International Journal of Information Technology & Decision Making
Vol. 9, No. 5 (2010) 831–845

DOI: 10.1142/S0219622010004044



# MECHANISM DESIGN FOR OPTIMAL AUCTION OF DIVISIBLE GOODS

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In this paper, the auction of divisible goods is investigated and a mechanism design method for optimal auction of divisible goods is presented. First, the definitions of feasible allocations and divisible goods auctions are given based on several important assumptions of divisible goods auction. Second, an optimal auction mechanism of divisible goods is designed, and a method of how to use the uniform price auction to implement the optimal auctions is discussed under the background of allocating the total permitted pollution discharge capacity (TPPDC). Lastly, this method is applied to the environmental planning of Wuhan City Circle in Hubei Province, China.

*Keywords*: Divisible goods auction; optimal auction; uniform price; allocation of TPPDC.

## 1. Introduction

With the development of auction market of the world, multi-object auctions have attracted increasing attention.[1–7] Multi-object auctions can be classified as auctions of indivisible goods and auctions of divisible goods. In an auction of indivisible goods, the goods are indivisible, which means each good is an independent unit (e.g., auctions of mineral rights on federal land, offshore drilling rights, procurement contracts, estate collections of stamps, coins or antiques, fish, flowers, wine, etc.).[8] However, the goods in auctions of divisible goods are homogeneous and divisible, which means one unit good can be divided into many smaller units. For example, the auctions of emission rights, stocks, treasury bills, and spectrum are all the auctions of divisible goods.

Many scholars have studied multi-object auctions. Myerson[9] considered the problem faced by a seller who has a single unit of an indivisible good to sell to one of several possible buyers, and designed an optimal auction for a wide class of auction design problems. Fernando[8] gave the characterization of optimal selling procedures for a seller that has several units of an indivisible good to be sold extending the analysis of a single unit model in Ref. 9. Ortega Reichert[10] analyzed the properties of sequential English auction, discriminatory price auction, and uniform price auction, and presented revenue equivalence theorem of multi-object auctions for the first time. Harris and Raviv[11] first gave an optimal conclusion, i.e., if the bidders' valuations are independent, and follow uniform distribution, then the auction mechanism in Ref. 10 is optimal. Maskin and Riloy[12] gave a complete characterization for the multi-object auctions and generalized the conclusions in Ref. 11 to any valuation distribution. Especially, for the auctions of divisible goods, Back and Zender[13,14] compared the single object auction with divisible multi-object auctions, and designed a special uniform price auction mechanism of divisible goods. This is a new idea for studying the auction of divisible goods. Thenceforward, Wang and Zender[15] derived equilibrium bidding strategies in divisible goods auctions for asymmetrically informed risk neutral and risk averse bidders when there is random noncompetitive demand. Kremer and Nyborg[16] studied the impact of different allocation rules in divisible goods, uniform price auction. Damianov[17] concluded that low-price equilibrium in the uniform price auction with endogenous supply does not exist if the seller employs the proportional rationing rule and is consistent when selecting among profit-maximizing quantities. Indranil and Richard[18] studied the asymptotic price in the uniform price auction; the results showed that the expected price becomes large depending only on the aggregate of the marginal distributions of each bidder's marginal values, and not on the correlation between the marginal values. Sade *et al.*[19] pointed out that asymmetry in bidders' capacity constraints plays an important role in inhibiting collusion and promoting competitive outcomes in multi-unit auctions in which the final value of the goods is common knowledge. These are all the important research results in multi-object auctions theory in the past few years. However, most of these results are obtained based on some simple and special conditions, such as unitary demand for every bidder's valuation follows uniform distribution, the bid price of bidders are discrete, the bidders are symmetrical, and so on. When these conditions are changed, the corresponding conclusions need to be reconsidered.

In this paper, under several generic conditions, i.e., the auction goods are divisible, the bid price of bidders is continuous and the bidders are asymmetrical, a mechanism design method of optimal auctions that maximize the seller's expected utility is studied for a kind of divisible goods. We will try to provide a universally applicable method for auctioning and allocating the emission rights, stocks, treasury bills, network bandwidth, land, and so on.

The rest of this paper is organized as follows. Section 2 gives some basic assumptions and defines the feasible allocations, and then designs an optimal auction mechanism of divisible goods. Section 3 proposes a method of how to use the uniform price auction to implement the optimal auctions under the background of allocating the total permitted pollution discharge capacity (TPPDC). Based on the statistical data of chemical oxygen demand (COD) in nine cities of Wuhan City Circle in

Hubei Province, China, Sec. 4 applies our method to the environmental planning of Wuhan City Circle. Section 5 concludes this paper.

