SOME RESULTS FOR FRACTIONAL IMPULSIVE BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS*

Xiangkui Zhao, Weigao Ge, Beijing

(Received April 10, 2009)

Abstract. In this paper, we consider a fractional impulsive boundary value problem on infinite intervals. We obtain the existence, uniqueness and computational method of unbounded positive solutions.

Keywords: fractional derivative, impulsive equations, positive solutions, fixed point theorem, monotone iterative method

MSC 2010: 26A33, 34A37

1. Introduction

Fractional operators have a long history, having been mentioned by Leibnitz in a letter to L'Hospital in 1695. Early mathematicians who contributed to fractional differential operators include Liouville, Riemann, and Holmgren. For three centuries, the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. However, in the last few decades, many authors pointed out that fractional calculus is very suitable for the description of memory and hereditary properties of various materials and processes, such effects are in fact neglected in classical models. Nowadays, fractional differential equations are increasingly used to model problems in acoustics and thermal systems, materials and mechanical systems, control and robotics, and other areas of application. For example, nonlocal epidemics can be modeled with fractional derivatives [2]. The results are relevant to foot-and-mouth disease, SARS, avian flu. The constitutive

* Supported by National Natural Science Foundation of China (10671012), the Doctoral Program Foundation of Education Ministry of China (20050007011), and the Fundamental Research Funds for the Central Universities (06108024).

equation of viscoelastic fluid is given by $\tau = \eta_0(\mathrm{d}^\alpha\gamma/\mathrm{d}t^\alpha)$ in [5], where $\mathrm{d}^\alpha\gamma/\mathrm{d}t^\alpha$ is the fractional derivative. The fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [6]. Based on experimental data, fractional partial differential equations for seepage flow in porous media are suggested in [7]. A review of some applications of fractional derivatives in continuum and statistical mechanics is given by Mainardi [11]. Recently, there are some papers dealing with the existence and multiplicity of solution (or positive solution) to fractional boundary value problems (see [3], [12], [13], and the references therein).

The theory of impulsive differential equations provides a natural framework for mathematical modeling of many real world phenomena. Significant progress has been made in the theory of impulsive differential equations in recent years (see [1], [4], [8]–[10], [14], and the references therein). However, to the best knowledge of the authors, there is no paper concerned with fractional impulsive boundary value problems on infinite intervals. Inspired by the above-mentioned works, in this paper, we will consider the following boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,\infty),\ t \neq t_k,\ k = 1,2,\ldots,m, \\ u(t_k^+) - u(t_k^-) = -I_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = 0, \quad D_{0+}^{\alpha-1}u(\infty) = 0, \end{cases} \tag{1.1}$$

where α is a real number with $1 < \alpha \leqslant 2$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $t_0 = 0$, $1 < t_1 < t_2 < \ldots < t_m < \infty$, $u(t_k^+) = \lim\limits_{h\to 0^+} u(t_k + h)$, $u(t_k^-) = \lim\limits_{h\to 0^+} u(t_k - h)$, $D_{0+}^{\alpha-1}u(\infty) = \lim\limits_{t\to\infty} D_{0+}^{\alpha-1}u(t)$.

The main features of this paper are as follows. First the existence and uniqueness of unbounded positive solutions for fractional boundary value problems are considered. Second a computational method is given. Third the solutions of boundary value problem (1.1) are unbounded.

Now we list some conditions in this section for convenience.

Let

$$F(t,u) = f(t,(1+t^\alpha)u).$$

(H1) $F\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ is continuous;

(H2) $|F(t,u)| \leqslant \varphi(t)\kappa(|u|)$ with $\kappa \in C([0,\infty),[0,\infty))$ nondecreasing and

$$0 < \int_0^\infty \varphi(r)\,\mathrm{d}r < \infty;$$

(H3) $I_k\colon [0,\infty) \to [0,\infty)$ $(k = 1,2,\ldots,m)$ are continuous;

(H4) there exist constants c_k $(k = 1,2,\ldots,m)$ such that $|I_k(u)| \leqslant c_k$.

2. Preliminaries

For the convenience of the readers, we provide some background material in this section.

Let $u\colon [0,\infty) \to \mathbb{R}$, $J_0 = [0, t_1]$, $J_m = (t_m, \infty)$, $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m-1$. For $k = 1, 2, \dots, m$, define the function $u_k\colon J_k \to \mathbb{R}$ by $u_k(t) = u(t)$. Consider the Banach spaces

$$PC = \Big\{u\colon\ u_k \in C(J_k, \mathbb{R}),\ k = 0, 1, \dots, m,\\ u(t_k^+) \text{ and } u(t_k^-) \text{ exist},\ u(t_k) = u(t_k^-),\ \lim_{t\to\infty} \frac{u(t)}{1+t^\alpha} \text{ exists}\Big\}$$

with the norm

$$\|u\|_{PC} = \sup_{t\in[0,\infty)} \left|\frac{u(t)}{1+t^\alpha}\right|$$

and

$$PC_l = \big\{u\colon\ u_k \in C(J_k, \mathbb{R}),\ k = 0, 1, \dots, m,\\ u(t_k^+) \text{ and } u(t_k^-) \text{ exist},\ u(t_k) = u(t_k^-),\ \lim_{t\to\infty} u(t) \text{ exists}\big\}$$

with the norm

$$\|u\|_l = \sup_{t\in[0,\infty)} |u(t)|.$$

Lemma 2.1 ([15])**.** *Let $\Omega_1 \subseteq PC_l$. Then Ω_1 is relatively compact in PC_l if the following conditions hold*

(a) *Ω_1 is bounded in PC_l;*
(b) *the functions belonging to Ω_1 are piecewise equicontinuous on any interval of $[0,\infty)$;*
(c) *the functions belonging to Ω_1 are equiconvergent, that is, given $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $|u(t) - u(\infty)| < \varepsilon$ for any $t > T(\varepsilon)$ and $u \in \Omega_1$.*

Definition 2.1. A function u is said to be a positive solution of (1.1) if $u \in PC$ satisfies (1.1), $u(t) \geqslant 0$, $t \in [0,\infty)$ and $u(t)$ is not identically zero on $[0,\infty)$.

