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An improved stability criterion for fixed-point state-space digital filters using two's complement arithmetic[☆]

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ABSTRACT

A new stability criterion for fixed-point state-space digital filters using two's complement arithmetic is presented. The effectiveness of the results obtained is shown by using a numerical example.

1. Introduction

The system under consideration is given by

$$\begin{aligned} x(r+1) &= \boldsymbol{f}(y(r)) \\ &= [f_1(y_1(r)) \ \cdots \ f_n(y_n(r))]^{\mathrm{T}}, \end{aligned} \tag{1}$$

where

$$y(r) = [y_1(r) \cdots y_n(r)]^{\mathrm{T}} = Ax(r), \tag{2}$$

$x(r)$ is an n-vector state, $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, and A satisfies the following conditions:

$$k_i > 1, \quad i = 1, \ldots, m, \tag{3}$$

$$k_i \leq 1, \quad i = m+1, \ldots, n, \tag{4}$$

where $k_i = \sum_{j=1}^{n} |a_{ij}|$, and m is an integer between 1 and n. The nonlinearities characterized by

$$\left\{\begin{matrix} f_i(y_i(r)) = y_i(r), & \text{if } -1 \leq y_i(r) \leq 1 \\ -1 \leq f_i(y_i(r)) \leq 1, & \text{if } |y_i(r)| > 1 \end{matrix}\right\}, \quad i = 1, \ldots, n, \tag{5}$$

are under consideration.

In recent years, digital filters have been widely studied by many authors. Stability for digital filters using saturation arithmetic has drawn considerable attention (Ebert, Mazo, & Taylor, 1969; Hu & Lin, 2001; Johnson & Sandberg, 1995; Kar & Singh, 2004, 2005; Liu & Michel, 1992; Ooba, 2003, 2010; Singh, 1985, 1990, 2006, 2007, 2008a,b, 2011). It is well known that the hardware implementation of the saturation arithmetic adder is more expensive than that of two's complement arithmetic adder (Sandberg, 1979). So, it is worth investigating the stability of digital filters using two's complement arithmetic. In the literature, the stability problem of this class of digital filters has been studied (Kar, 2010; Mills, Mullis, & Roberts, 1978; Shen & Yuan, 2010; Shen, Yuan, & Wang, 2011; Singh, 1986, 1990, 2010a,b; Vaidyanathan & Liu, 1987). Now, we will recall some previously presented stability criteria.

The well-known Mills–Mullis–Roberts criterion takes the form

$$D - A^{\mathrm{T}}DA > 0, \tag{6}$$

where D is a positive diagonal matrix.

The stability criterion (Vaidyanathan & Liu, 1987) takes the form

$$D - A^{\mathrm{T}}DA \geq 0, \tag{7}$$

where D is a positive diagonal matrix and $\rho(A) < 1$ is satisfied.

By using the results (Shen & Yuan, 2010), the stability criterion proposed by Singh (2010a) may be written as follows:

$$\tilde{P} - A^{\mathrm{T}}\tilde{P}A > 0, \tag{8}$$

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where

$$\tilde{P} = \begin{bmatrix} \tilde{D} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & \tilde{Q} \end{bmatrix},$$

$\tilde{D} \in \mathbb{R}^{m\times m}$ is a positive-definite diagonal matrix, and $\tilde{Q} \in \mathbb{R}^{(n-m)\times(n-m)}$ is a positive-definite symmetric matrix.

Recently, a new stability criterion has been proposed (Shen et al., 2011), by estimating the region which the system trajectory will enter and remain in.

However, for system (1)–(5), the criteria (Shen et al., 2011; Singh, 2010a) are equivalent to the Mills–Mullis–Roberts criterion when $m = n$. Hence, the main purpose of this note is to present a new stability criterion, which may be less conservative than these previous results when $m = n$. Without loss of generality, we assume that m is an integer, with $1 \le m \le n$.

2. Preliminaries

In view of (1) and (5), one has

$$|x_i(r)| \le 1, \quad i = 1, \dots, n. \tag{9}$$

Note that even if the system trajectory lies out of the region (9) initially, (1) and (5) ensure that the trajectory will enter region (9) and will not leave this region once it enters it. Then, we assume that the system trajectory lies in region (9) initially.

In the following, for real symmetric matrices X and Y, the notation $X > Y$ means that the matrix $X - Y$ is positive definite.

Lemma 1. *For a given system (1)–(5), it holds that*

$$[d_i y_i(r) + \alpha_i][y_i(r) - f_i(y_i(r))] \ge 0, \quad i = 1, \dots, n, \tag{10}$$

where d_i, $i = 1, \dots, n$, are positive constants, and α_i are arbitrary constants satisfying $-d_i \le \alpha_i \le d_i$, $i = 1, \dots, n$.

