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An implicit numerical method for the two-dimensional fractional percolation equation

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ABSTRACT

In this paper, an implicit numerical method for the two-dimensional fractional percolation equation without the assumption of continued and rigid body motion is proposed. Consistency, stability and convergence of the implicit finite difference method are established. Finally, some numerical examples are given. The numerical results demonstrate the effectiveness of the theoretical analysis.

1. Introduction

Percolation flow problems have been analyzed in many research fields including seepage hydraulics, groundwater hydraulics, groundwater dynamics and fluid dynamics in porous media [8,23,21,4]. The equation governing the two-dimensional flow in porous media under the hypotheses of continuity and Darcy's law is given by Bear and Verruijt [1]

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial p}{\partial y}\right)=S_0\frac{\partial p}{\partial t},\quad (x,y)\in\Omega, \tag{1}$$

where p is the pressure head; S_0 is the specific storativity, which is often approximated by the specific yield or effective porosity; Ω denotes the percolation domain; $K=\frac{k\rho g}{\mu}$ is the hydraulic conductivity with components k_x, k_y in the x and y directions; k the intrinsic permeability of the media; ρ the density of water; μ the viscosity of water; and g the gravitational acceleration.

While these models have been applied in many important applications of seepage, the underlying assumption is not adequate in describing the movement of solute in a non-homogeneous porous medium because of the large deviation from standard Gaussian diffusion. A more realistic model is obtained by considering the use of fractional-order density gradient to recover the non-homogeneities of the porous medium. He [7] and Ochoa-Tapia [19] proposed the following modified Darcy law, or generalized Darcy law, with Riemann–Liouville fractional derivative:

$$q_x=k_x\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}},\quad 0<\alpha_1<1, \tag{2}$$

$$q_y=k_y\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}},\quad 0<\alpha_2<1. \tag{3}$$

The usual Darcy law is recovered in the special case that $\alpha_1=\alpha_2=1$. Here, the Riemann–Liouville fractional derivative is defined as [17,20,22]

$$\frac{\partial^{\alpha}p}{\partial x^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x p(s,y,t)(x-s)^{-\alpha}ds, \tag{4}$$

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where $0 < \alpha < 1$. Under the simplifying assumption of continuity of seepage flow, we have the following fractional differential equation:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}\right) + h(x,y,t) = S_0 \frac{\partial p}{\partial t}, \quad (x,y) \in \Omega, \tag{5}$$

where $h = h(x, y, t)$ is the source term. Eq. (5) is an important special case treated in the literature [7]. If we further simplify Eq. (5) by making the additional assumption that seepage flow may be considered as a rigid body motion, then the continuity equation can be written as

$$\frac{\partial^0}{\partial x^0}\left(k_x \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}\right) + \frac{\partial^0}{\partial y^0}\left(k_y \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}\right) + h(x,y,t) = S_0 \frac{\partial p}{\partial t}, \quad (x,y) \in \Omega, \tag{6}$$

with the definition $\frac{\partial^0 u}{\partial x^0} \equiv u$. In fact, the seepage flow is neither continued nor rigid, so a more general equation for seepage flow can be expressed as

$$\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}\right) + \frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}\right) + h(x,y,t) = S_0 \frac{\partial p}{\partial t}, \quad (x,y) \in \Omega, \tag{7}$$

where $0 \leqslant \beta_1,\ \beta_2 \leqslant 1$. Moreover, in non-uniform media, the equation can be written as

$$\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x(x,y) \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}\right) + \frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y(x,y) \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}\right) + q(x,y,t) = \frac{\partial p}{\partial t}, \tag{8}$$

where $(x,y) \in \Omega,\ k_x(x,y) = k_x/S_0,\ k_y(x,y) = k_y/S_0,\ q(x,y,t) = h(x,y,t)/S_0$. The above equation, known as the two-dimensional fractional percolation equation, is the focus of this paper.

As is well-known, analytic solutions of most fractional differential equations cannot be obtained explicitly, so many authors resort to numerical solution strategies based on convergence and stability analysis [2,3,5–7,9–11,13–15,18,24,25]. Liu et al. [12] developed some modified alternating direction methods for solving a two-dimensional non-continuous seepage flow equation with fractional derivatives in uniform media. However, numerical methods for the two-dimensional fractional percolation equation without the assumption of continued and rigid body motion are still limited. This motivates us to consider a computationally effective implicit finite difference method for this equation.

The structure of the paper is as follows. In Section 2, an implicit numerical method for the fractional percolation equation without the assumptions of continued and rigid body motion is proposed and its consistency is analyzed. The stability and convergence of the implicit numerical method are discussed in Section 3. Two numerical results for the fractional percolation equation are given in Section 4 to evaluate the performance of the method.

2. An implicit numerical method and its consistency

In this section we discuss the following two-dimensional fractional percolation equation without the assumption of continued and rigid body motion:

$$\frac{\partial p}{\partial t} = \frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x(x,y) \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}\right) + \frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y(x,y) \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}\right) + q(x,y,t), \quad (x,y) \in \Omega, \quad 0 \leqslant t \leqslant T, \tag{9}$$

subject to the initial condition

$$p(x,y,0) = \varphi(x,y), \quad (x,y) \in \Omega, \tag{10}$$

and the Dirichlet boundary conditions

$$p(a_1,y,t) = p(x,b_1,t) = 0, \tag{11}$$

$$p(a_2,y,t) = u(y,t), \quad p(x,b_2,t) = v(x,t), (x,y) \in \Omega, \quad 0 \leqslant t \leqslant T, \tag{12}$$

where $\Omega = \{(x,y)|a_1 \leqslant x \leqslant a_2, b_1 \leqslant y \leqslant b_2\}$.