Definition 2.2. The Riemann-Liouville fractional integral of order α of a function f is defined as

$$I^\alpha_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad \alpha > 0,$$

provided that the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function y is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, \quad \alpha > 0,$$

where $n = [\alpha] + 1$, provided that the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.4. Let E be a real Banach space. A nonempty closed set $K \subset E$ is a cone provided that

(1) $au + bv \in K$ for all $u, v \in K$ and all $a \geqslant 0$, $b \geqslant 0$,
(2) $u, -u \in K$ implies $u = 0$.

Every cone $K \subset E$ induces an ordering in E given by $x \leqslant y$ if and only if $y - x \in K$.

Definition 2.5. The map α is a nonnegative continuous concave functional on a cone K of a real Banach space E, provided that $\alpha\colon K \to [0, \infty)$ is continuous and

$$\alpha(tu + (1-t)v) \geqslant t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in K$, $0 \leqslant t \leqslant 1$.

Let $E = (E, \|\cdot\|)$ be a Banach space, $K \subset E$ a cone, α a nonnegative continuous concave functional on K and $a, b, c > 0$ constants. Define

$$K_c = \{x \in K\colon \|x\| < c\},$$
$$K(\alpha, a, b) = \{x \in K\colon a \leqslant \alpha(x),\ \|x\| \leqslant b\}.$$

To prove our results, we need the following fixed point theorem.

Theorem 2.1 ([3]). *Let K be a cone in a Banach space E. Suppose that there exist a positive number c, and a nonnegative continuous concave functional α on K with $\alpha(x) \leqslant \|x\|$ for $x \in \overline{K}_c$. Moreover, assume that $T\colon \overline{K}_c \to \overline{K}_c$ is completely continuous and there exist a, b, c, d with $0 < a < b < d \leqslant c$ such that*

(S1) *$\{x \in K(\alpha, b, d)\colon \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for all $x \in K(\alpha, b, d)$;*
(S2) *$\|Tu\| < a$ for all $x \in \overline{K}_a$;*
(S3) *$\alpha(Tx) > b$ for all $x \in K(\alpha, b, c)$ with $\|Tu\| > d$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in K$ such that

$$\|x_1\| < a, \quad \alpha(x_2) > b, \quad \|x_3\| > a \quad \text{with } \alpha(x_3) < b.$$

Remark 2.1. If $d = c$, then the condition (S1) of Theorem 2.1 implies the condition (S3) of Theorem 2.1.

3. Related lemmas

Lemma 3.1. *Let $y \in C[0,\infty)$, with $\int_0^\infty y(s)\,\mathrm{d}s$ convergent, and $1 < \alpha \leqslant 2$. If u is a solution of the equation*

$$u(t) = \int_0^\infty G(t,s)y(s)\,\mathrm{d}s + \sum_{k=1}^m W_k(t,u(t_k)), \tag{3.1}$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant \infty, \\ \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s < \infty, \end{cases} \tag{3.2}$$

and

$$W_k(t,u(t_k)) = \begin{cases} \dfrac{I_k(u(t_k))t^{\alpha-1}}{t_k^{\alpha-1} - t_k^{\alpha-2}}, & 0 \leqslant t \leqslant t_k, \\ \dfrac{I_k(u(t_k))t^{\alpha-2}}{t_k^{\alpha-1} - t_k^{\alpha-2}}, & t_k < t < \infty, \end{cases} \tag{3.3}$$

then u is a solution of the following boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) + y(t) = 0, & t \in (0,\infty),\ t \neq t_k,\ k = 1,2,\ldots,m, \\ u(t_k^+) - u(t_k^-) = -I_k(u(t_k)), & k = 1,2,\ldots,m, \\ u(0) = 0, \quad D_{0+}^{\alpha-1}u(\infty) = 0. \end{cases} \tag{3.4}$$

P r o o f. If u satisfies the integral equation (3.1), then $u(0) = D_{0+}^{\alpha-1}u(\infty) = 0$. Let

$$t \in (0,\infty) \setminus \{t_1, t_2 \ldots, t_m\},$$

then we have

$$\begin{aligned} D_{0+}^\alpha u(t) &= D_{0+}^\alpha\Big(\int_0^\infty G(t,s)y(s)\,\mathrm{d}s + \sum_{k=1}^m W_k(t,u(t_k))\Big) \\ &= D_{0+}^\alpha\Big(\int_0^t \frac{t^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + \int_t^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s \\ &\qquad + \sum_{k=1}^m W_k(t,u(t_k))\Big) \\ &= D_{0+}^\alpha\Big(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(s)\,\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + \sum_{k=1}^m W_k(t,u(t_k))\Big) \\ &= -y(t). \end{aligned}$$

Clearly we have

$$u(t_k^+) - u(t_k^-) = -I_k(u(t_k)), \quad k = 1, 2 \ldots, m.$$

The proof is complete. $\square$

Let

$$M = \frac{1}{\Gamma(\alpha)} \sup_{0 \leqslant t < \infty} \frac{t^{\alpha-1}}{1 + t^{\alpha}}.$$

Lemma 3.2. *The function $G(t,s)$ defined by Eq.* (3.2) *satisfies the following conditions*

(1) $0 \leqslant \dfrac{G(t,s)}{1+t^{\alpha}} \leqslant \dfrac{1}{\Gamma(\alpha)} \dfrac{t^{\alpha-1}}{1+t^{\alpha}} \leqslant M$ *for* $t, s \in (0, \infty)$;

(2) *there exists a positive function* $\gamma \in C(0, \infty)$ *such that*

$$\min_{k \leqslant t \leqslant l} G(t,s) = \gamma(s) \sup_{0 \leqslant t \leqslant \infty} G(t,s) = \gamma(s) G(s,s) \quad \textit{for } 0 < s < \infty, \tag{3.5}$$

where $0 < k < l < \infty$.