## 2. The Model

### 2.1. *Assumptions and definitions*

First of all, we give some important assumptions and definitions to describe the kind of optimal auction design problems which this paper will consider. It is supposed that there is one risk neutral seller who wants to sell $Q_0$ units of a divisible good and his objective is to maximize the expected revenue. The seller faces $n \geq 2$ potential bidders, numbered $1, 2, \ldots, n$. The set of bidders is denoted as $N = \{1, 2, \ldots, n\}$. All bidders are risk neutral, and all want to maximize their expected profits. The allocated quantity for bidder $i$ is denoted as $q_i$, where $q_i$ satisfies $q_i \geq 0$ and $\sum_{i=1}^{n} q_i \leq Q_0$. Supposedly, the bidder $i$ must pay $T_i$ to obtain the quantity $q_i$. The allocation outcome space and the payment space are denoted as $q = \{q_1, q_2, \ldots, q_n\}$ and $T = \{T_1, T_2, \ldots, T_n\}$, respectively. From these assumptions, we define an auction of a divisible good as $A(q, T)$.

Each potential bidder has his own value estimate for the per unit good, $s_i$, which means the maximum amount which bidder $i$ would be willing to pay for the per unit good given his current information about it. $s_i$ is only known by bidder $i$, other bidders do not know the real value of $s_i$ and treat it as a draw from a cumulative distribution $F_i(s_i) \cdot F_i(s_i)$ is defined on a support $\Omega_i = [h_i, l_i]$ and with a density function $f_i(s_i)$, where $h_i$ and $l_i$ represent the highest possible value and the lowest possible value, respectively, which bidder $i$ might assign to the per unit good. Any two variables $s_i$ and $s_j$ with $i \neq j$ are independent.

In the following text, $\Omega$ is used to denote the set of all possible combinations of bidders' value estimates, that is,

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n = [l_1, h_1] \times [l_2, h_2] \times \cdots \times [l_n, h_n],$$

where $\Omega$ is called a signal space. For any bidder $i$, we let $\Omega_{-i}$ denote the set of all possible combinations of value estimates which might be held by bidders other than $i$, so we have:

$$\begin{aligned}\Omega_{-i} &= \Omega_1 \times \Omega_2 \times \cdots \times \Omega_{i-1} \times \Omega_{i+1} \times \cdots \times \Omega_n \\ &= [l_1, h_1] \times [l_2, h_2] \times \cdots \times [l_{i-1}, h_{i-1}] \times [l_{i+1}, h_{i+1}] \times \cdots \times [l_n, h_n].\end{aligned}$$

In an auction, the bidder $i$'s final value estimate for the per unit good is not only determined by his own valuation $s_i$, but also influenced by other bidders' value estimates. Because of the preference uncertainty and quality uncertainty, every bidder might tend to revise his valuation of the per unit good after learning about other bidders' value estimates.[8] Let function $e_j(s_j)$, $j = 1, 2, \ldots, n$ be a revision effect functions, which means that if bidder $i$ learned that $s_j$ was bidder $j$'s value estimate for the per unit good, then bidder $i$ would revise his value estimate for the per unit good from $s_i$ to $s_i + e_j(s_j)$ with $i \neq j$. Thus, if bidder $i$ learned that $s = (s_1, s_2, \ldots, s_n)$ was the vector of value estimates initially held by the $n$ bidders, then he would revise his own valuation of the per unit good to:

$$r_i(s) = s_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} e_j(s_j).$$

Therefore, the value of the quantity $q_i$ of the good for potential bidder $i$ satisfies:

$$E[r_i(q_i, s)] = k_i s_i q_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} k_j \int_0^{q_i} e_j(x, s_j)dx,$$

where the functions $e_j(x, s_j)$ $(j = 1, 2, \ldots, n)$ are nonincreasing in $q_i$ and increasing in $s_j$. $E[r_i(q_i, s)]$ represents the expectation of $r_i(q_i, s)$, and the real number $k_j$ $(j = 1, 2, \ldots, n)$ is an influence coefficient that the bidder $j$'s information will affect bidder $i$, they are non-negative. If $e_j(x, s_j) = 0$ $(j = 1, 2, \ldots, n,\ j \neq i)$, then the auction model is a private values model. If for any $i, j$ $(j \neq i)$, we have $k_i = k_j$ and $s_i = e_j(s_j)$ $(j = 1, 2, \ldots, n,\ j \neq i)$, then we can obtain a common value model. In practical auction, we regard all bidder's information plays an equally important influence for bidder $i$ to determine the final value estimate for the per unit good, that is, $k_i = 1/n$. In addition, $s_i = e_i(s_i)$ is noted. So the expectation of $r_i(q_i, s)$ can be rewritten as:

$$E[r_i(q_i, s)] = \frac{1}{n} \sum_{j=1}^{n} \int_0^{q_i} e_j(x, s_j)dx.$$

In auction $A(q, T)$, the seller's utility function (goal function) is denoted as $u_0(q, T)$, which is equal to the bidders' total expected payments, i.e.,

$$u_0 = E\left[\sum_{i=1}^{n} T_i\right].$$

In addition, the bidders' utilities are also influenced by the allocated quantities and bidder $i$'s utility can be expressed by:

$$u_i(s_i) = E[r_i(q_i, s)] - T_i(s_i) \quad i = 1, 2, \ldots, n. \tag{1}$$

The assumptions and definitions above define the basic environment in which optimal auctions will be characterized. Based on this information, the mechanism design is studied for optimal auctions of divisible goods in the next section.

### 2.2. *Optimal auctions design*

Based on the density function $f_i(s_i)$ and the revision effect function $e_j(\cdot, s_j)$, and utility function $u_0(q, T)$ given above, we focus on the mechanism design of optimal auctions. The seller's objective is to find a mechanism to maximize his expected revenue from the auction of the goods. In order to realize this objective, the seller must decide the allocation that should be implemented. From the results of Refs. 8, 20, 22, we conclude that an optimal auction mechanism must be a feasible auction. Thus, we start the analysis with a characterization of feasible allocations.

Weber[21] pointed out that if an auction $A(q, T)$ was a feasible auction, it must be incentive compatible and individually rational based on revelation mechanisms.

First, we will discuss the individual rationality. We assume that every bidder voluntarily participates in the auction. If the bidder does not participate in the auction, he will not get the goods and not pay any money, so his utility is zero. Thus, to stimulate the bidders to participate in the auction actively, the following individual rationality conditions must be satisfied:

$$u_i(s_i, s_i) = E[r_i(q_i(s_i, s_{-i}), s)] - T_i(s_i, s_{-i}) \geq 0 \quad i = 1, 2, \ldots, n, \tag{2}$$

where

$$\begin{aligned} s_{-i} &= \{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n), \\ E[r_i(q_i(s_i, s_{-i}), s)] &= \frac{1}{n} \sum_{j=1}^{n} \int_0^{q_i(s_i, s_{-i})} e_j(x, s_j)\, dx \\ &= \frac{1}{n} \int_0^{q_i(s_i, s_{-i})} e_i(x, s_i) dx + \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_0^{q_i(s_i, s_{-i})} e_j(x, s_j)\, dx, \end{aligned}$$

where $u_i(s_i, s_i)$ denotes the expected utility of bidder $i$ from announcing type $s_i$ when his true type is $s_i$. The individual rationality implies that every type of bidder must receive an expected payoff at least as high as he would not participate in the auction.

Second, we suppose that the seller cannot prevent any bidder from lying about his value estimate. Myerson[9] shows that the revelation mechanism can be implemented only if no bidder ever expects to gain from lying. That is, honest responses must form a Nash-equilibrium in the auction. Because all players are risk neutral, so the expected utility of bidder $i$ from announcing type $\hat{s}_i$, when his true type is $s_i$, is:

$$u_i(s_i, \hat{s}_i) = E[r_i(q_i(\hat{s}_i, s_{-i}), s)] - T_i(\hat{s}_i, s_{-i}) \quad \forall\, i \in N, \;\; \forall\, s_i, \;\; \hat{s}_i \in \Omega_i,$$

where

$$\begin{aligned} E[r_i(q_i(\hat{s}_i, s_{-i}), s)] = \frac{1}{n} & \int_0^{q_i(\hat{s}_i, s_{-i})} e_i(x, s_i) dx \\ & + \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_0^{q_i(\hat{s}_i, s_{-i})} e_j(x, s_j) dx \quad \forall\, i \in N, \;\; \forall\, s_i, \;\; \hat{s}_i \in \Omega_i. \end{aligned}$$

Our objective is to find rules such that it is a best response for each bidder to truthfully announce his signals, given that all others are doing likewise. This requirement is called as incentive compatibility, and is equivalent to the condition that:

$$u_i(s_i, s_i) \geq u_i(s_i, \hat{s}_i) \quad \forall\, i \in N, \quad \forall\, s_i,\ \ \hat{s}_i \in \Omega_i. \tag{3}$$

We say that $A(q, T)$ is a feasible auction mechanism if and only if condition (2) and (3) are satisfied. That is, when the seller plans to allocate the goods according to $s = (s_1, s_2, \ldots, s_n)$ and $T = (T_1, T_2, \ldots, T_n)$, the mechanism can be implemented with all bidders willing to participate honestly, if and only if conditions (2) and (3) are satisfied. Thus, a simplified characterization of the feasible auction mechanism is deduced as follows.