To derive a numerical approximation, we define $t_n = n(\Delta t)$ to be the integration time $t_n > 0,\ \Delta x = (a_2 - a_1)/m_1$ is the grid size in the x-direction, where m_1 is a positive integer, with $x_i = a_1 + i(\Delta x)$ for $i = 0, \ldots, m_1;\ \Delta y = (b_2 - b_1)/m_2$ is the grid size in the y-direction, where m_2 is a positive integer, with $y_j = b_1 + j(\Delta y)$ for $j = 0, \ldots, m_2$. Define $p^n_{i,j}$ as the numerical approximation to $p(x_i, y_j, t_n)$. The initial conditions are defined by $p^0_{i,j} = \varphi(x_i, y_j)$. Similarly, we define $k_{i,j} = k_x(x_i, y_j),\ \bar{k}_{i,j} = k_y(x_i, y_j)$, $\varphi_{i,j} = \varphi(x_i, y_j)$ and $q^n_{i,j} = q(x_i, y_j, t_n)$. The Dirichlet boundary conditions are defined by $p^n_{0,j} = p^n_{i,0} = 0,\ p^n_{m_1,j} = u(y_j, t_n),\ p^n_{i,m_2} = v(x_i, t_n),\ 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ n > 0$.

To approximate Eq. (9) we use the backward Euler scheme for the first order time derivative given by

$$\left.\frac{\partial p}{\partial t}\right|_{(x_i,y_j,t_n)} \sim \frac{p^n_{i,j} - p^{n-1}_{i,j}}{\Delta t} + O(\Delta t). \tag{13}$$

Similarly, using Theorem 1 in [3] and the Dirichlet boundary condition, the mixed fractional spatial derivatives in Eq. (9) can be described as

$$\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x(x,y)\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}}\right)\bigg|_{(x_i,y_j,t_n)} = \frac{1}{(\Delta x)^{\alpha_1+\beta_1}}\sum_{v=1}^{i+1}\left[\sum_{u=0}^{i+1-v} g_{\beta_1,u}g_{\alpha_1,i+1-v-u}k_{i-u,j}\right]p^n_{v,j} + O(\Delta x),$$
$$\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y(x,y)\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}}\right)\bigg|_{(x_i,y_j,t_n)} = \frac{1}{(\Delta y)^{\alpha_2+\beta_2}}\sum_{v=1}^{j+1}\left[\sum_{u=0}^{j+1-v} g_{\beta_2,u}g_{\alpha_2,j+1-v-u}\bar{k}_{i,j-u}\right]p^n_{i,v} + O(\Delta y), \tag{14}$$

where, for example,

$$g_{\beta_1,u} = (-1)^u\binom{\beta_1}{u} = (-1)^u\frac{\Gamma(\beta_1+u+1)}{\Gamma(\beta_1+1)\Gamma(u+1)}. \tag{15}$$

Therefore the implicit numerical method for the two-dimensional fractional percolation equation is determined by the finite difference equation

$$\frac{p^n_{i,j}-p^{n-1}_{i,j}}{\Delta t} = \frac{1}{(\Delta x)^{\alpha_1+\beta_1}}\sum_{v=1}^{i+1}\left[\sum_{u=0}^{i+1-v} g_{\beta_1,u}g_{\alpha_1,i+1-v-u}k_{i-u,j}\right]p^n_{v,j} + \frac{1}{(\Delta y)^{\alpha_2+\beta_2}}\sum_{v=1}^{j+1}\left[\sum_{u=0}^{j+1-v} g_{\beta_2,u}g_{\alpha_2,j+1-v-u}\bar{k}_{i,j-u}\right]p^n_{i,v} + q^n_{i,j},$$
$$i \leqslant m_1-1, \qquad 1 \leqslant j \leqslant m_2-1. \tag{16}$$

The method defined by (16) is consistent with order $O(\Delta t)+O(\Delta x)+O(\Delta y)$.

We next define the following difference operators:

$$\delta_x p^n_{i,j} = \frac{1}{(\Delta x)^{\alpha_1+\beta_1}}\sum_{v=1}^{i+1}\left[\sum_{u=0}^{i+1-v} g_{\beta_1,u}g_{\alpha_1,i+1-v-u}k_{i-u,j}\right]p^n_{v,j}, \tag{17}$$
$$\delta_y p^n_{i,j} = \frac{1}{(\Delta y)^{\alpha_2+\beta_2}}\sum_{v=1}^{j+1}\left[\sum_{u=0}^{j+1-v} g_{\beta_2,u}g_{\alpha_2,j+1-v-u}\bar{k}_{i,j-u}\right]p^n_{i,v}. \tag{18}$$

Then the implicit Euler method may be written in the operator form

$$(1-\Delta t\delta_x-\Delta t\delta_y)p^n_{i,j} = p^{n-1}_{i,j}+\Delta t q^n_{i,j}, \quad 1 \leqslant i \leqslant m_1-1, \quad 1 \leqslant j \leqslant m_2-1. \tag{19}$$

A standard method for the numerical solution of classical multi-dimensional PDEs is the alternating-direction implicit (ADI) method, where the difference equations are specified and solved in alternating directions. For the ADI method (and, in a similar fashion, the splitting method), the operator form is rewritten as a directional separation product form:

$$(1-\Delta t\delta_x)(1-\Delta t\delta_y)p^n_{i,j} = p^{n-1}_{i,j}+\Delta t q^n_{i,j}, \tag{20}$$

which introduces an additional perturbation error equal to $(\Delta t)^2(\delta_x\delta_y)p^n_{i,j}$. Following the argument of [16], we can conclude that the additional perturbation error is not larger than the approximation errors for the other terms in (9). Eq. (20), which is called the ADI-Euler method, is consistent with order $O(\Delta t)+O(\Delta x)+O(\Delta y)$.

