Applied Mathematics and Computation 216 (2010) 819–821

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on approximation properties of q-Durrmeyer operators

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ARTICLE INFO

Keywords:
q-Bernstein polynomials
Positive linear operators
q-Durrmeyer operators
Approximation process
Continuous functions

ABSTRACT

In this paper, the approximation properties of q-Durrmeyer operators $D_{n,q}(f;x)$ for $f \in C[0,1]$ are discussed. The exact class of continuous functions satisfying approximation process $\lim_{n\to\infty} D_{n,q}(f;x) = f(x)$ is determined. The results of the paper provide an elaboration of the previously-known ones on operators $D_{n,q}$.

1. Introduction

Recently, Gupta [3] introduced q-Durrmeyer type operators $D_{n,q}$ and $D_{\infty,q}$. These operators are defined respectively as follows:

Definition 1. Let $f \in C[0,1],\ 0 < q \leqslant 1$ and $0 \leqslant x \leqslant 1$, q-Durrmeyer operators $D_{n,q}(f;x)$ are defined by

$$D_{n,q}(f;x) = [n+1]\sum_{k=0}^{n} q^{-k} p_{nk}(q;x)\int_0^1 f(t)p_{nk}(q;qt)d_qt, \tag{1}$$

where $p_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1-q^s x)$ are q-Bernstein basis functions.

Definition 2. Let $f \in C[0,1],\ 0 < q < 1$ and $0 \leqslant x \leqslant 1$, the operator $D_{\infty,q}(f;x)$ is defined by

$$D_{\infty,q}(f;x) = \frac{1}{1-q}\sum_{k=0}^{\infty} p_{\infty k}(q;x)q^{-k}\int_0^1 f(t)p_{\infty k}(q;\ qt)d_qt, \tag{2}$$

where $p_{\infty k}(q;x) = \frac{x^k}{(1-q)^k[k]!}\prod_{s=0}^{\infty}(1-q^s x)$.

When $q = 1,\ D_{n,q}(f;x)$ reduces to the well-known Durrmeyer operators $D_n(f;x)$ (cf. [1]),

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_0^1 f(t)p_{nk}(t)\,dt.$$

We recall some important polynomials and operator which are closely related to the operators $D_{n,q}$ and $D_{\infty,q}$. Let $B_{n,q}(f;x)$, (n =1, 2,...) be the q-Bernstein polynomials of a function $f \in C[0,1]$. These polynomials have been studied by a lot of authors, see [2,4,7–14]. In the case $0 < q < 1$, a sequence $\{B_{n,q}(f;x)\}$ generates a positive linear operator $B_{\infty,q}$ on $C[0,1]$. For a function $f \in C[0,1]$, the operator $B_{\infty,q}$ is defined by

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doi:10.1016/j.amc.2010.01.086

$$B_{\infty,q}(f;x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)p_{\infty k}(q;x) & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1. \end{cases} \tag{3}$$

Operator (3) is called the limit q-Bernstein operator, whose nature is similar to that of operator $D_{\infty,q}$. This operator was introduced for the first time in [4], and studied in detail in [6].

For $f = t^j,\ j = 0,1,\ldots$, operators $D_{n,q}$ and $D_{\infty,q}$ can be written with the help of the q-Beta function (cf. [5]) as follows:

$$D_{n,q}(t^j;x) = \sum_{k=0}^{n} c_{nk}p_{nk}(q;x),$$

where

$$c_{nk} = [n+1]\begin{bmatrix} n \\ k \end{bmatrix} B_q(j+k+1, n-k+1)$$

and

$$D_{\infty,q}(t^j;x) = \sum_{k=0}^{n} c_{\infty k}p_{\infty k}(q;x),$$

where

$$c_{\infty k} = \frac{1}{1-q} \cdot \frac{1}{(1-q)_q^k} B(j+k+1, \infty).$$

Using these formulae we obtain by plain calculations:

$$D_{\infty,q}(1;x) = 1, \quad D_{\infty,q}(t;x) = 1 + q(x-1), \tag{4}$$

$$D_{\infty,q}(t^2;x) = (1-q)^2(1+q) + q(1+2q)(1-q)x + q^3(1-q)x + q^4x^2, \tag{5}$$

for all $x \in [0,1]$.

