $j = 1, \ldots, J$ and the initial and boundary conditions

$$\begin{aligned}
&u_{i,j}^0 = u_0(r_i, \theta_j, 0), \qquad u_{a,j}^n = u_0(a, \theta_j, t_n), \qquad u_{i,0}^n = u_{i,J}^n, \\
&v_{i,j}^0 = v_0(r_i, \theta_j, 0), \qquad v_{a,j}^n = v_0(a, \theta_j, t_n), \qquad v_{i,0}^n = u_{i,J}^n,
\end{aligned}$$

where

$$u_{i,j}^{n+\frac{1}{2}} = \frac{1}{2}(u_{i,j}^{n+1} + u_{i,j}^n), \qquad v_{i,j}^{n+\frac{1}{2}} = \frac{1}{2}(v_{i,j}^{n+1} + v_{i,j}^n).$$

Obviously, the above systems of equations cannot be solved uniquely since the equations are less than the unknowns. Thus the two artificial boundary conditions are introduced to make the systems complete. Since the artificial boundary conditions are nonlocal and very complicated, we need to deal with them carefully. On the artificial boundary Γ, we use the trapezoid formula to discretize the integrals and manipulate the $\omega(R, \phi, \tau)$ as follows which is very feasible, one can refer to [4,25] for indications:

$$\begin{aligned}
&\int_0^t \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R, \phi, \tau) \mathrm{d}\phi \frac{H_0(t - \tau)}{\sqrt{t - \tau}} \mathrm{d}\tau = \sum_{l=0}^{n} \sum_{s=0}^{J-1} \frac{\Delta y}{2} (\mathrm{e}^{\omega_{I,s}^{l+1}} G_{I,s}^{l+1} + \mathrm{e}^{\omega_{I,s}^{l}} G_{I,s}^{l}) \int_{t_l}^{t_{l+1}} \frac{H_0(t_{n+1} - \tau)}{\sqrt{t_{n+1} - \tau}} \mathrm{d}\tau, \\
&\int_0^t \int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,\tau)} G(R, \phi, \tau) \cos m(\phi - \theta) \mathrm{d}\phi \frac{H_m(t - \tau)}{\sqrt{t - \tau}} \mathrm{d}\tau \\
&\quad = \sum_{l=0}^{n} \sum_{s=0}^{J-1} \frac{1}{2\Delta y m^2} (\mathrm{e}^{\omega_{I,s}^{l+1}} G_{I,s}^{l+1} + \mathrm{e}^{\omega_{I,s}^{l}} G_{I,s}^{l}) [2\cos m(\theta_s - \theta_j) \\
&\qquad - \cos m(\theta_{s+1} - \theta_j) - \cos m(\theta_{s-1} - \theta_j)] \int_{t_l}^{t_{l+1}} \frac{H_m(t_{n+1} - \tau)}{\sqrt{t_{n+1} - \tau}} \mathrm{d}\tau, \\
&\int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \cos m(\phi - \theta) \mathrm{d}\phi = \sum_{s=0}^{J-1} \frac{1}{\Delta y m^2} \mathrm{e}^{\omega_{I,s}^{n+1}} [2\cos m(\theta_s - \theta_j) \\
&\qquad - \cos m(\theta_{s+1} - \theta_j) - \cos m(\theta_{s-1} - \theta_j)],
\end{aligned}$$

$$\int_0^{2\pi} \mathrm{e}^{\omega(R,\phi,t)} \sin m(\phi-\theta)\mathrm{d}\phi = \sum_{s=0}^{J-1} \frac{1}{\Delta y m^2} \mathrm{e}^{\omega_{I,s}^{n+1}} [2\sin m(\theta_s-\theta_j) - \sin m(\theta_{s+1}-\theta_j) - \sin m(\theta_{s-1}-\theta_j)],$$

where

$$\omega_{I,s}^{n+1} = \omega_{I,s}^{n} + \frac{\Delta t}{2}(G_{I,s}^{n+1} + G_{I,s}^{n})$$

and

$$G_{I,s}^{l} = \frac{u_{I+1,s}^{l} - u_{I-1,s}^{l}}{2\Delta r}\cos\phi_s - \frac{u_{I,s+1}^{l} - u_{I,s-1}^{l}}{2R\Delta\theta}\sin\phi_s + \frac{v_{I+1,s}^{l} - v_{I-1,s}^{l}}{2\Delta r}\sin\phi_s + \frac{v_{I,s+1}^{l} - u_{I,s-1}^{l}}{2R\Delta\theta}\cos\phi_s + (u_{I,s}^{l})^2 + (v_{I,s}^{l})^2.$$