Computationally, the ADI-Euler method defined by (20) can be solved using the following iterative scheme. At time t_n:

(1) First, we solve the problem in the x-direction (for each fixed y_j) to obtain an intermediate solution $p^*_{i,j}$:

$$(1-\Delta t\delta_x))p^*_{i,j} = p^{n-1}_{i,j}+\Delta t q^n_{i,j}, \tag{21}$$

(2) then solve in the y-direction (for each fixed x_i) to obtain $p^n_{i,j}$:

$$(1-\Delta t\delta_y))p^n_{i,j} = p^*_{i,j}. \tag{22}$$

The initial and boundary conditions for the numerical solution $p^n_{i,j}$ and $p^{n-1}_{i,j}$ are defined from the given initial and Dirichlet boundary conditions. Prior to carrying out step one of solving (21), the boundary conditions for the intermediate solution $p^*_{i,j}$ should be set from Eq. (22) which incorporates the values of $p^n_{i,j}$ at the boundary; otherwise the order of convergence will be affected. Specifically, we assume that the Dirichlet boundary conditions are given by the function $u(y,t)$ and $v(x,t)$ on the boundary of the rectangular region Ω. For example, on the right boundary we write $p^n_{m_1,j} = u(y_j,t_n)$ and compute the boundary values for p^* from

$$p^*_{m_1,j} = (1-\Delta t\delta_{\beta,y})p^n_{m_1,j}, \tag{23}$$

for use in setting up and solving the sets of Eq. (21). See the proofs of Theorem 1 and 2 in Section 3 for more details.

3. Stability and convergence of the implicit numerical method

In this section, we discuss the stability and convergence of the implicit finite difference equation (16). Let $X = [x_1,x_2,\ldots,x_m]^T \in \mathbb{R}^m$, $\|X\|_\infty = \max_{1\leqslant i\leqslant m}|x_i|$.

Lemma. *Let the matrix $D = (d_{i,j})_{m\times m} \in \mathbb{R}^{m\times m}$ satisfy the conditions*

$$\sum_{l=1,l\neq i}^{m} |d_{i,l}| \leqslant |d_{i,i}| - 1, \quad (i = 1, 2, \ldots, m) \tag{24}$$

Then

$$\|X\|_\infty \leqslant \|DX\|_\infty. \tag{25}$$

Proof. Suppose that there exists s $(1 \leqslant s \leqslant m)$ such that $\|X\|_\infty = |x_s|$. Using the conditions (24), we obtain

$$\begin{aligned}\|X\|_\infty &\leqslant \|X\|_\infty + (|d_{s,s}| - 1)\|X\|_\infty - \sum_{v=1,v\neq s}^{m} |d_{s,v}|\|X\|_\infty \leqslant |d_{s,s}||x_s| - \sum_{v=1,v\neq s}^{m} |d_{s,v}||x_v| \leqslant |d_{s,s}x_s| - \left|\sum_{v=1,v\neq s}^{m} d_{s,v}x_v\right| \\ &\leqslant \left|d_{s,s}x_s + \sum_{v=1,v\neq s}^{m} d_{s,v}x_v\right| = \left|\sum_{v=1}^{m} d_{s,v}x_v\right| \leqslant \|DX\|_\infty.\end{aligned}$$

This completes the proof. □

To prove the stability and convergence of the numerical method, we need to rewrite (21), (22) and (20) in the matrix form. Let

$$r_1 = \frac{\Delta t}{(\Delta x)^{\alpha_1+\beta_1}}, \quad r_2 = \frac{\Delta t}{(\Delta y)^{\alpha_2+\beta_2}}, \tag{26}$$

then Eq. (21) may be written as

$$A^{(u)}p_u^* = p_u^{n-1} + \Delta t q_u^n + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r_1 k_{m_1-1,u} p_{m_1,u}^* \end{pmatrix}, \quad 1 \leqslant u \leqslant m_2 - 1, \tag{27}$$

where $p_u^* = (p_{1,u}^*, p_{2,u}^*, \ldots, p_{m_1-1,u}^*)^T$, $q_u^n = (q_{1,u}^n, q_{2,u}^n, \ldots, q_{m_1-1,u}^n)^T$,
$A^{(u)} = (a_{i,j}^{(u)})_{(m_1-1)\times(m_1-1)}$,

$$a_{i,j}^{(u)} = \begin{cases} 0, & \text{for } j > i+1; \\ -r_1 k_{i,u}, & \text{for } j = i+1; \\ 1 - r_1\left[\sum_{s=0}^{1} g_{\beta_1,s} g_{\alpha_1,1-s} k_{i-s,u}\right], & \text{for } j = i; \\ -r_1\left[\sum_{s=0}^{i+1-j} g_{\beta_1,s} g_{\alpha_1,i+1-j-s} k_{i-s,u}\right], & \text{for } j < i. \end{cases} \tag{28}$$