P r o o f. Looking at the expression for $G(t,s)$, it is clear that

$$0 \leqslant \frac{G(t,s)}{1+t^{\alpha}} \leqslant \frac{1}{\Gamma(\alpha)} \sup_{0 \leqslant t < \infty} \frac{t^{\alpha-1}}{1+t^{\alpha}} \quad \text{for } s, t \in (0, \infty).$$

In the following, we consider the existence of the positive function γ. Set

$$g_1(t,s) = \frac{t^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad g_2(t,s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

First for given $s \in (0, \infty)$, $G(t,s)$ is decreasing with respect to t for $s \leqslant t$ and increasing with respect to t for $t \leqslant s$. Consequently, we have

$$\begin{aligned}
\min_{k \leqslant t \leqslant l} G(t,s) &= \begin{cases} g_1(l,s), & s \in (0,k], \\ \min\{g_1(l,s), g_2(k,s)\}, & s \in [k,l], \\ g_2(k,s), & s \in [l,\infty), \end{cases} \\
&= \begin{cases} g_1(l,s), & s \in (0,r], \\ g_2(k,s), & s \in [r,\infty), \end{cases} \\
&= \begin{cases} \dfrac{l^{\alpha-1} - (l-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \in (0,r], \\ \dfrac{k^{\alpha-1}}{\Gamma(\alpha)}, & s \in [r,\infty), \end{cases}
\end{aligned}$$

where r is the unique solution of the equation

$$l^{\alpha-1} - (l-s)^{\alpha-1} = k^{\alpha-1}.$$

Specially if $k=1$, $l=4$, $\alpha=\frac{3}{2}$, then $s=3$ is the unique solution of the equation

$$2 - \sqrt{4-s} = 1, \quad \text{i.e. } r = 3.$$

By the monotonicity of $G(t,s)$, we have

$$\sup_{0\leqslant t\leqslant\infty} G(t,s) = G(s,s) = \frac{s^{\alpha-1}}{\Gamma(\alpha)}, \quad s\in(0,\infty).$$

Thus setting

$$\gamma(s) = \begin{cases} \dfrac{l^{\alpha-1}-(l-s)^{\alpha-1}}{s^{\alpha-1}}, & 0<s\leqslant r,\\ \Big(\dfrac{k}{s}\Big)^{\alpha-1}, & r\leqslant s<\infty, \end{cases}$$

the proof is complete. □

Define the cone $K\subset PC$ by

$$K = \{u\in PC\colon\ u \text{ is nonnegative on } [0,\infty)\}.$$

For $u\in K$, define the operator T by

$$(Tu)(t) = \int_0^\infty G(t,s)f(s,u(s))\,\mathrm{d}s + \sum_{k=1}^m W_k(t,u(t_k)). \tag{3.6}$$

Clearly the boundary value problem (1.1) has a solution $u=u(t)$ if and only if u solves the operator equation $u=Tu$. Since the Arzela-Ascoli theorem fails to work in the space PC, we need a modified compactness criterion to prove T is compact. Similarly to Theorem 2.5 in [16], we give the following lemma.

Lemma 3.3. *Let $\Omega_2\subseteq PC$. Then Ω_2 is relatively compact in PC if the following conditions hold*

(a) *Ω_2 is bounded in PC;*
(b) *the functions belonging to $\{u(t)/(1+t^\alpha)\colon\ u\in\Omega_2\}$ are piecewise equicontinuous on any interval of $[0,\infty)$,*
(c) *the functions belonging to $\{u(t)/(1+t^\alpha)\colon\ u\in\Omega_2\}$ are equiconvergent at infinity.*

Proof. It is easy to see that $\Omega_2' = \{y\colon\ y(t)=u(t)/(1+t^\alpha),\ u\in\Omega_2\}\subseteq PC_l$ satisfies the conditions of Theorem 2.1. So there exists a sequence $\{y_n\}\subseteq\Omega_2'$ and $y_0\in PC_l$ such that $\lim\limits_{n\to\infty}\|y_n-y_0\|_l = 0$. Let $u_n(t)=(1+t^\alpha)y_n(t)$, $n=1,2,\ldots$ and $u_0(t)=(1+t^\alpha)y_0(t)$. Obviously $\{u_n\}\subseteq\Omega_2$, $u_0\in PC$ and $\lim\limits_{n\to\infty}\|u_n-u_0\|_{PC} = \lim\limits_{n\to\infty}\|y_n-y(0)\|_l = 0$. □

Lemma 3.4. *If* (H1)–(H4) *hold, then* $T\colon K\to K$ *is completely continuous.*

P r o o f. We divide the proof into two steps.

Step 1: We prove that $T\colon K\to K$ is continuous.

First we show that $T\colon K\to K$. From Lemma 3.2, we have $(Tu)(t)\geqslant 0$ for $u\in K$. By (H2), (H4), we get

$$\int_0^\infty \frac{G(t,s)}{1+t^\alpha}F\Big(s,\frac{u(s)}{1+s^\alpha}\Big)\,\mathrm{d}s\leqslant\frac{\kappa(\|u\|_{PC})t^{\alpha-1}}{\Gamma(\alpha)(1+t^\alpha)}\int_0^\infty\varphi(s)\,\mathrm{d}s<\infty \tag{3.7}$$

and

$$\frac{1}{1+t^\alpha}\sum_{k=1}^m W_k(t,u(t_k))\leqslant\frac{1}{1+t^\alpha}\sum_{k=1}^m\frac{I_k(u(t_k))t^{\alpha-1}}{t_k^{\alpha-1}-t_k^{\alpha-2}}\leqslant\frac{t^{\alpha-1}}{1+t^\alpha}\sum_{k=1}^m\frac{c_k}{t_k^{\alpha-1}-t_k^{\alpha-2}}.$$

Thus the Dominated Convergence Theorem guarantees that

$$\begin{aligned}\lim_{t\to\infty}&\frac{(Tu)(t)}{1+t^\alpha}\\ &=\lim_{t\to\infty}\Big(\int_0^\infty\frac{G(t,s)}{1+t^\alpha}F\Big(s,\frac{u(s)}{1+s^\alpha}\Big)\,\mathrm{d}s+\frac{1}{1+t^\alpha}\sum_{k=1}^m W_k(t,u(t_k))\Big)=0.\end{aligned} \tag{3.8}$$