Proof. If $y_i(r) = 0$, then we can obtain that (10) holds. Hence, we only need to consider the case when $y_i(r) \ne 0$. It follows that

$$\begin{aligned}&[d_i y_i(r) + \alpha_i][y_i(r) - f_i(y_i(r))]\\ &\quad = d_i y_i^2(r)[1 + \beta_i(r)][1 - \gamma_i(r)], \quad i = 1, \dots, n,\end{aligned} \tag{11}$$

where $\beta_i(r) = \alpha_i/[d_i y_i(r)]$ and $\gamma_i(r) = f_i(y_i(r))/y_i(r)$.

When $|y_i(r)| \le 1$, it can be obtained by (5) that $f_i(y_i(r)) = y_i(r)$ and $\gamma_i(r) = 1$. Thus, we have $[d_i y_i(r) + \alpha_i][y_i(r) - f_i(y_i(r))] = 0$.

When $|y_i(r)| > 1$, it follows from (5) that $|\beta_i(r)| < 1$ and $|\gamma_i(r)| < 1$. Then, we can get that $d_i y_i^2(r)[1 + \beta_i(r)][1 - \gamma_i(r)] > 0$. This completes the proof of Lemma 1. □

3. Main results

Theorem 1. *The null solution of system (1)–(5) is globally asymptotically stable, if there are a positive-definite symmetric matrix $P = [p_{ij}] \in \mathbb{R}^{n\times n}$, a diagonal matrix $L = \mathrm{diag}[l_1, \dots, l_n] \in \mathbb{R}^{n\times n}$, and two matrices $M = [m_{ij}]$, $N = [n_{ij}] \in \mathbb{R}^{n\times n}$ such that*

$$\Pi_1 + \Pi_2 < 0 \tag{12}$$

and

$$l_i \ge \sum_{j=1}^{n} |m_{ji}| + \sum_{j=1}^{n} |n_{ji}|, \quad i = 1, \dots, m \tag{13}$$

hold, where

$$\Pi_1 = \begin{bmatrix} -P & 0_{n\times n} \\ 0_{n\times n} & P \end{bmatrix}$$

and

$$\Pi_2 = \begin{bmatrix} 2A^{\mathrm{T}}LA + MA + A^{\mathrm{T}}M^{\mathrm{T}} & A^{\mathrm{T}}N^{\mathrm{T}} - M - A^{\mathrm{T}}L \\ NA - M^{\mathrm{T}} - LA & -N - N^{\mathrm{T}} \end{bmatrix}.$$

Proof. Construct the Lyapunov function

$$V(x(r)) = x^{\mathrm{T}}(r)Px(r), \tag{14}$$

and let $\Delta V(x(r)) = V(x(r+1)) - V(x(r))$. Then, for system (1)–(5), it follows that

$$\Delta V(x(r)) = \zeta^{\mathrm{T}}(r)\Pi_1\zeta(r), \tag{15}$$

where $\zeta(r) = [x^{\mathrm{T}}(r),\ x^{\mathrm{T}}(r+1)]^{\mathrm{T}}$.

In the following, we will show that $\zeta^{\mathrm{T}}(r)\Pi_2\zeta(r) \ge 0$ holds.

It follows that

$$\begin{aligned}\zeta^{\mathrm{T}}(r)\Pi_2\zeta(r) &= 2y^{\mathrm{T}}(r)Ly(r) + 2x^{\mathrm{T}}(r)My(r) + 2\boldsymbol{f}(y(r))Ny(r)\\ &\quad - 2y^{\mathrm{T}}(r)L\boldsymbol{f}(y(r)) - 2x^{\mathrm{T}}(r)M\boldsymbol{f}(y(r))\\ &\quad - 2\boldsymbol{f}^{\mathrm{T}}(y(r))N\boldsymbol{f}(y(r)).\end{aligned} \tag{16}$$

Then, we can write the above equation as follows:

$$\begin{aligned}\zeta^{\mathrm{T}}(r)\Pi_2\zeta(r) &= 2\sum_{i=1}^{n}[l_i y_i(r) + x^{\mathrm{T}}(r)M_i\\ &\quad + \boldsymbol{f}^{\mathrm{T}}(y(r))N_i][y_i(r) - f_i(y_i(r))],\end{aligned} \tag{17}$$

where M_i and N_i denote the ith column of matrices M and N, respectively.