Similarly, Eq. (20) may be written in the matrix form

$$B^{(v)}\bar{p}_v^n = \bar{p}_v^* + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r_2 \bar{k}_{v,m_2-1} p_{v,m_2}^n \end{pmatrix}, \quad 1 \leqslant v \leqslant m_1 - 1, \tag{29}$$

where $\bar{p}_v^n = (p_{v,1}^n, p_{v,2}^n, \ldots, p_{v,m_2-1}^n)^T$, $\bar{p}_v^* = (p_{v,1}^*, p_{v,2}^*, \ldots, p_{v,m_2-1}^*)^T$,
$B^{(v)} = (b_{i,j}^{(v)})_{(m_2-1)\times(m_2-1)}$,

$$b_{i,j}^{(v)} = \begin{cases} 0, & \text{for } j > i+1; \\ -r_2 \bar{k}_{v,i}, & \text{for } j = i+1; \\ 1 - r_2\left[\sum_{s=0}^{1} g_{\beta_2,s} g_{\alpha_2,1-s} \bar{k}_{v,i-s}\right], & \text{for } j = i; \\ -r_2\left[\sum_{s=0}^{i+1-j} g_{\beta_2,s} g_{\alpha_2,i+1-j-s} \bar{k}_{v,i-s}\right], & \text{for } j < i. \end{cases} \tag{30}$$

Eq. (20) may be written as

$$STP^n = P^{n-1} + R^n, \tag{31}$$

where the matrices S and T represent the operators $(1-\Delta t\delta_x)$ and $(1-\Delta t\delta_y)$,

$$p^n = \left[p^n_{1,1}, p^n_{2,1}, \ldots, p^n_{m_1-1,1}, \ldots, p^n_{1,m_2-1}, \ldots, p^n_{m_1-1,m_2-1}\right]^T \tag{32}$$

and the vector R^n absorbs the source term and the boundary conditions in the discretized equation. Therefore the matrix S is a block diagonal matrix of $(m_2-1)\times(m_2-1)$ blocks which are the square $(m_1-1)\times(m_1-1)$ matrices $A^{(u)}$ resulting from Eq. (27). We may write $S = diag(A^{(1)}, A^{(2)}, \ldots, A^{(m_2-1)})$. The matrix T is a block diagonal matrix of $(m_2-1)\times(m_2-1)$ blocks which are the square $(m_1-1)\times(m_1-1)$ diagonal matrices resulting from Eq. (29). That is, we may write $T = [T_{i,j}]$, where each $T_{i,j}$ is an $(m_1-1)\times(m_1-1)$ matrix, such that for $j > i+1$ each $T_{i,j} = 0$, and for $j \leqslant i+1$ each $T_{i,j}$ is a diagonal matrix $T_{i,j} = diag(b^{(1)}_{i,j}, b^{(2)}_{i,j}, \ldots, b^{(m_1-1)}_{i,j})$, where $b^{(v)}_{i,j}$ is the (i,j)th entry of the matrix $B^{(v)}$ defined previously.

To discuss the stability of the numerical method, we suppose that $\bar{p}^n_{i,j}(0 \leqslant i \leqslant m_1; 0 \leqslant j \leqslant m_2)$ is the approximation of the corresponding difference scheme, and $\varepsilon^n_{i,j} = \bar{p}^n_{i,j} - p^n_{i,j}(0 \leqslant i \leqslant m_1; 0 \leqslant j \leqslant m_2)$ denotes the corresponding error, with

$$\varepsilon^n = \left[\varepsilon^n_{1,1}, \ldots, \varepsilon^n_{m_1-1,2}, \ldots, \varepsilon^n_{1,m_2-1}, \ldots, \varepsilon^n_{m_1-1,m_2-1}\right]^T. \tag{33}$$

For the convergence of the numerical method, we assume that $e^n_{i,j} = p(x_i, y_j, t_n) - p^n_{i,j}(0 \leqslant i \leqslant m_1; 0 \leqslant j \leqslant m_2)$, with

$$e^n = \left[e^n_{1,1}, e^n_{2,1}, \ldots, e^n_{m_1-1,1}, \ldots, e^n_{1,m_2-1}, \ldots, e^n_{m_1-1,m_2-1}\right]^T. \tag{34}$$

Firstly, we establish stability and convergence of the implicit finite difference method under the assumption of continuity of seepage flow ($\beta_1 = \beta_2 = 1$).