**Proposition 2.1.** *For an auction $A(q, T)$, if the following conditions hold:*

(i) $$\frac{\partial q_i(s)}{\partial s_i} \geq 0, \quad \forall\, i \in N,\ \ \forall\, s_i \in \Omega_i;$$

(ii) $$u_i(l_i, l_i) \geq 0 \quad \forall\, i \in N,\ \ \forall\, l_i \in \Omega_i;$$

(iii) $$u_i(s_i, s_i) = u_i(l_i, l_i) + \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i(x, s_{-i})} \frac{\partial e_i(y, x)}{\partial s_i} dy dx \quad \forall\, i \in N,\ \ \forall\, s_i,\ l_i \in \Omega_i;\ \textit{and}$$

(iv) $$\sum_{i=1}^{n} q_i \leq Q_0,\ \textit{where}\ q_i \geq 0, \quad \forall\, i \in N.$$

*Then $A(q, T)$ is a feasible auction mechanism.*

**Proof.** For any $i \in N,\ s_i, \hat{s}_i \in \Omega_i$,

$$\begin{aligned} u_i(s_i, \hat{s}_i) &= E[r_i(q_i(\hat{s}_i, s_{-i}), s)] - T_i(\hat{s}_i, s_{-i}) \\ &= \frac{1}{n}\int_0^{q_i(\hat{s}_i, s_{-i})} e_i(x, s_i) dx + \frac{1}{n}\sum_{\substack{j=1 \\ j \neq i}}^{n} \int_0^{q_i(\hat{s}_i, s_{-i})} e_j(x, s_j) dx - T_i(\hat{s}_i, s_{-i}). \end{aligned}$$

So we have:

$$\frac{\partial u_i(s_i, \hat{s}_i)}{\partial s_i} = \frac{1}{n}\int_0^{q_i(\hat{s}_i, s_{-i})} \frac{\partial e_i(x, s_i)}{\partial s_i} dx.$$

Then

$$\frac{\partial u_i(s_i, s_i)}{\partial s_i} = \left.\frac{\partial u_i(s_i, \hat{s}_i)}{\partial s_i}\right|_{\hat{s}_i = s_i} = \frac{1}{n}\int_0^{q_i(s_i, s_{-i})} \frac{\partial e_i(x, s_i)}{\partial s_i} dx.$$

Using the envelope theorem, we have:

$$u_i(s_i, s_i) = u_i(\hat{s}_i, \hat{s}_i) + \frac{1}{n}\int_{\hat{s}_i}^{s_i}\int_0^{q_i(x, s_{-i})} \frac{\partial e_i(y, x)}{\partial s_i} dy dx \quad \forall\, i \in N, \quad \forall\, s_i,\ \ \hat{s}_i \in \Omega_i. \tag{4}$$

It just means (iii) hold.

On the one hand, from Eq. (4), when $\hat{s}_i = l_i$, so we have:

$$u_i(s_i, s_i) = u_i(l_i, l_i) + \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i(x,s_{-i})} \frac{\partial e_i(y,x)}{\partial s_i} dydx \quad \forall\, i \in N,\ \ \forall\, s_i,\ \ l_i \in \Omega_i. \tag{5}$$

By (ii), $u_i(l_i, l_i) \geq 0\ \forall\, i \in N,\ \forall\, l_i \in \Omega_i$, and $\frac{\partial e_i(y,x)}{\partial s_i} \geq 0$, then

$$u_i(s_i, s_i) \geq 0 \quad \forall\, i \in N,\ \ \forall\, s_i \in \Omega_i,$$

which means the condition of individual rationality is satisfied.

On the other hand, use conditions (4) and (i), (iv) to write:

$$\begin{aligned}
u_i(s_i, s_i) &= u_i(\hat{s}_i, \hat{s}_i) + \frac{1}{n}\int_{\hat{s}_i}^{s_i}\int_0^{q_i(x,s_{-i})} \frac{\partial e_i(y,x)}{\partial s_i} dydx \\
&\geq \frac{1}{n}\int_0^{q_i(\hat{s}_i,s_{-i})} e_i(y,\hat{s}_i)dy + \frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\int_0^{q_i(\hat{s}_i,s_{-i})} e_j(y,s_j)dy - T_i(\hat{s}_i, s_{-i}) \\
&\quad + \frac{1}{n}\int_{\hat{s}_i}^{s_i}\int_0^{q_i(\hat{s}_i,s_{-i})} \frac{\partial e_i(y,x)}{\partial s_i} dydx \\
&= \frac{1}{n}\int_0^{q_i(\hat{s}_i,s_{-i})} e_i(y,\hat{s}_i)dy + \frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\int_0^{q_i(\hat{s}_i,s_{-i})} e_j(y,s_j)dy \\
&\quad + \frac{1}{n}\int_0^{q_i(\hat{s}_i,s_{-i})} [e_i(y,s_i) - e_i(y,\hat{s}_i)]dy - T_i(\hat{s}_i, s_{-i}) \\
&= \frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\int_0^{q_i(\hat{s}_i,s_{-i})} e_j(y,s_j)dy + \frac{1}{n}\int_0^{q_i(\hat{s}_i,s_{-i})} e_i(y,s_i)dy - T_i(\hat{s}_i, s_{-i}) \\
&= u_i(s_i, \hat{s}_i).
\end{aligned}$$