The above scheme is implicit, we need to use an iterative method to solve it numerically. Some strategies are given to iterate the nonlinear term as references by

$$u_{i,j}^{n+\frac{1}{2}}\cos\theta_j \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r} = (u_{i,j}^{n+\frac{1}{2}})^{(k)} \left(\cos\theta_j \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r}\right)^{(k+1)},$$

$$v_{ij}^{n+\frac{1}{2}}\sin\theta_j \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r} = (v_{ij}^{n+\frac{1}{2}})^{(k)} \left(\sin\theta_j \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{2\Delta r}\right)^{(k+1)},$$

where the superscript k denotes the kth iteration to solve the nonlinear difference equations at each time step. The initial iteration is given as $(u_{i,j}^{n+1})^{(0)} = u_{i,j}^{n}$, $(v_{i,j}^{n+1})^{(0)} = v_{i,j}^{n}$. We can use the same strategy to cope with the iteration on the artificial boundary by

$$\mathrm{e}^{\omega_{I,s}^{n+1}} G_{I,s}^{n+1} = (\mathrm{e}^{\omega_{I,s}^{n+1}})^{(k)} (G_{I,s}^{n+1})^{(k+1)}.$$

The last difficulty is to handle the integral kernel

$$\int_{t_l}^{t_{l+1}} \frac{H_m(\tau)}{\sqrt{\tau}}\mathrm{d}\tau = \frac{4}{\sqrt{\pi^3}} \int_0^{\infty} \frac{\int_{t_l}^{t_{l+1}} \mathrm{e}^{-\mu^2\tau}\mathrm{d}\tau}{J_m^2(\mu R) + Y_m^2(\mu R)} \frac{\mathrm{d}\mu}{\mu},$$

which decays so slow that it has a great influence on the computational efficiency. However the integral is independent of the variables u and v, hence we can make some tables first to enhance the computational efficiency before starting the numerical computing.

5. Numerical examples

In order to demonstrate the effectiveness of the artificial boundary conditions given in this paper, we present some numerical examples in this section.

Example 5.1. Consider the following initial boundary value problem:

$$\mathbf{u}_t = \nabla^2\mathbf{u} + 2(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{F}, \quad \text{in } \Omega_i,\ t > 0,$$
$$\mathbf{u}(a,\theta,t) = \mathbf{g}(\theta,t),$$
$$\mathbf{u}(r,\theta,0) = 0,$$

with $a = 2$, $\mathbf{F} = 0$ and

$$g_1(\theta,t) = -\frac{\mathrm{e}^{-\frac{(a\cos\theta - x_0)^2 + (a\sin\theta - y_0)^2}{4t}}(a\cos\theta - x_0)}{(2t\mathrm{e}^{-\frac{(a\cos\theta - x_0)^2 + (a\sin\theta - y_0)^2}{4t}} + 2t^2)},$$

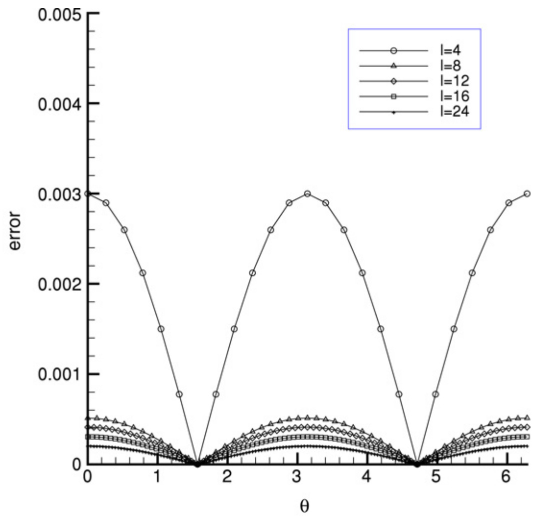

Fig. 1. Errors of the difference solution $|u(R,\theta,1) - u_h(R,\theta,1)|$ with $x_0 = 0$, $y_0 = 0$, $M = 4$ and $t = 1$.