Theorem 1. *If $\beta_1 = \beta_2 = 1$, $k_x(x,y)$ decreases monotonically with respect to x and $k_y(x,y)$ decreases monotonically with respect to y in the domain Ω, then the implicit finite difference method defined by (31) is unconditionally stable and convergent, and there exists a positive constant $C > 0$ such that*

$$\|e^n\|_\infty \leqslant C(\Delta t + \Delta x + \Delta y). \tag{35}$$

Proof. By definition, for $\beta_1 = 1$, we have $g_{\beta_1,0} = 1$, $g_{\beta_1,1} = -1$, and $g_{\beta_1,n} = 0$ $(n \geqslant 2)$. Hence Eq. (28) can be rewritten as

$$a^{(u)}_{i,j} = \begin{cases} 0, & \text{for } j > i+1; \\ -r_1 k_{i,u}, & \text{for } j = i+1; \\ 1 + r_1\left[\alpha_1 k_{i,u} + k_{i-1,u}\right], & \text{for } j = i; \\ -r_1\left[g_{\alpha_1,i+1-j}k_{i,u} - g_{\alpha_1,i-j}k_{i-1,u}\right], & \text{for } j < i. \end{cases} \tag{36}$$

Since $0 < \alpha_1 < 1$, we get $g_{\alpha_1,0} = 1$, $g_{\alpha_1,1} = -\alpha_1$, $g_{\alpha_1,n} < 0$ $(n \geqslant 1)$, and $\sum_{k=0}^{\infty} g_{\alpha_1,k} = 0$. Now $g_{\alpha_1,k} = (1 - \frac{\alpha_1+1}{k})g_{\alpha_1,k-1}$, so $g_{\alpha_1,k-1} < g_{\alpha_1,k} < 0$, for $k \geqslant 2$. Since $k_x(x,y)$ decreases monotonically with respect to x, then $0 < k_{i,u} < k_{i-1,u}$ $(i = 1, 2, \ldots, m_1-1)$. Therefore $g_{\alpha_1,i+1-j}k_{i,u} - g_{\alpha_1,i-j}k_{i-1,u} > 0$ for $j < i$. Then

$$\begin{aligned}
\sum_{l=1,l\neq i}^{m_1-1} |a^{(u)}_{i,l}| &= r_1\sum_{j=1}^{i-1}\left[g_{\alpha_1,i+1-j}k_{i,u} - g_{\alpha_1,i-j}k_{i-1,u}\right] + r_1 k_{i,u} = r_1 k_{i,u}\sum_{j=1}^{i-1} g_{\alpha_1,i+1-j} - r_1 k_{i-1,u}\sum_{j=1}^{i-1} g_{\alpha_1,i-j} + r_1 k_{i,u} \\
&= r_1 k_{i,u}\sum_{s=0,s\neq 1}^{i} g_{\alpha_1,s} - r_1 k_{i-1,u}\sum_{s=1}^{i-1} g_{\alpha_1,s} = r_1 k_{i,u}\left(-g_{\alpha_1,1} - \sum_{s=i+1}^{\infty} g_{\alpha_1,s}\right) - r_1 k_{i-1,u}\sum_{s=1}^{i-1} g_{\alpha_1,s} \\
&= r_1 k_{i,u}\alpha_1 - r_1 k_{i,u}\sum_{s=i+1}^{\infty} g_{\alpha_1,s} - r_1 k_{i-1,u}\sum_{s=1}^{i-1} g_{\alpha_1,s} < r_1 k_{i,u}\alpha_1 - r_1 k_{i-1,u}\sum_{s=i+1}^{\infty} g_{\alpha_1,s} - r_1 k_{i-1,u}\sum_{s=1}^{i-1} g_{\alpha_1,s} \\
&= r_1 k_{i,u}\alpha_1 - r k_{i-1,u}\sum_{s=1,s\neq i}^{\infty} g_{\alpha_1,s} < r_1 k_{i,u}\alpha_1 + r_1 k_{i-1,u} = a^{(u)}_{i,i} - 1.
\end{aligned} \tag{37}$$

For $\beta_2 = 1$, Eq. (30) can be rewritten as

$$b^{(v)}_{i,j} = \begin{cases} 0, & \text{for } j > i+1; \\ -r_2\bar{k}_{v,i}, & \text{for } j = i+1; \\ 1 + r_2\left[\alpha_2\bar{k}_{v,i} + \bar{k}_{v,i-1}\right], & \text{for } j = i; \\ -r_2\left[g_{\alpha_2,i+1-j}\bar{k}_{v,i} - g_{\alpha_2,i-j}\bar{k}_{v,i-1}\right], & \text{for } j < i. \end{cases} \tag{38}$$

Since $k_y(x,y)$ decreases monotonically with respect to y, similarly, we have

$$\sum_{l=1,l\neq i}^{m_2-1} |b_{i,l}^{(v)}| < b_{i,i}^{(v)} - 1. \tag{39}$$

(1) For the implicit finite difference method defined by (31), its error satisfies

$$ST\varepsilon^n = \varepsilon^{n-1}. \tag{40}$$

Because $A^{(u)}$ and $B^{(v)}$ satisfy the condition of Lemma, according to the relationship between the matrices S and $A^{(u)}$ and the relationship between the matrices T and $B^{(v)}$, we can see that S and T both also satisfy the condition of Lemma; therefore

$$\|\varepsilon^n\|_\infty \leqslant \|T\varepsilon^n\|_\infty \leqslant \|ST\varepsilon^n\|_\infty = \|\varepsilon^{n-1}\|_\infty. \tag{41}$$

Applying (41) repeatedly n times, we obtain

$$\|\varepsilon^n\|_\infty \leqslant \|\varepsilon^0\|_\infty. \tag{42}$$

Therefore the implicit numerical method defined by (31) is unconditionally stable.

