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Multipartite Entanglement in Heisenberg Model*

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Abstract *The effects of anisotropy and magnetic field on multipartite entanglement of ground state in Heisenberg XY model are investigated. The multipartite entanglement increases as a function of the inverse strength of the external field when the degree of anisotropy is finite. There are two peaks when the degree of anisotropy is $\gamma = \pm 1$. When the degree of anisotropy increases further, the multipartite entanglement will decrease and tend to a constant. The threshold of the inverse strength of the external field for generating multipartite entanglement generally decreases with the increasing of qubits.*

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1 Introduction

The entanglement is an important resource in the fields of quantum computation and quantum information.[1−3] Due to its potential applications, the pairwise entanglement of anisotropic Heisenberg model has been extensively studied in recent years.[4,5] The entanglement of thermal states was introduced. Its properties, including threshold temperature, magnetic field dependence and anisotropic effects, were studied. The entanglement properties of ground state are very important. Some properties of the ground state were studied.[6] The pairwise entanglement in one-dimensional infinite-lattice anisotropic XY model was introduced.[7] The bipartite entanglement is well understood, while the multipartite entanglement is still under intensive research. To understand the multipartite entanglement, the distributed entanglement has been presented.[8] The residual entanglement is generalized to the multipartite entanglement.[9] The multipartite entanglement in Ising model is also studied.[10]

In this paper, the multipartite entanglement of ground state in a Heisenberg XY model with an external magnetic field is investigated. In Sec. 2, the basic measures of the multipartite entanglement are presented. In Sec. 3, the multipartite entanglement in Heisenberg XY model is studied when an external magnetic field is presented. A discussion concludes the paper.

2 Measures of Multipartite Entanglement

The anisotropic Heisenberg XY model of a one-dimensional lattice with N sites in a transverse field can be described by the Hamiltonian[10] of

$$H = -\sum_{i=1}^{N}\Big\{\frac{\lambda}{2}[(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y] + \sigma_i^z\Big\}\,, \quad (1)$$

where γ is the degree of anisotropy, λ is the inverse strength of the external magnetic field, σ_i^β $(\beta = x, y, z)$ are the Pauli matrices at qubit of i. The cyclic boundary conditions of $\sigma_{N+1}^\beta = \sigma_1^\beta$ $(\beta = x, y, z)$ is assumed.

The quantity tangle τ[8] is introduced to measure the tripartite entanglement of a pure state $|\psi\rangle$. For a tripartite two-level system, the residual entanglement is referred to as

$$\tau_{ABC} = C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2\,, \quad (2)$$

where C_{AB} and C_{AC} are the concurrence of the original pure state ρ_{ABC} with tracing over the qubits C and B, respectively, $C_{A(BC)}$ is the concurrence of $\rho_{A(BC)}$ with qubits B and C regarded as a single object. It is shown that the residual entanglement of a three-qubit state $|\psi\rangle = \sum_{i,j,k} a_{ijk}|ijk\rangle$ can be obtained,[8]

$$\tau_{ABC} = 2\Big|\sum a_{ijk}a_{i'j'm}a_{npk'}a_{n'p'k'}\epsilon_{ii'}\epsilon_{jj'}\epsilon_{kk'}\epsilon_{mm'}\epsilon_{nn'}\epsilon_{pp'}\Big|\,, \quad (3)$$

where the sum is taken over all the indices, and $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = \delta_{\alpha\beta}$.

The residual entanglement can be generalized to the multipartite entanglement.[9] The residual entanglement $\tau_{ABCD\cdots N}$ of N-particle system $\rho_{ABCD\cdots N}$ is defined as

$$\tau_{ABC\cdots N} = \min\Big\{\tau_\alpha|\alpha = 1, 2, 3, \ldots, \sum_{i=1}^{[n/2]} C_N^i\Big\}\,, \quad (4)$$

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where τ corresponds to all possible foci, $C_N^i = N/i(N-i)$ and $[N/2]$ is $N/2$ when N is even, $[N/2]$ is $(N-1)/2$ when N is odd. When the focus is A, the residual entanglement is

$$\tau_{A(BC\cdots N)} = C^2_{A(BC\cdots N)} - C^2_{AB} - C^2_{AC} - \cdots - C^2_{AN} \,. \quad (5)$$

If the focus is changed, one will obtain the other $N-1$ equations. It is worth while noting that AB, ABC and so on can be considered as foci. So there are $\sum_{i=1}^{[N/2]} C_N^i$ focus. The multipartite entanglement of the well-known Greenberger–Horne–Zeilinger (GHZ) state

$$(1/\sqrt{2})(|00\cdots 0\rangle + |11\cdots 1\rangle)$$

and W state

$$(1/\sqrt{N})(|0\cdots 01\rangle + |0\cdots 10\rangle + |1\cdots 00\rangle)$$

correspond to 1 and 0 respectively.