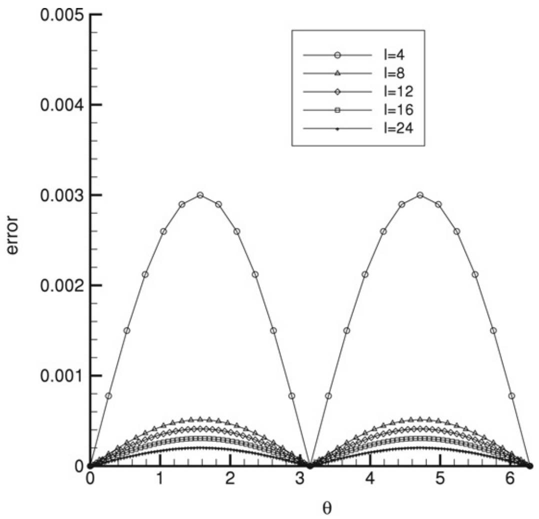

Fig. 2. Errors of the difference solution $|v(R,\theta,1) - v_h(R,\theta,1)|$ with $x_0 = 0$, $y_0 = 0$, $M = 4$ and $t = 1$.

$$g_2(\theta,t) = -\frac{\mathrm{e}^{-\frac{(a\cos\theta - x_0)^2 + (a\sin\theta - y_0)^2}{4t}}(a\sin\theta - y_0)}{(2t\mathrm{e}^{-\frac{(a\cos\theta - x_0)^2 + (a\sin\theta - y_0)^2}{4t}} + 2t^2)}.$$

The exact solutions of the problem are

$$u(r,\theta,t) = -\frac{\mathrm{e}^{-\frac{(r\cos\theta - x_0)^2 + (r\sin\theta - y_0)^2}{4t}}(r\cos\theta - x_0)}{(2t\mathrm{e}^{-\frac{(r\cos\theta - x_0)^2 + (r\sin\theta - y_0)^2}{4t}} + 2t^2)},$$

$$v(r,\theta,t) = -\frac{\mathrm{e}^{-\frac{(r\cos\theta - x_0)^2 + (r\sin\theta - y_0)^2}{4t}}(r\sin\theta - y_0)}{(2t\mathrm{e}^{-\frac{(r\cos\theta - x_0)^2 + (r\sin\theta - y_0)^2}{4t}} + 2t^2)}.$$

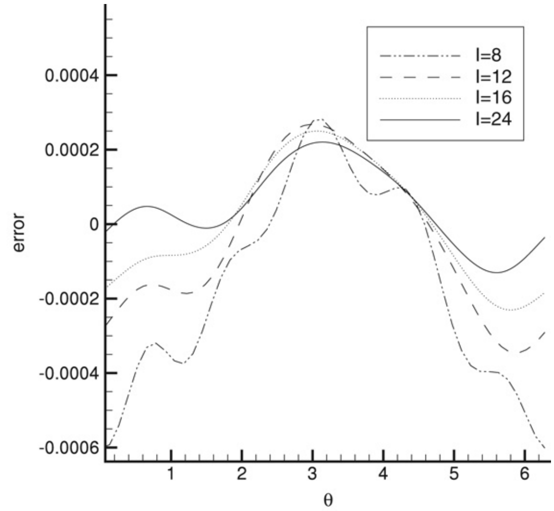

Fig. 3. Errors of the difference solution $u(R, \theta, 1) - u_h(R, \theta, 1)$ with $x_0 = 0.25$, $y_0 = 0.25$, $M = 4$ and $t = 1$.