(2) According to the results on the consistency of the method (31) in Section 2, we obtain the following error equation:

$$STe^n = e^{n-1} + \Delta t R^n \tag{43}$$

with

$$e^0 = 0, \tag{44}$$

where $\|R^n\|_\infty \leqslant C_1(\Delta t + \Delta x + \Delta y)$, C_1 being a positive constant. Using (43) we obtain

$$\|e^n\|_\infty \leqslant \|Te^n\|_\infty \leqslant \|STe^n\|_\infty = \|e^{n-1} + \Delta t R^n\|_\infty \leqslant \|e^{n-1}\|_\infty + \Delta t\|R^n\|_\infty \leqslant \|e^{n-1}\|_\infty + \Delta t C_1(\Delta t + \Delta x + \Delta y). \tag{45}$$

Combining (45) and (44) we conclude that

$$\|e^1\|_\infty \leqslant \Delta t C_1(\Delta t + \Delta x + \Delta y). \tag{46}$$

Using (45) repeatedly n times, we obtain

$$\|e^n\|_\infty \leqslant n\Delta t C_1(\Delta t + \Delta x + \Delta y). \tag{47}$$

Since $n\Delta t \leqslant T$, we get

$$\|e^n\|_\infty \leqslant TC_1(\Delta t + \Delta x + \Delta y) = C(\Delta t + \Delta x + \Delta y). \tag{48}$$

Therefore the implicit numerical method defined by (31) is convergent. □

Next, we discuss the stability and convergence of the implicit numerical method under the following assumptions:

Theorem 2. *If $1 \leqslant \alpha_1 + \beta_1 < 2, 1 \leqslant \alpha_2 + \beta_2 < 2$, the function $k_x(x,y) = C_1$ for $(x,y) \in \Omega$, where C_1 is a positive constant and the function $k_y(x,y) = C_2$ for $(x,y) \in \Omega$, where C_2 is a positive constant, then the implicit numerical method defined by (31) is unconditionally stable and convergent, and there exists a positive constant $C > 0$ such that*

$$\|e^n\|_\infty \leqslant C(\Delta t + \Delta x + \Delta y). \tag{49}$$

Proof. With the function $k_x(x,y) = C_1$ for $(x,y) \in \Omega$, Eq. (28) can be rewritten as

$$a_{i,j}^{(u)} = \begin{cases} 0, & \text{for } j > i+1; \\ -r_1 C_1, & \text{for } j = i+1; \\ 1 - r_1 C_1 \left[\sum_{u=0}^{1} g_{\beta_1,u} g_{\alpha_1,1-u}\right], & \text{for } j = i; \\ -r_1 C_1 \left[\sum_{u=0}^{i+1-j} g_{\beta_1,u} g_{\alpha_1,i+1-j-u}\right], & \text{for } j < i. \end{cases} \tag{50}$$

Now $(1-x)^{\alpha_1} = \sum_{k=0}^{\infty} g_{\alpha_1,k} x^k$, $(1-x)^{\beta_1} = \sum_{k=0}^{\infty} g_{\beta_1,k} x^k$, and $(1-x)^{\alpha_1+\beta_1} = \sum_{k=0}^{\infty} g_{\alpha_1+\beta_1,k} x^k$, for $-1 \leqslant x \leqslant 1$, according to $(1-x)^{\alpha_1}(1-x)^{\beta_1} = (1-x)^{\alpha_1+\beta_1}$, we have

$$\left(\sum_{k=0}^{\infty} g_{\alpha_1,k} x^k\right)\left(\sum_{k=0}^{\infty} g_{\beta_1,k} x^k\right) = \sum_{k=0}^{\infty} g_{\alpha_1+\beta_1,k} x^k. \tag{51}$$

Comparing the coefficients on either side of the above equation, we have

$$\sum_{k=0}^{n} g_{\alpha_1,k} g_{\beta_1,n-k} = g_{\alpha_1+\beta_1,n} \quad n = 0, 1, \ldots. \tag{52}$$

Therefore Eq. (50) can be rewritten as

$$a_{i,j}^{(u)} = \begin{cases} 0, & \text{for } j > i+1; \\ -r_1 C_1, & \text{for } j = i+1; \\ 1 - r_1 C_1 g_{\alpha_1+\beta_1,1}, & \text{for } j = i; \\ -r_1 C_1 g_{\alpha_1+\beta_1,i+1-j}, & \text{for } j < i. \end{cases} \tag{53}$$

Similarly, with the function $k_y(x,y) = C_2$ for $(x,y) \in \Omega$, Eq. (30) can be rewritten as

$$b_{i,j}^{(v)} = \begin{cases} 0, & \text{for } j > i+1; \\ -r_2 C_2, & \text{for } j = i+1; \\ 1 - r_2 C_2 g_{\alpha_2+\beta_2,1}, & \text{for } j = i; \\ -r_2 C_2 g_{\alpha_2+\beta_2,i+1-j}, & \text{for } j < i. \end{cases} \tag{54}$$

For $1 < \alpha_1 + \beta_1 < 2$, since $g_{\alpha_1+\beta_1,0} = 1,\ g_{\alpha_1+\beta_1,1} = -(\alpha_1+\beta_1) < 0,\ g_{\alpha_1+\beta_1,n} > 0\ (n \neq 1)$, and $\sum_{k=0}^{\infty} g_{\alpha_1+\beta_1,k} = 0$, so $\sum_{l=0,l\neq 1}^{i} g_{\alpha_1+\beta_1,l} < -g_{\alpha_1+\beta_1,1}$. Therefore

$$\sum_{l=1,l\neq i}^{m_1-1} |a_{i,l}^{(u)}| = r_1 C_1 \sum_{l=0,l\neq 1}^{i} g_{\alpha_1+\beta_1,l} < -r_1 C_1 g_{\alpha_1+\beta_1,1} = a_{i,i}^{(u)} - 1. \tag{55}$$