That is to say, the incentive compatibility is satisfied. Therefore, $A(q,T)$ is a feasible auction mechanism. □

Based on the results of Proposition 1.1, we can analyze the optimal auction mechanism.

**Proposition 2.2.** *In an auction* $A(q^*, T^*)$, *suppose that the seller's revenue satisfies*:

$$\text{Max } u_0 = E\left[\sum_{i=1}^{n} T_i\right], \tag{6}$$

*subject to the following constraints*

$$u_i(l_i, l_i) \geq 0 \quad \forall\, i \in N,\ \ \forall\, l_i \in \Omega_i, \tag{7}$$

$$u_i(s_i, s_i) \geq u_i(s_i, \hat{s}_i) \quad \forall\, i \in N,\ \ \forall\, s_i,\ \ \hat{s}_i \in \Omega_i, \tag{8}$$

$$\frac{\partial q_i(s)}{\partial s_i} \geq 0, \quad \forall\, i \in N,\ \ \forall\, s_i \in \Omega_i, \tag{9}$$

$$\sum_{i=1}^{n} q_i \leq Q_0, \tag{10}$$

$$q_i \geq 0 \quad \forall\, i \in N. \tag{11}$$

*Suppose also that*

$$T_i^*(s_i, s_{-i}) = E[r_i(q_i^*(s_i, s_{-i}), s)] - \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i^*(x, s_{-i})} \frac{\partial e_i(y, x)}{\partial s_i} dy dx. \tag{12}$$

*Then $A(q^*, T^*)$ represents an optimal auction.*

**Proof.** One the one hand, from Eqs. (1) and (5), the seller's revenue can be written as:

$$\begin{aligned} u_0 &= E\left[\sum_{i=1}^{n} T_i\right] \\ &= \sum_{i=1}^{n} E[r_i(q_i, s)] - \sum_{i=1}^{n} E[r_i(q_i, s) - T_i(s_i)] \\ &= \sum_{i=1}^{n} E[r_i(q_i, s)] - \sum_{i=1}^{n} E[u_i(s_i)] \\ &= \sum_{i=1}^{n} E[r_i(q_i, s)] - \sum_{i=1}^{n} E\left[u_i(l_i, l_i) + \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i(x, s_{-i})} \frac{\partial e_i(y, x)}{\partial s_i} dy dx\right] \\ &= E\left[\sum_{i=1}^{n} r_i(q_i, s) - \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i(x, s_{-i})} \frac{\partial e_i(y, x)}{\partial s_i} dy dx\right] - \sum_{i=1}^{n} u_i(l_i, l_i). \end{aligned}$$

Because the bidder's payment $T_i$ is only included in $u_i(l_i, l_i)$, $i = 1, 2, \ldots, n$. Thus, to maximize the seller's revenue $u_0$ is equivalent to minimize the utility $u_i(l_i, l_i)$, $i = 1, 2, \ldots, n$. In addition, the individual rationality requires that $u_i(l_i, l_i) \geq 0$. Therefore, if the seller's revenue $u_0$ reaches optimal, then the utility $u_i(l_i, l_i)$ must satisfy $u_i(l_i, l_i) = 0$.

Using Eqs. (1) and (5) again, we have:

$$\begin{aligned} u_i(l_i, l_i) &= u_i(s_i, s_i) - \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i(x,s_{-i})} \frac{\partial e_i(y,x)}{\partial s_i} dy dx \\ &= E[r_i(q_i(s_i, s_{-i}), s)] - T_i(s_i, s_{-i}) - \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i(x,s_{-i})} \frac{\partial e_i(y,x)}{\partial s_i} dy dx. \end{aligned}$$

When $u_i(l_i, l_i) = 0$, we obtain:

$$T_i^*(s_i, s_{-i}) = E[r_i(q_i^*(s_i, s_{-i}), s)] - \frac{1}{n}\int_{l_i}^{s_i}\int_0^{q_i^*(x,s_{-i})} \frac{\partial e_i(y,x)}{\partial s_i} dy dx.$$

On the other hand, if Eq. (12) is satisfied, then $u_i(l_i, l_i) = 0$ is hold, so the seller's revenue $u_0$ reaches optimal. In addition, from Eqs. (7)–(11), we can conclude that the auction $A(q^*, T^*)$ is a feasible auction mechanism. Therefore, $A(q^*, T^*)$ is an optimal auction. □

## 3. Implementing the Optimal Auctions Based on Uniform Price Auction

In this section, we give the method of how to use the uniform price auction to implement the optimal auctions based on the background of allocating the total permitted pollution discharge capacity (TPPDC).