2. Results

For approximation properties of the operators $D_{n,q}(f;x)$ and $D_{\infty,q}(f;x)$ Gupta [3] presented the following results.

Theorem A ([3, Theorem 3]). *Let $0 < q < 1$, then for each $f \in C[0,1]$ the sequence $\{D_{n,q}(f;x)\}$ converges to $D_{\infty,q}(f;x)$ uniformly on $[0,1]$. Furthermore,*

$$\|D_{n,q}(f) - D_{\infty,q}(f)\| \leqslant C_q\omega(f, q^n). \tag{6}$$

Theorem B ([3, Theorem 4]). *Let $0 < q < 1$ be fixed and let $f \in C[0,1]$. Then $D_{\infty,q}(f;x) = f(x)$ for all $x \in [0,\ 1]$ if and only if f is linear.*

Theorem C ([3, Theorem 5]). *For any $f \in C[0,1]$, $\{D_{\infty,q}(f;x)\}$ converges to f uniformly on* [0, 1] *as $q \to 1-$.*

These Theorems are important for the research of q-Durrmeyer type operators. However, there are some inaccuracies in Theorems B and C. In fact, from (4), we may observe that the operator $D_{\infty,q}(f;x)$ does not reproduce the linear functions. Hence, the class of continuous functions satisfying the approximation process $\lim_{n\to\infty} D_{n,q}(f;x) = f(x)$, for $x \in [0,1]$ is not the one given in Theorem B. Then, what class of continuous functions will exactly satisfy approximation process $\lim_{n\to\infty} D_{n,q}(f;x) = f(x)$ for $x \in [0,1]$? Undoubtedly, this is an important problem. In this note we will solve this problem by giving the following theorem and corollary.

Theorem 1. *Let $0 < q < 1$ be fixed and $f \in C[0,1]$, then $D_{\infty,q}(f;x) = f(x)$ for all $x \in [0,1]$ if and only if f is constant.*

To establish Theorem 1 we need to prove the following Lemma 1.

Lemma 1. *Let L be a positive linear operator on $C[0,1]$ which reproduces constant functions. If $L(t;x) > x$ for all $x \in [0,1)$, then $L(f) = f$ if and only if f is a constant function.*

Proof. Since L reproduces constant functions, it is sufficient to prove that f is constant if $L(f) = f$. Let $g(x) = f(x) - f(1)$. Then $g(1) = 0$ and $Lg = g$. We will show that $g = 0$. Assume that $g \neq 0$. Without loss of generalization we may assume that there exits an $x_0 \in [0,1)$ such that $g(x_0) > 0$. Then, it must exist a negative number α, such that $\frac{\alpha}{2} > \alpha(x_0 - \frac{1}{2}) - g(x_0)$. Now let $h(x) = \alpha(x - \frac{1}{2}) - g(x)$. $h(x)$ is continuous on [0,1] and $h(1) > h(x_0)$. Let m be the minimum of $h(x)$ on [0,1], then there exists a $\xi \in [0,1)$ such that $h(\xi) = \alpha(\xi - \frac{1}{2}) - g(\xi) = m$. Thus for all $x \in [0,1]$

$$\alpha\left(x - \frac{1}{2}\right) - g(x) \geqslant m = \alpha\left(\xi - \frac{1}{2}\right) - g(\xi),$$

that is

$$g(x) \leqslant \alpha(x-\xi)+g(\xi). \tag{7}$$

Since L is a positive linear operator on $C[0,1]$, it follows from (5) that

$$L(g(x);\xi) \leqslant \alpha L((x-\xi);\xi)+g(\xi)=\alpha(L(x;\xi)-\xi)+g(\xi). \tag{8}$$

Note that $L(g(x);\xi)=g(\xi)$ and $L(x;\xi)>\xi$, inequality (8) derives $\alpha \geqslant 0$, which leads to a contradiction. Hence $g=0$. Lemma 1 is proved. □

Remark 1. The condition "If $L(t;x)>x$ for all $x \in [0,1)$" in Lemma 1 can be replaced by the condition "If $L(t;x)>x$ for all $x \in (0,1]$". The proof is similar, we omit the details here.