Therefore, $TK\subseteq K$. Second we show that $T\colon K\to K$ is continuous. In fact, suppose $\{u_n\}\subset K$, $u_0\in K$ and $u_0=\lim\limits_{n\to\infty}u_n$, then there exists $B>0$ such that $\|u_n\|_{PC}\leqslant B$, $n=0,1,2,\ldots$. By (H1), (H3), we have

$$\lim_{n\to\infty}F\Big(t,\frac{u_n(t)}{1+t^\alpha}\Big)=F\Big(t,\frac{u_0(t)}{1+t^\alpha}\Big),\quad t\in[0,\infty),$$

and

$$\lim_{n\to\infty}\sum_{k=1}^m W_k(t,u_n(t_k))=\sum_{k=1}^m W_k(t,u_0(t_k)),\quad t\in[0,\infty).$$

According to the Dominated Convergence Theorem,

$$\begin{aligned}\lim_{n\to\infty}\|Tu_n-Tu_0\|_{PC}\leqslant\lim_{n\to\infty}M\Big(&\int_0^\infty\Big|F\Big(s,\frac{u_n(s)}{1+s^\alpha}\Big)-F\Big(s,\frac{u_0(s)}{1+s^\alpha}\Big)\Big|\,\mathrm{d}s\\ &+\Gamma(\alpha)\sum_{k=1}^m\Big|\frac{I_k(u_n(t_k))-I_k(u_0(t_k))}{t_k^{\alpha-1}-t_k^{\alpha-2}}\Big|\Big)=0.\end{aligned}$$

Thus $T\colon K\to K$ is continuous.

Step 2: We show that $T\colon K \to K$ is relatively compact.

Let Ω be any bounded subset of K. Then there exists $L > 0$ such that

$$\sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t^\alpha} = \|u\|_{PC} \leqslant L.$$

First we will show that $T\Omega$ is uniformly bounded. For $u \in \Omega$, it is easy to prove that

$$\begin{aligned}
\|Tu\|_{PC} &= \sup_{t\in[0,\infty)} \frac{(Tu)(t)}{1+t^\alpha} \\
&= \sup_{t\in[0,\infty)} \biggl(\int_0^\infty \frac{G(t,s)}{1+t^\alpha} F\Bigl(s, \frac{u(s)}{1+s^\alpha}\Bigr)\,\mathrm{d}s + \frac{1}{1+t^\alpha} \sum_{k=1}^m W_k(t, u(t_k)) \biggr) \\
&\leqslant M\biggl(\kappa(L) \int_0^\infty \varphi(s)\,\mathrm{d}s + \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} \biggr) < \infty.
\end{aligned}$$

Hence, $T\Omega$ is uniformly bounded. Second we show that the functions belonging to $\{(Tu)(t)/(1+t^\alpha)\colon u \in \Omega\}$ are locally equicontinuous on $[0,\infty)$. For any $u \in \Omega$, $\bar{t}, \tilde{t} \in J_k$, $\bar{t} < \tilde{t}$, $k = 0, 1, 2, \ldots, m$, we have

Case 1. $\bar{t}, \tilde{t} \in J_0$.

$$\begin{aligned}
&\Bigl| \frac{(Tu)(\tilde{t})}{1+\tilde{t}^\alpha} - \frac{(Tu)(\bar{t})}{1+\bar{t}^\alpha} \Bigr| \\
&\quad\leqslant \int_0^\infty \frac{|G(\tilde{t},s) - G(\bar{t},s)|}{1+\tilde{t}^\alpha} F\Bigl(s, \frac{u(s)}{1+s^\alpha}\Bigr)\,\mathrm{d}s \\
&\qquad + (\tilde{t}^\alpha - \bar{t}^\alpha) \int_0^\infty \frac{G(\bar{t},s)}{1+\bar{t}^\alpha} F\Bigl(s, \frac{u(s)}{1+s^\alpha}\Bigr)\,\mathrm{d}s \\
&\qquad + \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} |\bar{t}^{\alpha-1}(1+\tilde{t}^\alpha) - \tilde{t}^{\alpha-1}(1+\bar{t}^\alpha)| \\
&\quad\leqslant \kappa(L) \int_0^\infty \frac{|G(\tilde{t},s) - G(\bar{t},s)|}{1+\tilde{t}^\alpha} \varphi(s)\,\mathrm{d}s \\
&\qquad + M\kappa(L)(\tilde{t}^\alpha - \bar{t}^\alpha) \int_0^\infty \varphi(s)\,\mathrm{d}s \\
&\qquad + \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} |\bar{t}^{\alpha-1}(1+\tilde{t}^\alpha) - \tilde{t}^{\alpha-1}(1+\bar{t}^\alpha)| < \infty.
\end{aligned}$$

Case 2. $\bar{t}, \tilde{t} \in J_k$, $k = 1, 2, \dots, m$.

$$\begin{aligned}
&\Big|\frac{(Tu)(\tilde{t})}{1+\tilde{t}^{\alpha}} - \frac{(Tu)\bar{t}}{1+\bar{t}^{\alpha}}\Big| \\
&\quad \leqslant \int_0^{\infty} \frac{|G(\tilde{t}, s) - G(\bar{t}, s)|}{1+\tilde{t}^{\alpha}} F\Big(s, \frac{u(s)}{1+s^{\alpha}}\Big)\,\mathrm{d}s \\
&\qquad + (\tilde{t}^{\alpha} - \bar{t}^{\alpha}) \int_0^{\infty} \frac{G(\bar{t}, s)}{1+\bar{t}^{\alpha}} F\Big(s, \frac{u(s)}{1+s^{\alpha}}\Big)\,\mathrm{d}s \\
&\qquad + \sum_{k=1}^{m} \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} |\bar{t}^{\alpha-1}(1+\tilde{t}^{\alpha}) - \tilde{t}^{\alpha-1}(1+\bar{t}^{\alpha})| \\
&\qquad + \sum_{k=1}^{m} \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} |\bar{t}^{\alpha-2}(1+\tilde{t}^{\alpha}) - \tilde{t}^{\alpha-2}(1+\bar{t}^{\alpha})| \\
&\quad \leqslant \kappa(L) \int_0^{\infty} \frac{|G(\tilde{t}, s) - G(\bar{t}, s)|}{1+\tilde{t}^{\alpha}} \varphi(s)\,\mathrm{d}s \\
&\qquad + M\kappa(L)(\tilde{t}^{\alpha} - \bar{t}^{\alpha}) \int_0^{\infty} \varphi(s)\,\mathrm{d}s \\
&\qquad + \sum_{k=1}^{m} \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} |\bar{t}^{\alpha-1}(1+\tilde{t}^{\alpha}) - \tilde{t}^{\alpha-1}(1+\bar{t}^{\alpha})| \\
&\qquad + \sum_{k=1}^{m} \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} |\bar{t}^{\alpha-2}(1+\tilde{t}^{\alpha}) - \tilde{t}^{\alpha-2}(1+\bar{t}^{\alpha})| < \infty.
\end{aligned}$$