TPPDC is the pollution discharge capacity of a certain pollutant which is set by the environmental management department in a certain period based on the factors of technology, economy, environment, and management. It provides an important basis for effectively implementing the pollutant permit system and the total emission control system. TPPDC may be classed as homogeneous divisible goods, whose allocation usually involves complicated private information. Its values depend on location, enterprise, and pollutant discharge period.

It is given that there is an assigner (environmental management department) who wants to allocate the quantity $Q_0$ of TPPDC to $n$ polluters. The assigner is risk neutral, and his objective is to maximize his expected revenue. In this problem, we suppose the polluter $i$'s valuation of the per unit TPPDC, $e_i(s_i)$, is just the polluter $i$'s marginal cost of pollutant treatment, which is regarded as a polluter's private information. Let $v_i(x)$ and $g_i(x)$ denote the polluter $i$'s actual marginal treatment cost function and declared marginal treatment cost function, respectively. The polluter $i$'s actual marginal treatment cost function $v_i(x)$ is only known by bidder $i$, and other bidders do not observe the realization of $v_i(x)$. Any two functions, $v_i(x)$ and $v_j(x)$, with $i \neq j$ are independent. For polluter $i$, the actual marginal treatment cost function $v_i(x)$ is greater than or equal to the declared marginal treatment cost function $g_i(x)$.

Let $G_i$ and $q_i \in [0, \infty)$ $(i = 1, 2, \ldots, n)$ denote the polluter $i$'s actual pollutant discharge capacity and the permitted pollution discharge capacity, respectively, and $x = G_i - q_i$ denotes the surplus pollution treatment. The cost function of treating surplus pollutant $x = G_i - q_i$ of the $i$th polluter is given by $F_i(x)$. Then the marginal cost function $v_i(x) = dF_i(x)/dx \geq 0$, which means the treatment cost rises as the permitted pollution discharge capacity rises. The allocation of TPPDC, $Q_0$, may be described by $\sum_{i=1}^{n} q_i \leq Q_0$, where $0 \leq q_i \leq G_i,\ i = 1, 2, \ldots n$.

The assigner allocates TPPDC, $Q_0$, under a uniform price $p = g_i(G_i - q_i)$, $(i = 1, 2, \ldots, n)$ in which the marginal treatment cost $g_i(x)$ is declared by the $i$-th polluter. Because of the stimulation of declared information, the polluter $i$ must pay $T_i = pq_i$ to obtain the permitted pollution discharge capacity $q_i$.

The assigner's goal is to maximize the goal function $u_0 = E[\sum_{i=1}^{n} T_i] = \sum_{i=1}^{n} pq_i = pQ$, raised from TPPDC allocation by choosing a specific total capacity $Q \leq Q_0$ and a uniform price $p > 0$. Hence,

$$
(\mathrm{M}_1) \qquad
\begin{aligned}
&\text{Max} \quad u_0 = E\left[\sum_{i=1}^{n} T_i\right] = \sum_{i=1}^{n} pq_i = pQ \\
&\text{S.T.} \quad \begin{cases} \displaystyle\sum_{i=1}^{n} q_i = Q \leq Q_0 \\ g_i(G_i - q_i) = p, & i = 1, 2, \ldots, n \\ 0 \leq q_i \leq G_i, & i = 1, 2, \ldots, n. \end{cases}
\end{aligned}
$$

Because all polluters are risk neutral, the expected utility of polluter $i$ from announcing treatment cost $\hat{s}_i = g_i(x)$ when his true marginal treatment cost function is $s_i = v_i(x)$ can be expressed as:

$$
\begin{aligned}
u_i(s_i, \hat{s}_i) &= E[r_i(q_i(\hat{s}_i, s_{-i}), s)] - T_i(\hat{s}_i, s_{-i}) \\
&= \int_{G_i - q_i}^{G_i} v_i(x)dx - pq_i.
\end{aligned}
$$

The goal of the $i$-th polluter is to maximize revenue:

$$
\text{Max} \quad u_i(s_i, \hat{s}_i) = \text{Max} \int_{G_i - q_i}^{G_i} v_i(x)dx - pq_i, \tag{13}
$$

by declaring his smart marginal treatment cost function $\hat{s}_i = g_i(x)$.