Table 1
L_1-norms for different meshes and M at ($x_0 = y_0 = 0.0$) and ($x_0 = y_0 = 0.25$)

M	$I \times J$	Error	M	$I \times J$	Error	M	$I \times J$	Error	M	$I \times J$	Error
0	4 × 24	1.03e−2	0	4 × 24	2.18e−2	3	4 × 24	6.57e−3	3	4 × 24	6.86e−3
	8 × 48	5.30e−3		8 × 48	2.03e−2		8 × 48	2.08e−3		8 × 48	3.20e−3
	12 × 72	4.33e−3		12 × 72	1.90e−2		12 × 72	1.01−3		12 × 72	1.36e−3
	16 × 96	3.72e−3		16 × 96	1.87e−2		16 × 96	5.77e−4		16 × 96	7.11e−4
	24 × 144	3.09e−3		24 × 144	1.77e−2		24 × 144	2.92e−4		24 × 144	3.60e−4
1	4 × 24	6.57e−3	1	4 × 24	8.11e−3	4	4 × 24	6.57e−3	4	4 × 24	6.86e−3
	8 × 48	2.08e−3		8 × 48	5.46e−3		8 × 48	2.08e−3		8 × 48	3.20e−3
	12 × 72	1.01−3		12 × 72	3.95e−3		12 × 72	1.01e−3		12 × 72	1.17e−3
	16 × 96	5.77e−4		16 × 96	3.72e−3		16 × 96	5.77e−4		16 × 96	7.06e−4
	24 × 144	2.92e−4		24 × 144	3.53e−3		24 × 144	2.92e−4		24 × 144	3.49e−4
2	4 × 24	6.57e−3	2	4 × 24	6.89e−3	5	4 × 24	6.57e−3	5	4 × 24	6.86e−3
	8 × 48	2.08e−3		8 × 48	3.30e−3		8 × 48	2.08e−3		8 × 48	3.20e−3
	12 × 72	1.01−3		12 × 72	1.36e−3		12 × 72	1.01e−3		12 × 72	1.17e−3
	16 × 96	5.77e−4		16 × 96	9.66e−4		16 × 96	5.77e−4		16 × 96	7.06e−4
	24 × 144	2.92e−4		24 × 144	6.81e−4		24 × 144	2.92e−4		24 × 144	3.49e−4

Let the artificial boundary $\Gamma = \{(r, \theta) : r = R, 0 \le \theta \le 2\pi\}$ with $R = 3$. Taking $\Delta r = \frac{R-a}{I}$, $J = 6I$, $N = I$, $\Delta\theta = \frac{2\pi}{J}$, $\Delta t = T/N$, and denote the L_1-norm E_1 by

$$E_1 = \frac{1}{(I+1)(J+1)(N+1)} \sum_{n=0}^{N} \sum_{i=0}^{I} \sum_{j=0}^{J} (|u(r_i, \theta_j, t^n) - u_{ij}^n| + |v(r_i, \theta_j, t^n) - v_{ij}^n|).$$

Table 1 gives L_1 errors for different M at points ($x_0 = 0.0$ and $y_0 = 0.0$) and ($x_0 = 0.25$ and $y_0 = 0.25$), respectively. The numerical errors decrease fast with the increasing of M judged from Table 1. When $x_0 = y_0 = 0$ the exact solution is reduced to

$$u(r, \theta, t) = -\frac{e^{-\frac{r^2}{4t}} r \cos\theta}{(2te^{-\frac{r^2}{4t}} + 2t^2)},$$

$$v(r, \theta, t) = -\frac{e^{-\frac{r^2}{4t}} r \sin\theta}{(2te^{-\frac{r^2}{4t}} + 2t^2)}.$$

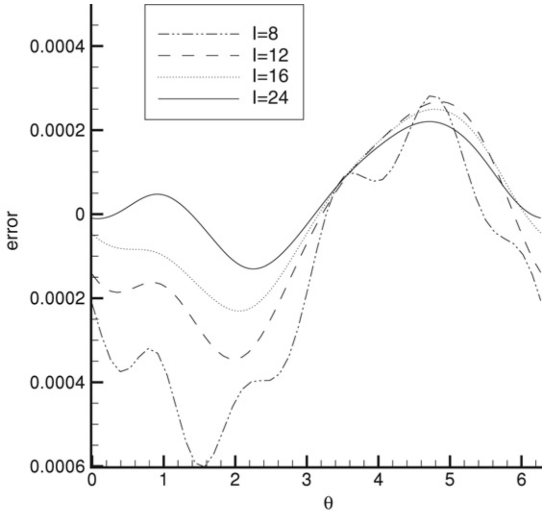

Fig. 4. Errors of the difference solution $v(R, \theta, 1) - v_h(R, \theta, 1)$ with $x_0 = 0.25$, $y_0 = 0.25$, $M = 4$ and $t = 1$.