Hence,

$$\Big|\frac{(Tu)(\tilde{t})}{1+\tilde{t}^{\alpha}} - \frac{(Tu)\bar{t}}{1+\bar{t}^{\alpha}}\Big| \to 0 \quad \text{uniformly as } \bar{t} \to \tilde{t}.$$

Therefore, the functions belonging to $\{(Tu)(t)/(1+t^{\alpha})\colon u \in \Omega\}$ are piecewise equicontinuous on any interval of $[0, \infty)$.

Finally by (H2), (H4), and Lemma 3.2, for any $u \in \Omega$ we have

$$\lim_{t\to\infty}\Big|\frac{(Tu)(t)}{1+t^{\alpha}}\Big| \leqslant \Big(\frac{\kappa(L)}{\Gamma(\alpha)} \int_0^{\infty} \varphi(s)\,\mathrm{d}s + \sum_{k=1}^{m} \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}}\Big) \lim_{t\to\infty} \frac{t^{\alpha-1}}{1+t^{\alpha}} = 0.$$

Hence, $T\Omega$ is equiconvergent at infinity. By using Lemma 3.3, we obtain that $T\Omega$ is relatively compact, that is, T is a compact operator. As a consequence of Step 1 and Step 2, we can prove that $T\colon K \to K$ is completely continuous. The proof is complete. $\square$

4. Existence of three positive solutions

We define the nonnegative continuous concave functional

$$\alpha(u) = \min_{t\in[k,l]} \frac{|u(t)|}{1+t^{\alpha}}, \quad \forall\, u \in K.$$

Let

$$N_1 = \frac{1}{M\int_0^{\infty}\varphi(s)\,\mathrm{d}s}, \quad N_2 = \frac{1+l^{\alpha}}{\int_k^l \gamma(s)G(s,s)\,\mathrm{d}s}.$$

Theorem 4.1. *Assume that* (H1)–(H4) *hold and there exist constants a, b, c, d with $0 < (1/\Gamma(\alpha))\sum\limits_{k=1}^{m} c_k/(t_k^{\alpha-1} - t_k^{\alpha-2}) < a < b < c = d$ such that the following conditions hold*

(B1) $\kappa(u) < N_1\Big(a - \sum\limits_{k=1}^{m} c_k/(t_k^{\alpha-1} - t_k^{\alpha-2})\Big)$ *for all* $u \in [0,a]$;

(B2) $F(t,u) \geqslant N_2 b$ *for all* $(t,u) \in [k,l]\times[b,c]$;

(B3) $\kappa(u) \leqslant N_1\Big(c - \sum\limits_{k=1}^{m} c_k/(t_k^{\alpha-1} - t_k^{\alpha-2})\Big)$ *for all* $u \in [a,c]$.

Then the boundary value problem (1.1) *has at least three positive solutions* u_1, u_2, *and* u_3 *such that*

$$\sup_{t\in[0,\infty)}\Big|\frac{u_1(t)}{1+t^{\alpha}}\Big| < a, \qquad b < \min_{t\in[k,l]}\Big|\frac{u_2(t)}{1+t^{\alpha}}\Big| < \sup_{t\in[0,\infty)}\Big|\frac{u_2(t)}{1+t^{\alpha}}\Big| \leqslant c,$$

$$a < \sup_{t\in[0,\infty)}\Big|\frac{u_3(t)}{1+t^{\alpha}}\Big| \leqslant c \quad \textit{with} \quad \min_{t\in[k,l]}\Big|\frac{u_3(t)}{1+t^{\alpha}}\Big| < b.$$

Proof. First we show that (S2) of Theorem 2.1 holds.

If $u \in \overline{K}_c$ then $\|u\|_{PC} \leqslant c$. By Lemma 3.2 and (B3), we get

$$\begin{aligned}
\|Tu\|_{PC} &= \sup_{t\in[0,\infty)} \frac{(Tu)(t)}{1+t^{\alpha}}\\
&= \sup_{t\in[0,\infty)} \Big(\int_0^{\infty} \frac{G(t,s)}{1+t^{\alpha}} F\Big(s, \frac{u(s)}{1+s^{\alpha}}\Big)\,\mathrm{d}s + \frac{1}{1+t^{\alpha}}\sum_{k=1}^{m} W_k(t,u(t_k))\Big)\\
&\leqslant M\int_0^{\infty}\varphi(s)\kappa\Big(\frac{u(s)}{1+s^{\alpha}}\Big)\,\mathrm{d}s + \sum_{k=1}^{m}\frac{c_k}{t_k^{\alpha-1}-t_k^{\alpha-2}}\\
&\leqslant M\int_0^{\infty}\varphi(s)N_1\Big(c - \sum_{k=1}^{m}\frac{c_k}{t_k^{\alpha-1}-t_k^{\alpha-2}}\Big)\,\mathrm{d}s + \sum_{k=1}^{m}\frac{c_k}{t_k^{\alpha-1}-t_k^{\alpha-2}} \leqslant c.
\end{aligned}$$

Hence, the condition (S2) of Theorem 2.1 is satisfied. In the same way, we can show that if (B1) holds, then $T\overline{K}_a \subset K_a$.