For $1 < \alpha_2 + \beta_2 < 2$, similarly, we have

$$\sum_{l=1,l\neq i}^{m_2-1} |b_{i,l}^{(v)}| < b_{i,i}^{(v)} - 1. \tag{56}$$

For $\alpha_1 + \beta_1 = 1$, since $g_{\alpha_1+\beta_1,0} = 1,\ g_{\alpha_1+\beta_1,1} = -1 < 0,\ g_{\alpha_1+\beta_1,n} = 0\ (n \neq 1)$, so

$$\sum_{l=1,l\neq i}^{m_1-1} |a_{i,l}^{(u)}| = a_{i,i}^{(u)} - 1. \tag{57}$$

For $\alpha_2 + \beta_2 = 1$, similarly, we have

$$\sum_{l=1,l\neq i}^{m_2-1} |b_{i,l}^{(v)}| = b_{i,i}^{(v)} - 1. \tag{58}$$

Therefore $A^{(u)}$ and $B^{(v)}$ satisfy the condition of Lemma. Similar to the proof of Theorem 1, we can conclude the results of Theorem 2. □

Remark. For the case $1 \leqslant \alpha_1 + \beta_1 < 2$ and $1 \leqslant \alpha_2 + \beta_2 < 2$, where the functions $k_x(x,y)$ and $k_y(x,y)$ are not constant in the domain Ω, many numerical experiments show that the implicit numerical method defined by (31) is also unconditionally stable and convergent. We provide evidence in the numerical examples in the following section.

4. Numerical examples

To demonstrate the effectiveness of the numerical method for solving the fractional percolation equation, we consider two examples. We set $\Delta t = \Delta x = \Delta y = h$ in the following two examples. Let $\|e_h^n\|_\infty$ express the maximum error of the numerical solution for $\Delta t = \Delta x = \Delta y = h$; and let $G_h = log_2(\|e_{2h}^n\|_\infty / \|e_h^{2n}\|_\infty)$ reflect the convergence order of the method.

Example 4.1. Consider the following two-dimensional fractional percolation equation:

$$\begin{aligned} \frac{\partial p}{\partial t} = {} & \frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left((2-x^2)\frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}\right) + \frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left((2-y^2)\frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}\right) - x^2 y^2 e^{-t} \\ & - \frac{\Gamma(3)}{\Gamma(3-\alpha_1)} y^2 e^{-t}\left(\frac{2\Gamma(3-\alpha_1)}{\Gamma(3-\alpha_1-\beta_1)} x^{2-\alpha_1-\beta_1} - \frac{\Gamma(5-\alpha_1)}{\Gamma(5-\alpha_1-\beta_1)} x^{4-\alpha_1-\beta_1}\right) \\ & - \frac{\Gamma(3)}{\Gamma(3-\alpha_2)} x^2 e^{-t}\left(\frac{2\Gamma(3-\alpha_2)}{\Gamma(3-\alpha_2-\beta_2)} y^{2-\alpha_2-\beta_2} - \frac{\Gamma(5-\alpha_2)}{\Gamma(5-\alpha_2-\beta_2)} y^{4-\alpha_2-\beta_2}\right), (x,y) \in \Omega, \quad 0 \leqslant t \leqslant T, \end{aligned}$$

subject to the initial condition

$$p(x,y,0) = x^2y^2, \quad (x,y) \in \Omega,$$

and the Dirichlet boundary conditions

$$p(0,y,t) = p(x,0,t) = 0, \quad p(1,y,t) = y^2e^{-t}, \quad p(x,1,t) = x^2e^{-t}, \quad (x,y) \in \Omega, \quad 0 \leqslant t \leqslant T,$$

where $\Omega = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. The exact solution of the above problem is given by

$$p(x,y,t) = x^2y^2e^{-t},$$

which may be verified by direct differentiation and substitution in the fractional differential equation, using the formula

$$\frac{\partial^\alpha}{\partial x^\alpha}[x^v] = \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)}x^{v-\alpha}$$

for the Riemann–Liouville fractional derivative.

Firstly, Table 1 shows the maximum error for the numerical solution of Example 4.1 with $\alpha_1 = 0.5,\ \beta_1 = 1,\ \alpha_2 = 0.5,\ \beta_2 = 1$. The second column shows the maximum error, at time $t = 1$, between the exact solution and the numerical solution obtained by applying the implicit method (31). The third column shows that the convergence is of order $O(\Delta x) + O(\Delta y) + O(\Delta t)$ for $\beta_1 = \beta_2 = 1$, $k_x(x,y)$ decreasing monotonically with respect to x and $k_y(x,y)$ decreasing monotonically with respect to y in the domain Ω.

Secondly, Table 2 shows the maximum error for the numerical solution of Example 4.1 with $\alpha_1 = 0.7,\ \beta_1 = 0.8,\ \alpha_2 = 0.6,\ \beta_2 = 0.9$ at time $t = 1$ by applying the implicit method (31). It can be seen that the convergence is of order $O(\Delta x) + O(\Delta y) + O(\Delta t)$ for $\beta_1 \neq 1$ and $\beta_2 \neq 1$, $k_x(x,y)$ decreasing monotonically with respect to x and $k_y(x,y)$ decreasing monotonically with respect to y in the domain Ω. This shows that the convergence condition of the implicit method (31) can be weakened.