## 4. Application in Environmental Planning

Model $M_1$ and Eq. (13) describe an optimal auction of divisible goods under a uniform price. In this section, this model or method is applied to the environmental planning of Wuhan City Circle in Hubei Province, China.

Reaching out from the center of Wuhan City, a radius of 100 km covers an area of 6000 km$^2$ with a cluster of small towns or cities, i.e., Huangshi City, Ezhou City, Xiaogan City, Huanggang City, Xianning City, Xiantao City, Qianjiang City, and Tianmen City, which have formed Wuhan City Circle (see Fig. 1). Wuhan City Circle is a significant strategic measure made by Hubei province in China to

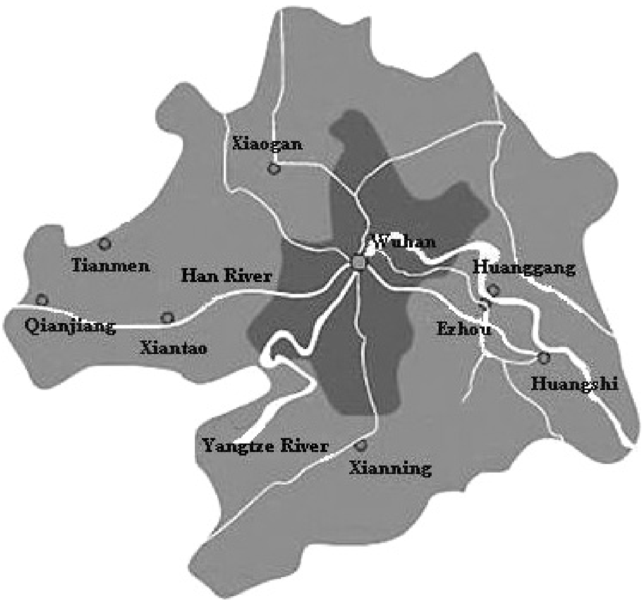


Fig. 1. The map of Wuhan City Circle.

promote its economy and give it impetus to rise in central area, while one urgent affair to promote the economy of Wuhan City Circle is its pollution control.

With the development of economy and society, the environment problem of Wuhan City Circle has become increasingly serious in recent years. In order to control and treat pollutant in Wuhan City Circle effectively, Hubei Province takes the measure of controlling TPPDC. Since implementing this measure, the environment of Wuhan City Circle has been significantly improved. In the following text, we will apply the model $M_1$ to do a case analysis for the environmental planning of Wuhan City Circle based on the statistical data of chemical oxygen demand (COD) in nine cities of Wuhan City Circle during "Tenth Five-Year" and "Eleventh Five-Year". These nine cities are called nine polluters.

Let the polluter $i$'s declared marginal treatment cost function be $g_i(x) = a_i x + b_i$, where $a_i \geq 0$ and $b_i \geq 0$ denote variable cost coefficient and fixed cost coefficient, respectively. $a_i$ is related to chemical medicine, electric cost, maintenance costs, processing level, and interrelated human resources; and $b_i$ may be determined by the construction cost and financing cost of every city's sewage treatment facility. $b_i$ can be calculated based on 20 years operation period depreciation.

In calculation, the fixed cost coefficient $b_i$ can be regarded as public information. If we keep statistics and calculate the construction cost of every city's sewage treatment facility, then we can obtain $b_i$. Thus, this information is known for the environmental management department and all polluters. In order to calculate the variable cost coefficient $a_i$, we can combine the integrated information of the current

Table 1. Allocation results of COD in 2010.

| Number | City | Actual Discharge Amount in 2005 ($10^4$ ton) | Controlling Quantity in 2010 ($10^4$ ton) | Predicted Discharge Amount in 2010 $G_i$ ($10^4$ ton) | Declared Marginal Cost Function $a_i$ (Yuan/ton$^2$) | $b_i$ (Yuan/ton) | Allocated Quantity in 2010 ($10^4$ ton) |
|---|---|---|---|---|---|---|---|
| 1 | Wuhan | 16.85 | 14.97 | 20.99 | 0.08 | 1050.00 | 18.57 |
| 2 | Huangshi | 3.28 | 3.22 | 4.08 | 0.13 | 556.00 | 2.21 |
| 3 | Ezhou | 1.40 | 1.33 | 1.74 | 0.17 | 699.00 | 0.40 |
| 4 | Xiaogan | 3.30 | 3.20 | 4.11 | 0.10 | 1551.00 | 2.68 |
| 5 | Huanggang | 3.95 | 3.73 | 4.92 | 0.11 | 871.00 | 3.00 |
| 6 | Xianning | 3.68 | 3.58 | 4.59 | 0.17 | 950.00 | 3.39 |
| 7 | Xiantao | 0.97 | 1.40 | 1.21 | 0.18 | 807.00 | 0.00 |
| 8 | Tianmen | 1.68 | 1.54 | 2.09 | 0.17 | 1623.00 | 1.29 |
| 9 | Qianjiang | 0.80 | 0.80 | 0.99 | 0.19 | 1431.00 | 0.17 |
| Total amount | | 35.91 | 33.77 | 44.72 | Equilibrium price $p^* = 2924.50$ Yuan/ton | | 31.71 |

situation of every city's sewage treatment, actual expenditure of related organizations and the reform trend of administrative organization in future five years.