Second we show that (S1) of Theorem 2.1 holds. To check the condition (S1) of Theorem 2.1, we choose

$$u_0(t) = \frac{1}{2}(b+c)(1+t^\alpha), \quad t \in [0,\infty).$$

It is easy to see that $u \in K$, $\|u\|_{PC} = \frac{1}{2}(b+c) \leqslant c$, $\alpha(u) = \frac{1}{2}(b+c) > b$. That is,

$$u_0 \in \{u \in K(\alpha,b,d)\colon \alpha(u) > b\} \neq \emptyset.$$

Moreover, for all $u \in K(\alpha,b,d)$ we have $b \leqslant u(t)/(1+t^\alpha) \leqslant c$ for $t \in [k,l]$. By (B2) and Lemma 3.2, we have

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[k,l]} \frac{|(Tu)(t)|}{1+t^\alpha} \\
&= \min_{t\in[k,l]} \Big(\int_0^\infty \frac{G(t,s)}{1+t^\alpha} F\Big(s, \frac{u(s)}{1+s^\alpha}\Big)\,\mathrm{d}s + \frac{1}{1+t^\alpha}\sum_{k=1}^m W_k(t,u(t_k))\Big) \\
&\geqslant \frac{1}{1+l^\alpha}\int_0^\infty \gamma(s)G(s,s)F\Big(s,\frac{u(s)}{1+s^\alpha}\Big)\,\mathrm{d}s \\
&> \frac{1}{1+l^\alpha}\int_k^l \gamma(s)G(s,s)F\Big(s,\frac{u(s)}{1+s^\alpha}\Big)\,\mathrm{d}s \\
&\geqslant \frac{1}{1+l^\alpha}\int_k^l \gamma(s)G(s,s)N_2 b\,\mathrm{d}s = b.
\end{aligned}$$

Hence, condition (S1) of Theorem 2.1 is satisfied. By Remark 2.1, condition (S3) of Theorem 2.1 is satisfied. To sum up, all the hypotheses of Theorem 2.1 are satisfied. The proof is complete. $\square$

5. Existence of successive iteration solutions

Theorem 5.1. *Assume* (H1)–(H4) *hold. If there exists a positive number* $\Lambda > \sum\limits_{k=1}^m c_k/(t_k^{\alpha-1} - t_k^{\alpha-2})$ *such that*

(C1) $F(t,\cdot)$, $I_k(\cdot)\colon [0,\Lambda] \to [0,\infty)$ $(k = 1,2,\dots,m)$ *are nondecreasing for all* $t \in [0,\infty)$;

(C2) $\kappa(\Lambda) \leqslant N_1\Big(\Lambda - \sum\limits_{k=1}^m c_k/(t_k^{\alpha-1} - t_k^{\alpha-2})\Big)$ *for all* $u \in [0,\Lambda]$;

(C3) $F(t,0)$ *is not identically zero on any compact subinterval of* $(0,\infty)$,

then the boundary value problem (1.1) *has two positive solutions* $w^*, v^* \in \overline{K}_\Lambda$ *with*

$$\|w^*\|_{PC} \leqslant \Lambda \quad \text{and} \quad \lim_{n\to\infty} T^n w_0 = w^*, \tag{5.1}$$

$$\|v^*\|_{PC} \leqslant \Lambda \quad \text{and} \quad \lim_{n\to\infty} T^n v_0 = v^*, \tag{5.2}$$

where

$$w_0(t) = \Lambda(1 + t^\alpha), \quad t \in [0, \infty),$$
$$v_0 = 0, \quad t \in [0, \infty).$$

P r o o f. We now show that $T\colon \overline{K}_\Lambda \to \overline{K}_\Lambda$. If $u \in \overline{K}_\Lambda$ then $\|u\|_{PC} \leqslant \Lambda$, and we have

$$0 \leqslant \frac{u(t)}{1+t^\alpha} \leqslant \sup_{t\in[0,\infty)} \frac{u(t)}{1+t^\alpha} = \|u\|_{PC} \leqslant \Lambda.$$

By (H2), (H4), (C1), (C2), and Lemma 3.2, we get

$$\begin{aligned}
\|Tu\|_{PC} &= \sup_{t\in[0,\infty)} \Bigl(\int_0^\infty \frac{G(t,s)}{1+t^\alpha} F\Bigl(s, \frac{u(s)}{1+s^\alpha}\Bigr)\,\mathrm{d}s + \frac{1}{1+t^\alpha}\sum_{k=1}^m W_k(t, u(t_k))\Bigr)\\
&\leqslant M\int_0^\infty \varphi(s)\kappa\Bigl(\frac{u(s)}{1+s^\alpha}\Bigr)\,\mathrm{d}s + \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}}\\
&\leqslant M\int_0^\infty \varphi(s)N_1\Bigl(\Lambda - \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}}\Bigr)\,\mathrm{d}s + \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} \leqslant \Lambda.
\end{aligned}$$

Thus we conclude that $T\colon \overline{K}_\Lambda \to \overline{K}_\Lambda$.

Let $w_n = T^n w_0$, $v_n = T^n v_0$. Then $w_n, v_n \in \overline{K}_\Lambda$, $n = 1, 2, \ldots$. Since T is completely continuous, we see that $\{w_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$ are relatively compact sets. By (H3), (C1), (C2), and Lemma 3.2, we get

$$\begin{aligned}
\frac{w_1(t)}{1+t^\alpha} &= \frac{(Tw_0)(t)}{1+t^\alpha}\\
&= \int_0^\infty \frac{G(t,s)}{1+t^\alpha} f(s, w_0(t))\,\mathrm{d}s + \frac{1}{1+t^\alpha}\sum_{k=1}^m W_k(t, w_0(t_k))\\
&\leqslant \int_0^\infty \frac{G(t,s)}{1+t^\alpha} F(s, \Lambda)\,\mathrm{d}s + \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}}\\
&\leqslant \int_0^\infty \frac{G(t,s)}{1+t^\alpha} \varphi(s)\kappa(\Lambda)\,\mathrm{d}s + \sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}}\\
&\leqslant M\int_0^\infty \varphi(s)N_1\Bigl(\Lambda - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}}\Bigr)\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^m \frac{c_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} = \Lambda.
\end{aligned}$$

Hence, $w_1(t) \leqslant \Lambda(1 + t^\alpha)$, i.e., $w_1(t) \leqslant w_0(t)$ for $t \in [0, \infty)$.

Similarly we get $w_2(t) = (Tw_1)(t) \leqslant (Tw_0)(t) = w_1(t)$ for $t \in [0, \infty)$.