3 Multipartite Entanglement

The generalized residual entanglement can be used to calculate the entanglement of an anisotropic Heisenberg XY model when there is an external magnetic field.

3.1 *Three and Four Qubits*

The ground state of the anisotropic Heisenberg XY model can be obtained by

$$|g\rangle = N_c\big[\big(2-\lambda+\sqrt{\lambda^2-4\lambda+4+3\lambda^2\gamma^2}\big)|000\rangle + \gamma\lambda|110\rangle + \gamma\lambda|011\rangle + \gamma\lambda|101\rangle\big]\,, \quad (6)$$

where N_c is a normalization constant. The multipartite entanglement τ of ground state in three qubits can be easily obtained by Eqs. (3) and (4). It is plotted as a function of the magnetic field λ and the degree of anisotropy γ in Fig. 1.

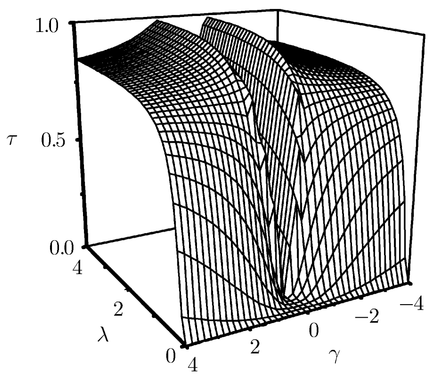

Fig. 1 The multipartite entanglement of ground state in three qubits is plotted as functions of the magnetic fields λ and degree of anisotropy γ.

It is shown that the multipartite entanglement of three qubits is symmetric about $\gamma = 0$. There is no multipartite entanglement of ground state in the isotropy Heisenberg XY model ($\gamma = 0$). It is found that there are two peaks located at $\gamma = \pm 1$. The peak value will reach 1.0 with the increase of λ. The ground state can be described by $|g\rangle = (1/2)(|000\rangle + |110\rangle + |011\rangle + |101\rangle)$. When $|\gamma| < 1$, the entanglement increases as a function of $|\gamma|$. After τ reaches the peak at $|\gamma| = 1$, the entanglement decreases and finally saturates to a constant value of about 0.77. The eigenvalues and eigenstates of isotropic Heisenberg XY model in an external magnetic field can be exactly solved by symmetric[11] or the Jordan–Wigner transformation.[12] For the eigenstates of three qubits, there is no entanglement in two states of $|000\rangle$ and $|111\rangle$. The other six states are just like W states. It means that all the states have no multipartite entanglement in the three-qubit isotropic Heisenberg XY model.

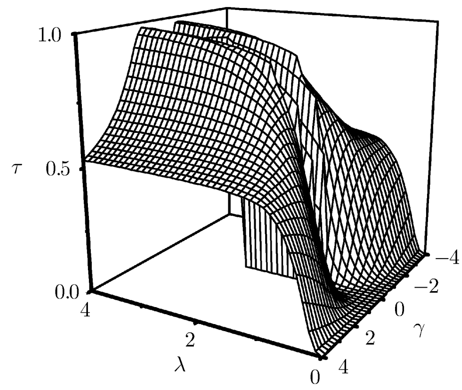

Fig. 2 The multipartite entanglement of ground state in four qubits is plotted as functions of the magnetic fields λ and degree of anisotropy γ.

The multipartite entanglement τ of the ground state in four qubits of an anisotropic Heisenberg model is plotted as a function of the magnetic field λ and the degree of anisotropy γ in Fig. 2. Similar result as that in Fig. 1 is obtained. The saturation value of τ is about 0.414 when $|\gamma|$ is very large. When $\gamma = 0$ and $\lambda > 2.414$, the multipartite entanglement still exists with a value of about 0.414. For four qubits isotropic XY model, it can also be exactly solved by symmetric[11] or the Jordan–Wigner transformation.[12] The eigenvalues of the ground states can be given by

$$E_0 = -4\,, \quad E_1 = -2\lambda - 2\,, \quad E_2 = -2\sqrt{2}\lambda\,, \quad (7)$$

and the corresponding eigenstates are

$$\begin{aligned} |\psi_0\rangle &= |0000\rangle\,,\\ |\psi_1\rangle &= \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle)\,,\\ |\psi_2\rangle &= \frac{\sqrt{2}}{4}(|0011\rangle + |0110\rangle + |1100\rangle + |1001\rangle)\\ &\quad + \frac{1}{2}(|0101\rangle + |1010\rangle)\,. \end{aligned} \quad (8)$$

When the inverse strength of the external field λ is small, the ground state is $|\psi_0\rangle$. There is no entanglement. When λ increases, the ground state changes to $|\psi_1\rangle$. It is just the W state. There is no multipartite entanglement either. However, if λ increases further, the ground state changes to $|\psi_2\rangle$. The multipartite entanglement exists.