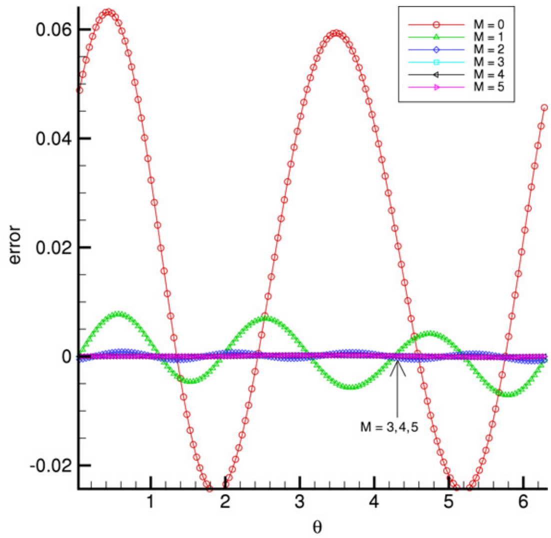

Fig. 5. $u(R, \theta, 1) - u_h(R, \theta, 1)$ for different M with mesh $= 24 \times 144$ and $x_0 = y_0 = 0.25$.

For $x_0 = y_0 = 0$, the solution is relatively simple, and from Table 1, we can see that $M = 1$ would be enough for the computation. For $x_0 = y_0 = 0.25$, the solution is relatively complicated. In this case, we need to take $M = 4$.

Figs. 1 and 2 plot the errors $|u(R, \theta, 1) - u_h(R, \theta, 1)|$ and $|v(R, \theta, 1) - v_h(R, \theta, 1)|$ with different meshes $I = 8, 12, 16, 24$ on the artificial boundary at time $t = 1$, $M = 4$, $x_0 = 0$ and $y_0 = 0$, respectively. Figs. 3 and 4 plot the errors $u(R, \theta, 1) - u_h(R, \theta, 1)$ with different meshes $I = 8, 12, 16, 24$ on the artificial boundary at time $t = 1$, $M = 4$, $x_0 = 0.25$ and $y_0 = 0.25$, respectively. Figs. 5 and 6 plot the errors $u(R, \theta, 1) - u_h(R, \theta, 1)$ and $v(R, \theta, 1) - v_h(R, \theta, 1)$ for different truncated numbers M from 0 to 5 with mesh $= 24 \times 144$ when $x_0 = y_0 = 0.25$.

6. Conclusion

By introducing an artificial boundary, we reduced the original problem to an equivalent problem defined on a bounded domain. Based on the Cole–Hopf transformation and Fourier series expansion, we obtained exact boundary conditions on the artificial boundary, and then provided a series of approximations to the artificial boundary conditions. In addition, we presented the stability analysis of the analytic solution and the numerical approximation scheme.

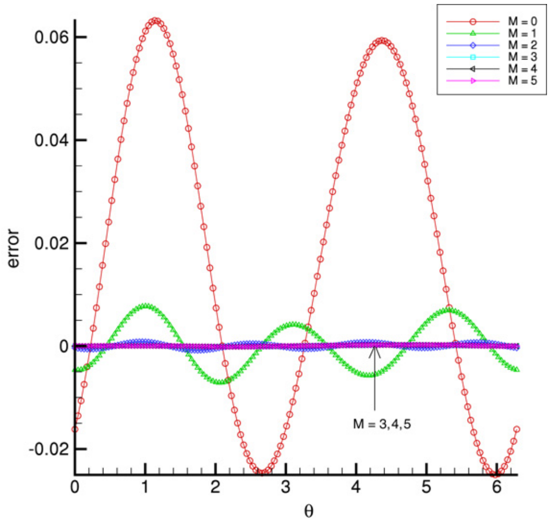

Fig. 6. $|v(R,\theta,1) - v_h(R,\theta,1)|$ for different M with mesh $= 24 \times 144$ and $x_0 = y_0 = 0.25$.

Finally, some numerical examples were given to demonstrate the effectiveness and feasibility of the artificial boundary conditions.

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