By induction, we get

$$w_{n+1}(t) = (Tw_n)(t) \leqslant (Tw_{n-1})(t) = w_n(t) \quad \text{for } t \in [0,\infty),\ n = 1,2,\ldots.$$

Hence, there exists $w^* \in \overline{K}_\Lambda$ such that $w^* = \lim_{n\to\infty} w_n$. Applying the continuity of T and $w_{n+1} = Tw_n$, we get $Tw^* = w^*$.

For $t \in [0,\infty)$, by (C3), we get

$$\begin{aligned} v_1(t) = (Tv_0)(t) &= \int_0^\infty G(t,s) f(s, v_0(s))\,\mathrm{d}s + \sum_{k=1}^m W_k(t, v_0(t_k)) \\ &\geqslant \int_0^\infty G(t,s) f(s,0)\,\mathrm{d}s > 0 = v_0(t) \end{aligned}$$

and $v_2(t) = (Tv_1)(t) \geqslant (Tv_0)(t) = v_1(t)$, $t \in [0,\infty)$.

By induction, we get

$$v_{n+1}(t) = (Tv_n)(t) \geqslant (Tv_{n-1})(t) = v_n(t), \quad t \in [0,\infty),\ n = 1,2,\ldots.$$

Hence, there exists $v^* \in \overline{K}_\Lambda$ such that $v^* = \lim_{n\to\infty} v_n$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$. The proof is complete. □

Theorem 5.2. *Assume that there exist a function $h \in L^1[0,\infty)$ and numbers d_k $(k = 1,2,\ldots,m)$ such that*

$$\begin{aligned} &|F(t,x) - F(t,y)| \leqslant h(t)|x-y| \quad \text{for } t \in [0,\infty),\ 0 < x,y < \Lambda, \\ &|I_k(x) - I_k(y)| \leqslant d_k|x-y| \quad \text{for } k = 1,2,\ldots,m,\ 0 < x,y < \Lambda, \end{aligned}$$

and

$$M\int_0^\infty h(s)\,\mathrm{d}s + \sum_{k=1}^m \frac{d_k}{t_k^{\alpha-1} - t_k^{\alpha-2}} < 1.$$

Then T has a unique fixed point in $\overline{K}_\Lambda$, that is, $w^* = v^*$ (defined by Eq. (5.1), (5.2)).

P r o o f. Let $u_1, u_2 \in E$. Then we have

$$\begin{aligned} \left|\frac{(Tu_1)(t) - (Tu_2)(t)}{1+t^\alpha}\right| &= \int_0^\infty \frac{G(t,s)}{1+t^\alpha}\left|F\Big(s, \frac{u_1(s)}{1+s^\alpha}\Big) - F\Big(s, \frac{u_2(s)}{1+s^\alpha}\Big)\right|\mathrm{d}s \\ &\quad + \frac{1}{1+t^\alpha}\sum_{k=1}^m |W_k(t, u_1(t_k)) - W_k(t, u_2(t_k))| \\ &\leqslant \Big(M\int_0^\infty h(s)\,\mathrm{d}s + \sum_{k=1}^m \frac{d_k}{t_k^{\alpha-1} - t_k^{\alpha-2}}\Big)\|u_1 - u_2\|_{PC} \\ &< \|u_1 - u_2\|_{PC}. \end{aligned}$$

Thus

$$\|(Tu_1)(t) - (Tu_2)(t)\|_{PC} < \|u_1 - u_2\|_{PC}.$$

Consequently, T is a contraction. As a consequence of the Banach fixed point theorem, we deduce that T has a unique fixed point which is an unbounded iteration positive solution of the boundary value problem (1.1). Hence, $w^* = v^*$ (defined by Eq. (5.1), (5.2)). The proof is complete. □

6. Example

Example 6.1. Consider the problem

$$\begin{cases} D_{0+}^{3/2}u(t) + f(t,u) = 0, \quad t \in (0,\infty) \setminus \{t_1\}, \\ u(t_1^+) - u(t_1^-) = -I_1(u(t_1)), \\ u(0) = 0, \quad D_{0+}^{3/2}u(\infty) = 0, \end{cases} \tag{6.1}$$

where

$$t_1 = 2, \quad I_1(u) = \begin{cases} \frac{1}{20}\sqrt{\pi}\,u, & u \in [0,10]; \\ \frac{1}{2}\sqrt{\pi}, & u \in [10,\infty), \end{cases}$$

$$F(t,u) = f(t,(1+t^{3/2})u) = \Big(\frac{2}{5\sqrt{\pi}}\,u + 10\Big)\mathrm{e}^{-t}, \quad t \in [0,\infty).$$

Choose $\Lambda = 5\sqrt{\pi}$, $h(t) = \varphi(t) = \mathrm{e}^{-t}$, $\kappa(u) = \frac{2}{5}\frac{1}{\sqrt{\pi}}\,u + 10$, $c_1 = d_1 = \frac{1}{20}\sqrt{\pi}$, $w_0(t) = 5\sqrt{\pi}(1+t^{3/2})$, $v_0(t) = 0$, $t \in [0,\infty)$. By computing, we have $M = \frac{4}{3}\frac{1}{\sqrt[3]{2}\sqrt{\pi}}$, $N_1 = \frac{3}{4}\sqrt[3]{2}\sqrt{\pi}$. Hence,

(1) $F(t,\cdot)$, $I_1(\cdot)\colon [0,\infty) \to [0,\infty)$ are nondecreasing for all $t \in [0,\infty)$;
(2) $\kappa(5\sqrt{\pi}) \leqslant 12 < N_1\big(\Lambda - c_1\sqrt{t_1}/(t_1-1)\big)$ for all $t \in [0,\infty)$;
(3) $F(t,0) = 10\mathrm{e}^{-t}$ is not identically zero on any compact subinterval of $(0,\infty)$;
(4) $|F(t,x) - F(t,y)| \leqslant h(t)|x-y|$ for $t \in [0,\infty)$, $0 < x,y < \Lambda$,
$|I_1(x) - I_1(y)| \leqslant d_1|x-y|$, $0 < x,y < \Lambda$, $M\int_0^\infty h(s)\,\mathrm{d}s + d_1\sqrt{t_1}/(t_1-1) < 1$.