Example 4.2. Consider the following two-dimensional fractional percolation equation:

$$\frac{\partial p}{\partial t} = \frac{\partial^{0.8}}{\partial x^{0.8}}\left(\frac{\partial^{0.7}p}{\partial x^{0.7}}\right) + \frac{\partial^{0.9}}{\partial y^{0.9}}\left(\frac{\partial^{0.6}p}{\partial y^{0.6}}\right) - x^2y^2e^{-t} - \frac{\Gamma(3)}{\Gamma(1.5)}x^{0.5}y^2e^{-t} - \frac{\Gamma(3)}{\Gamma(1.5)}x^2y^{0.5}e^{-t}, \quad x \in \Omega, \quad t > 0,$$

subject to the initial condition

$$p(x,y,0) = x^2y^2, \quad (x,y) \in \Omega,$$

and the Dirichlet boundary conditions

$$p(0,y,t) = p(x,0,t) = 0, \quad p(1,y,t) = y^2e^{-t}, \quad p(x,1,t) = x^2e^{-t}, \quad (x,y) \in \Omega, \quad 0 \leqslant t \leqslant T,$$

where $\Omega = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. The exact solution of the above problem is given by

$$p(x,y,t) = x^2y^2e^{-t}.$$

Table 1
Maximum error behavior for the implicit numerical method (31) as the grid size is reduced for Example 4.1 with $\alpha_1 = 0.5,\ \beta_1 = 1,\ \alpha_2 = 0.5,\ \beta_2 = 1$ at time $T_{end} = 1$.

$\Delta t = \Delta x = \Delta y = h$	$\|e_h^n\|_\infty$	G_h
1/20	0.0044467	
1/40	0.0023029	0.9493
1/80	0.001173	0.9733
1/160	0.00059207	0.9864

Table 2
Maximum error behavior for the implicit numerical method (31) as the grid size is reduced for Example 4.1 with $\alpha_1 = 0.7,\ \beta_1 = 0.8,\ \alpha_2 = 0.6,\ \beta_2 = 0.9$ at time $T_{end} = 1$.

$\Delta t = \Delta x = \Delta y = h$	$\|e_h^n\|_\infty$	G_h
1/20	0.0054622	
1/40	0.0028556	0.9357
1/80	0.0014619	0.9659
1/160	0.00073961	0.9830

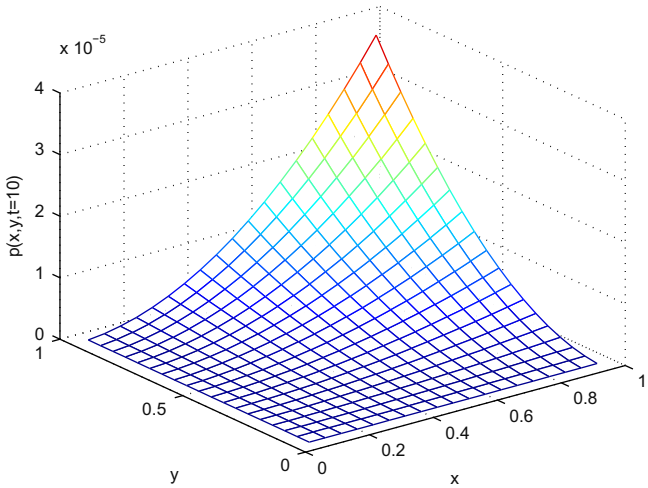

Fig. 1. Exact solution for Example 4.2 at time $t = 10$.

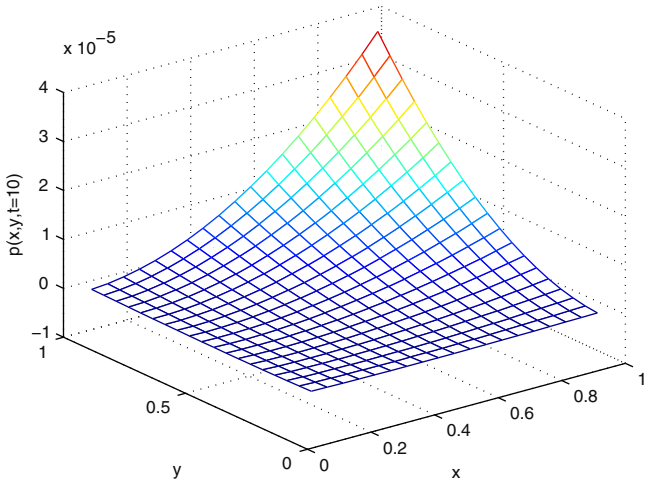

Fig. 2. Numerical solution of the difference scheme (31) for Example 4.2 at time $t = 10$.

Fig. 1 shows the exact solution for Example 4.2 at time $t = 10$. Fig. 2 shows the numerical solution of the difference scheme (31) for Example 4.2 at time $t = 10$ by setting $\triangle t = \triangle x = \triangle y = 1/20$. It is observed that the numerical solution of the difference scheme (31) is in good agreement with the exact solution.

5. Conclusions

We have presented a novel implicit finite difference method for the two-dimension fractional percolation equation. The consistency, stability, and convergence of the method has been established. Two numerical examples show that the method is effective and the behavior of the errors are analyzed to demonstrate the order of convergence of the numerical method.

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