3.2 *General Case of N Qubits*

The Hilbert space of N qubits in a one-dimensional Heisenberg chain is 2^N-dimensional and the corresponding Hamiltonian H_N has 2^N eigenvectors and eigenvalues.

When λ approaches zero, the ground state of the anisotropic Heisenberg XY model in a one-dimensional lattice with N sites in a transverse field becomes a product of spins pointing in the z-direction,

$$|g\rangle = |00\cdots 00\rangle\,. \tag{9}$$

When $\gamma = 1$ the model will be reduced to Ising model. When λ approaches infinity, the ground state becomes a GHZ state and is given by[13]

$$|g\rangle = N_c \sum_{\{i,j,\ldots\}_{\rm even}} |\{i,j,\ldots\}\rangle\,, \tag{10}$$

where N_c is a normalization constant, $\{i,j,\ldots\}_{\rm even}$ means that the even state is selected for summing up all the foci.

For general case of N qubits, the multipartite entanglement τ is quite similar to that shown in Figs. 1 and 2. There are also two peaks located at the degree of anisotropy equal to $\gamma = \pm 1$. When the degree of anisotropy increases further, the multipartite entanglement will saturate to a constant value. Although the three and four qubits are simple, they share many features of the general case of a chain with arbitrary number of N qubits.

The limiting cases of $\gamma = 0$ and $\gamma \to \infty$ need to be investigated when N is very large. When the degree of anisotropy $\gamma = 0$ and the inverse strength of the external field $\lambda \to \infty$, the Hamiltonian H_N has the following form

$$H_1 = -C_I \sum_{i=1}^{N} \bigl[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\bigr]\,. \tag{11}$$

When the degree of anisotropy $\gamma \to \infty$, the Hamiltonian H_N has the following form,

$$H_2 = -C_I \sum_{i=1}^{N} \bigl[\sigma_i^x \sigma_{i+1}^x - \sigma_i^y \sigma_{i+1}^y\bigr]\,. \tag{12}$$

The ground state is degenerate. The magnetic field can eliminate the degeneracy and the perturbation theory can be used. When N is even, it is easy to find

$$[H_1, H_2] = 0\,. \tag{13}$$

The result cannot be extended to odd qubits. So it is not strange that the multipartite entanglement for the cases of even and odd number of qubits are different.

For an isotropic Heisenberg model with $\gamma = 0$, the multipartite entanglement exists for finite values of λ when the number of qubits increases. The threshold of λ that the multipartite entanglement exists is shown in Table 1 when the number of qubits is varied. Small value of λ can induce multipartite entanglement if the number of qubits increases. When the inverse strength of the external field $\lambda \to \infty$, the multipartite entanglement will be stable. The stable value is shown in Table 2. Meanwhile, the stable value when $\gamma \to \infty$ is also shown. It is found that the stable values are different for odd and even qubits respectively when the number of qubits increases. For even number of qubits, the value of $\gamma = 0$, $\lambda \to \infty$ equals the value of $\gamma \to \infty$. While for odd qubits, these values are different.

Table 1 The threshold of the inverse strength of the external field when there exists multipartite entanglement of the ground state for N qubits when $\gamma = 0$.

N	3	4	5	6	7	8
λ	∞	2.42	1.62	1.37	1.25	1.18

Table 2 The multipartite entanglement of ground state of N qubits when $\gamma \to \infty$ and $\gamma = 0$, $\lambda \to \infty$.

N	3	5	7	4	6	8
$\gamma \to \infty$	0.7698	0.7740	0.7725	0.414	0.5463	0.5272
$\gamma = 0, \lambda \to \infty$	0	0.5683	0.6245	0.414	0.5463	0.5272

4 Discussion

In the paper, the multipartite entanglement of the ground state in an anisotropic Heisenberg model of three and four qubits is investigated. The effects of anisotropy and magnetic field are discussed. Some properties can be extended to the general case of N qubits. The multipartite entanglement is an increasing function of the inverse strength of the external field when the degree of anisotropy is not equal to zero. It is found that there are two peaks located at $\gamma = \pm 1$. When $\gamma = 1$, the model reduces to Ising model. When the degree of anisotropy increases further, the multipartite entanglement will saturate to a constant value. It is found that the constant value for the multipartite entanglement generally decreases with the increase of qubits. When the inverse strength of the external field approaches infinity, it is found that the value of $\gamma = 0$, $\lambda \to \infty$ equals the value of $\gamma \to \infty$ for even number of qubits. For odd qubits, these values are different.

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