Owing to (4), (5) and (9), we have

$$\begin{aligned}&[l_i y_i(r) + x^{\mathrm{T}}(r)M_i + \boldsymbol{f}^{\mathrm{T}}(y(r))N_i][y_i(r) - f_i(y_i(r))] = 0,\\ &i = m+1, \dots, n.\end{aligned} \tag{18}$$

It can be obtained by (5), (9) and (13) that

$$|x^{\mathrm{T}}(r)M_i + \boldsymbol{f}^{\mathrm{T}}(y(r))N_i| \le l_i, \quad i = 1, \dots, m. \tag{19}$$

By using Lemma 1, we have

$$\begin{aligned}&[l_i y_i(r) + x^{\mathrm{T}}(r)M_i + \boldsymbol{f}^{\mathrm{T}}(y(r))N_i][y_i(r) - f_i(y_i(r))] \ge 0,\\ &i = 1, \dots, m.\end{aligned} \tag{20}$$

Thus, it can be obtained that $\zeta^{\mathrm{T}}(r)\Pi_2\zeta(r) \ge 0$ holds, by (17), (18) and (20). Then, we can get that

$$\Delta V(x(r)) \le \zeta^{\mathrm{T}}(r)(\Pi_1 + \Pi_2)\zeta(r). \tag{21}$$

It follows from (12) and (21) that

$$\Delta V(x(r)) < 0 \quad \forall x^{\mathrm{T}}(r)x(r) \ne 0. \tag{22}$$

This completes the proof of Theorem 1. □

4. Comparative evaluation and numerical example

It is worth comparing the present approach with previously presented criteria.

First, consider the Mills–Mullis–Roberts criterion. Assume that, for system (1)–(5), there is a positive-definite diagonal matrix D such that (6) holds. Then, there can be ensured the existence of $\epsilon > 0$ satisfying

$$D - A^{\mathrm{T}}DA + \epsilon A^{\mathrm{T}}A > 0. \tag{23}$$

Thus, we can obtain there exist $L = D + \epsilon I_n$, $M = 0_{n\times n}$, $N = D + \epsilon I_n$ and $P = 2D$ such that (12) and (13) hold, where I_n denotes the $n \times n$ identity matrix.

Next, consider Singh's criterion. Assume that, for system (1)–(5), there are a positive-definite diagonal matrix $\tilde{D}$ and a positive-definite symmetric matrix $\tilde{Q}$ such that (8) holds. Then, there can be ensured the existence of $\epsilon > 0$ satisfying

$$\tilde{P} - A^{\mathrm{T}}\tilde{P}A + \epsilon A^{\mathrm{T}}A > 0, \tag{24}$$

where $\tilde{P}$ is defined as that in (8). Thus, we can obtain there is a solution to (12) and (13), i.e., $P = 2\tilde{P}$, $N = \tilde{P} + \epsilon I_n$, $M = A^{\mathrm{T}}\tilde{M}$,

$$L = \begin{bmatrix} \tilde{D} + \epsilon I_m & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & 0_{(n-m)\times(n-m)} \end{bmatrix},$$

and

$$\tilde{M} = \begin{bmatrix} 0_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & \tilde{Q} + \epsilon I_{(n-m)} \end{bmatrix}.$$

In the following, we will use a numerical example to show the effectiveness of our results.

Consider the third-order system (1)–(5) with

$$A = \begin{bmatrix} -0.02 & -0.73 & -0.35 \\ -0.07 & -0.85 & 0.17 \\ 0.21 & -0.71 & 0.18 \end{bmatrix}.$$

For this example, we can obtain that $m = n$ and the criteria (Shen & Yuan, 2010; Shen et al., 2011; Singh, 2010a) are equivalent to the Mills–Mullis–Roberts criterion. It can be verified that each of the stability criteria of (6)–(7) fails in this example.

Now, we will use the results of this note to analyze the stability of this example. For this example, there is a solution to (12) and (13), i.e., $L = \mathrm{diag}[71.884,\ 133.582,\ 53.929]$,

$$P = \begin{bmatrix} 15.249 & 9.314 & 5.358 \\ 9.314 & 256.555 & -10.192 \\ 5.358 & -10.192 & 50.288 \end{bmatrix},$$

$$M = \begin{bmatrix} -1.105 & 0.02 & -0.154 \\ 31.358 & 1.64 & 19.128 \\ 0.765 & -0.36 & -0.448 \end{bmatrix},$$

$$N = \begin{bmatrix} 17.096 & -0.042 & -0.069 \\ 5.084 & 130.899 & -3.981 \\ -0.003 & -0.236 & 29.534 \end{bmatrix}.$$

So, the method proposed by this note can ensure its stability.

It can be seen that for this example the results of this paper are less conservative than these criteria presented previously.

5. Conclusion

A new stability criterion for global asymptotic stability of fixed-point state-space digital filters using two's complement arithmetic is presented in this note. The effectiveness of the obtained results is verified by using a numerical example.

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