Remark 2. Lemma 1 supplies a sufficient condition for constant function to be the only fixed points of a positive linear operator on $C[0,1]$. Earlier, a result of similar type has been presented by Wang in [13, Theorem 9]. He proved that if a positive linear operator L on $C[0,1]$ leaves invariant linear functions and satisfies $L(t^2;x)>x^2$ for $x \in (0,1)$, then $Lf=f$ if and only if f is a linear function.

From (4) we have

$$D_{\infty,q}(t;x)=1+q(x-1)>x, \quad x \in [0,1).$$

Theorem 1 follows by this inequality and Lemma 1.

From Theorems 1 and A we obtain

Corollary 1. *For fixed $q \in (0,1)$ and $f \in C[0,1]$, the sequence $\{D_{n,q}(f;x)\}$ does not approximate $f(x)$ unless f is constant function. This is completely in contrast to the classical Durremeyer operators, by which $\{D_n(f;x)\}$ approximates $f(x)$ for any $f \in C[0,1]$.*

Remark 3. Since the positive linear operator $D_{\infty,q}(f;x)$ does not reproduce linear functions, the proof of Theorem C in article [3] needs to be corrected. It should add the condition $D_{\infty,q}(t;x)=1+q(x-1)\to x$ in the proof, and then use Korovkin theorem to derive the result.

Acknowledgements

This work is supported by the National Natural Science Foundation of China.

The authors thank the referee(s) for several important comments and suggestions which improve the quality of the paper.

References

[1] M.M. Derriennic, Sur approximation de functions integrables sur par des polynomes de Bernstein modifies, J. Approx. Theory 31 (1981) 325–343.
[2] M.M. Derriennic, Modified Bernstein polynomials and Jacobi polynomials in q-calculus, Rendiconti Del Circolo Matematico Di Palermo, Serie II, Suppl. 76, 2005, pp. 269-290.
[3] V. Gupta, Some approximation properties of q-Durrmeyer operators, Appl. Math. Comput. 197 (2008) 172–178.
[4] A. Il'inskii, S. Ostrovska, Convergence of generalized Bernstein polynomials, J. Approx. Theory 116 (2002) 100–112.
[5] V. Kac, P. Cheung, Quantum Calculus, Springer, New York, 2002.
[6] S. Ostrovska, On the improvement of analytic properties under the limit q-Bernstein operator, J. Approx. Theory 138 (2006) 37–53.
[7] S. Ostrovska, The first decade of the q-Bernstein polynomials: results and perspectives, J. Math. Anal. Approx. Theory 2 (1) (2007) 35–51.
[8] S. Ostrovska, q-Bernstein polynomials of the Cauchy kernel, Appl. Math. Comput. 198 (1) (2008) 261–270.
[9] G.M. Phillips, Bernstein polynomials based on the q-integers, Ann. Numer. Math. 4 (1997) 511–518.
[10] G.M. Phillips, Interpolation and Approximation by Polynomials, Springer, Berlin, 2003.
[11] V.S. Videnskii, On some classes of q-parametric positive operators, Oper. Theory: Adv. Appl. 158 (2005) 213–222.
[12] H. Wang, Voronovskaya type formulas and saturation of convergence for q-Bernstein polynomials for $0 < q < 1$, J. Approx. Theory 145 (2) (2007) 182–195.
[13] H. Wang, Properties of convergence for ω, q-Bernstein polynomials, J. Math. Anal. Appl. 340 (2008) 1096–1108.
[14] H. Wang, X. Wu, Saturation of convergence for q-Bernstein polynomials in the case $q > 1$, J. Math. Anal. Appl. 337 (1) (2008) 744–750.