According to the statistical data of nine cities' environmental protection "Eleventh Five-Year Plan", we can calculate the values of $a_i$ and $b_i$, $i = 1, 2, \ldots, 9$. The results are listed in Table 1. Furthermore, we can obtain the following statistical data:

(1) In 2005, nine cities' actual total discharge amount of COD is 359,100 tons, and the detailed statistical data are listed in Table 1.
(2) It is supposed that the total discharge amount is increasing with the annual growth rate of 4.5%. Based on this condition and nine cities' actual total discharge amount of COD in 2005, we can forecast the total discharge amount of COD in 2010. The results are listed in Table 1.
(3) By the year of 2010, the assigner's control objective of nine cities' total permitted emission quantity of COD is $Q_0$= 337,700 tons.

Based on the data in Table 1, we substitute related data into Model $M_1$ and solve the problem by using the software of Lingo. Then, we obtain the optimal total allocated quantity $Q^* = 317{,}100$ tons in 2010, and the nine cities' allocated quantity $q_i$, $i = 1, 2, \ldots, 9$ are listed in Table 1.

In practice, the assigner usually uses the free allocation method to allocate the COD. This free allocation method is simple and maneuverable. For any declared information of polluters, the quantity $Q_0$ of COD will be entirely allocated to the polluters by using the free allocation method. However, the polluter $i$'s declared marginal cost $g_i(x)$ is the polluter $i$'s private or partially private information. So the facticity of polluters' declared information and the validity of allocation results are not guaranteed. If we use the auction Model $M_1$ to allocate COD, the quantity $Q_0$ may be not entirely allocated to the polluters. From the stimulation of Model $M_1$ we know that if the polluters' declared information is not strictly true, then the actual total allocated capacity $Q^*$ is less than the fixed supply $Q_0$. And the realer the declared information is, the greater the actual total allocated capacity $Q^*$ is.

During the "Eleventh Five-Year Plan," if the assigner uses Model $M_1$ to auction COD or the other TPPDC, once the polluters observe the optimal allocating capacity $Q^* < Q_0$, then they will adjust their bids until $Q^* = Q_0$. Therefore, the declaration information and the validity of allocation are improved, and the allocation results are more reasonable, and the assigner's income will also be optimized.

In addition, because of the complexity of practical problems, the imbalance in regional development and the various support policies for different regions and different sectors, the results listed in Table 1 are only regarded as a reference for pollution charge or evaluating the credibility of environmental planning. The declared marginal treatment cost functions $g_i(x)$, $i = 1, 2, \ldots, 9$ given in Table 1 are not the results of direct and perfect competition, but only as a reference factor to make an environmental planning. There are two main reasons for the difference of polluters' biddings. The first is the regional difference, personnel difference and technique differences. Second, the environmental management department usually uses the multi-objective approaches (such as regional and industry support, the protection of key region, economic capacity, and the local environmental sustainability) to take into account the feasibility and fairness of allocation.

## 5. Conclusions

This paper studies an optimal auction mechanism that maximizes the seller's expected utility for a kind of divisible goods. Under the private signal conditions, the feasibility of optimal auction mechanism, i.e., incentive compatibility and individual rationality, is analyzed, and the necessary conditions of the optimal auction are given. Furthermore, we present the method of how to use the uniform price auction to implement the optimal auctions under the background of allocating the TPPDC. We apply our optimal auction mechanism to the environmental planning of Wuhan City Circle based on the statistical data of chemical oxygen demand (COD) in nine cities of Wuhan City Circle, and get satisfactory results. Therefore, this paper generalizes the auction models with single object and unit demand well. The optimal auction mechanism can be widely applied in some other allocations and auctions of divisible goods, such as stocks, treasury bills, network bandwidth, land, and power.

## Acknowledgments

This research is supported by the Postgraduate Science & Technology Innovation Fund of HUST (No. HF-06-007-08-184), the National Natural Science Foundation of China Grant (No. 70771041), and the Excellent Youth Project of Hubei Provincial

Department of Education (No. Q20102904). The authors are very grateful to Professor Yong Shi, Professor Gang Kou, and the two anonymous reviewers for their very valuable comments and suggestions to improve the quality of this research paper.



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