Thus by Theorem 5.2, the problem (6.1) has an unbounded iteration positive solution. By computing, we have

$$\|w_1 - w_0\|_{PC} = 7.85227, \quad \|w_2 - w_1\|_{PC} = 0.00975, \quad \|w_3 - w_2\|_{PC} = 0.000012.$$

Example 6.2. Consider the problem

$$\begin{cases} D_{0+}^{3/2}u(t) + f(t,u) = 0, \quad t \in (0,\infty) \setminus \{t_1\}, \\ u(t_1^+) - u(t_1^-) = -I_1(u(t_1)), \\ u(0) = 0, \quad D_{0+}^{3/2}u(\infty) = 0, \end{cases} \tag{6.2}$$

where

$$t_1 = 2, \quad I_1(u) = \begin{cases} \frac{1}{1000}\,u, & u \in [0,5], \\ \frac{1}{200}, & u \in [5,\infty), \end{cases}$$

$$F(t,u) = f(t,(1+t^{3/2})u) = \begin{cases} \left(\frac{1}{20} + 4u^4\right)\mathrm{e}^{-t}, & u \leqslant 1, \\ \left(3 + \frac{1}{20} + u\right)\mathrm{e}^{-t}, & u > 1. \end{cases}$$

Choose $a = \frac{1}{10}$, $b = 1$, $c = 12$, $k = 1$, $l = 4$. By computing, we have $N_1 = \frac{4}{3}\frac{1}{\sqrt[3]{2}\sqrt{\pi}} \approx 0.5971$, $N_2 = \frac{27}{14}\sqrt{\pi} \approx 3.4183$, $c_1 = \frac{1}{40}$. Hence,

(1) $\kappa(u) = \frac{1}{20} + 4u^4 \leqslant 0.0504 < N_1(a - c_1\sqrt{t_1}/(t_1-1))$ for $(t,u) \in [0,\infty) \times \left[0, \frac{1}{10}\right]$;

(2) $F(t,u) = 3 + \frac{1}{20} + u \geqslant 4.005 > N_2 b$ for $(t,u) \in [1,4] \times [1,12]$;

(3) $\kappa(u) = 3 + \frac{1}{20} + u \leqslant 15.05 < N_1(c - c_1\sqrt{t_1}/(t_1-1))$ for $(t,u) \in [0,\infty) \times [1,12]$.

By Theorem 4.1, we get that the fractional boundary value problem (6.2) has at least three positive solutions u_1, u_2, and u_3 with

$$\sup_{t\in[0,\infty)}\left|\frac{u_1(t)}{1+t^{3/2}}\right| < \frac{1}{10}, \quad 1 < \min_{t\in[1,4]}\left|\frac{u_2(t)}{1+t^{3/2}}\right| < \sup_{t\in[0,\infty)}\left|\frac{u_2(t)}{1+t^{3/2}}\right| \leqslant 12,$$
$$\frac{1}{10} < \sup_{t\in[0,\infty)}\left|\frac{u_3(t)}{1+t^{3/2}}\right| \leqslant 12 \quad \text{with} \quad \min_{t\in[1,4]}\left|\frac{u_3(t)}{1+t^{3/2}}\right| < 1.$$

References

[1] *R. P. Agarwal, D. O'Regan*: Multiple nonnegative solutions for second order impulsive differential equations. Appl. Math. Comput. *114* (2000), 51–59.

[2] *E. Ahmeda, A. S. Elgazzar*: On fractional order differential equations model for nonlocal epidemics. Physic A *379* (2007), 607–614.

[3] *Z. Bai, H. Lü*: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. *311* (2005), 495–505.

[4] *W. Ding, M. Han*: Periodic boundary value problem for the second order impulsive functional differential equations. Appl. Math. Comput. *155* (2004), 709–726.

[5] *J. H. He*: Nonlinear oscillation with fractional derivative and its applications. In: International Conference on Vibrating Engineering, Dalian, China (1998), 288–291.

[6] *J. H. He*: Some applications of nonlinear fractional differential equations and their approximations. Bull. Sci. Technol. *15* (1999), 86–90.

[7] *J. H. He*: Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Methods Appl. Mech. Eng. *167* (1998), 57–68.

[8] *S. G. Hristova, D. D. Bainov*: Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations. J. Math. Anal. Appl. *197* (1996), 1–13.

[9] *E. K. Lee, Y.-H. Lee*: Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equation. Appl. Math. Comput. *158* (2004), 745–759.

[10] *X. Liu, D. Guo*: Periodic boundary value problems for a class of second-order impulsive integro-differential equations in Banach spaces. J. Math. Anal. Appl. *216* (1997), 284–302.

[11] *F. Mainardi*: Fractional calculus: Some basic problems in continuum and statistical mechanics. In: Fractals and Fractional Calculus in Continuum Mechanics (A. Carpinteri, F. Mainardi, eds.). Springer, New York, 1997, pp. 291–348.

[12] *A. Ouahab*: Some results for fractional boundary value problem of differential inclusions. Nonlinear Anal., Theory Methods Appl. *69* (2008), 3877–3896.

[13] *X. Su*: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. *22* (2009), 64–69.

[14] *Z. Wei*: Periodic boundary value problems for second order impulsive integrodifferential equations of mixed type in Banach spaces. J. Math. Anal. Appl. *195* (1995), 214–229.

[15] *B. Yan*: Boundary value problems on the half-line with impulses and infinite delay. J. Math. Anal. Appl. *259* (2001), 94–114.

[16] *B. Yan, Y. Liu*: Unbounded solutions of the singular boundary value problems for second order differential equations on the half-line. Appl. Math. Comput. *147* (2004), 629–644.

Authors' addresses: *X. Zhao* (corresponding author), Department of Mathematics and Mechanics, School of Applied Science, University of Science and Technology Beijing, Beijing 100083, P. R. China, e-mail: `zhaoxiangkui@126.com`, and Department of Mathematics, Beijing Institute of Technology, Beijing 100081, P, R. China; *W. Ge*, Department of Mathematics, Beijing Institute of Technology, Beijing 100081, P. R. China, e-mail: `gew@bit.edu.cn`.