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Neumann inhomogeneous boundary value problem for the $n+1$ complex Ginzburg–Landau equation

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Abstract

We study the following Neumann inhomogeneous boundary value problem for the complex Ginzburg–Landau equation on $\Omega \subset \mathbb{R}^n (n \leqslant 3): u_t = (a + i\alpha)\Delta u - (b + i\beta)|u|^2 u (a, b, t > 0)$ under initial condition $u(x, 0) = h(x)$ for $x \in \Omega$ and Neumann boundary condition $\frac{\partial u}{\partial n} = K(x, t)$ on $\partial\Omega$ where h, K are given functions. Under suitable conditions, we prove the existence of a global solution in H^1.
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1. Introduction

This paper is the continuation of an earlier one [1] where the following Dirichlet type inhomogeneous boundary value problem for the complex Ginzburg–Landau equation is investigated:

$$u_t = (a + i\alpha)\Delta u - (b + i\beta)|u|^2 u, \quad t > 0, \quad x \in \Omega, \tag{1.1}$$

$$u(x, 0) = h(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x, t) = Q(x, t), \quad t > 0, \quad x \in \partial\Omega. \tag{1.3}$$

For this problem, it was assumed that $a, b > 0$, Ω is an open bounded domain in $\mathbb{R}^n$ with C^∞ boundary and h, Q are given smooth functions. Existence of a unique global solution in H^1 has been proved under the condition $-1 - \frac{\alpha\beta}{ab} < \sqrt{3}|\frac{\alpha}{a} - \frac{\beta}{b}|$, $\alpha \neq 0$. Further, this solution approaches to the solution of the corresponding NLS limit under identical initial and boundary conditions as $a, b \to 0^+$.

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The 1D Ginzburg–Landau equation $u_t = (a + i\alpha)\Delta u - (b + i\beta)|u|^2 u$ was originally proposed to describe nonlinear amplitude evolution of wave perturbation with a basic pattern when a control parameter R lies in the unstable region $O(\epsilon)$ away from the critical value R_ϵ for which the system loses stability. Here ϵ is a small parameter. The Ginzburg–Landau equation was found for a general class of nonlinear evolution problems in hydrodynamics and other applications of chemistry and physics. It was derived from the Navier–Stocks equations via multiple scaling methods in convection. This equation and its variations with additional nonlinear terms have been extensively studied. For example, a mathematically rigorous proof of the validity of this equation was given for a general solution of one space variable and a quadratic nonlinearity. (For Refs., see [2–11].) About the global existence for the Ginzburg–Landau equation: posed in a quarter-plane, see [12].

The objective of this paper is to prove global existence for the following Neumann type inhomogeneous boundary value problem for the Ginzburg–Landau equation in $1 \leqslant n \leqslant 3$ space dimensions:

$$u_t = (a + i\alpha)\Delta u - (b + i\beta)|u|^2 u, \quad t > 0,\ x \in \Omega, \tag{1.4}$$

$$u(x, 0) = h(x), \quad x \in \Omega, \tag{1.5}$$

$$\frac{\partial u}{\partial n} = K(x, t), \quad t > 0, \quad x \in \partial\Omega. \tag{1.6}$$

Let Ω be a bounded domain of $\mathbb{R}^n (1 \leqslant n \leqslant 3)$, $\vec{n}$ is the outer normal vector of $\partial\Omega$. For any $T > 0$, write $Q_T = \Omega \times (0, T]$. The following definition and properties of $W_p^{l,l/2}(Q_T)$ space can be found, for example, in [15].

Definition 1.1. Let l be a positive integer and $1 \leqslant p < \infty$.

If l is an even number, define

$$\|u\|_{W_p^{l,l/2}(Q_T)} = \left\{ \sum_{0 \leqslant r+2s \leqslant l} \|D_t^s D_x^r u\|_{L^p(Q_T)}^p \right\}^{\frac{1}{p}}. \tag{1.7}$$

If l is an odd number, define

$$\|u\|_{W_p^{l,l/2}(Q_T)} = \left\{ \sum_{0 \leqslant r+2s \leqslant l} \|D_t^s D_x^r u\|_{L^p(Q_T)}^p + \sum_{0 \leqslant r+2s \leqslant l-1} \left[D_t^s D_x^r u\right]_{L_{p,t}^{\frac{1}{2}}}^p (Q_T) \right\}^{\frac{1}{p}}. \tag{1.8}$$

One can verify that $W_p^{l,l/2}(Q_T)$ is a Banach space according to the norm defined above. Since we are particularly interested in the situation $p = 2$, we denote $H^{l,l/2}(Q_T)$ by $W_2^{l,l/2}(Q_T)$.

By the trace theorem in [15], there is a function $w(x, t) \in H^{2,1}(Q_T)$ such that $\frac{\partial w}{\partial n} = K$ on $\partial\Omega \times (0, T)$ for any $K(x, t) \in H^{\frac{1}{2},\frac{1}{4}}(\partial\Omega \times (0, T))$.

Now let $v = u - w$ and rewrite (1.4)–(1.6) as

$$\begin{aligned} v_t &= (a + i\alpha)\Delta v - (b + i\beta)|v + w|^2 (v + w) + (a + i\alpha)\Delta w - w_t \\ &= (a + i\alpha)\Delta v - (b + i\beta)|v|^2 v - (b + i\beta)G(v, w) + f(x, t), \end{aligned} \tag{1.9}$$

where

$$G(v, w) = 2|v|^2 w + \bar{v}w^2 + v^2\bar{w} + 2v|w|^2 + |w|^2 w, \tag{1.10}$$

$$f(x, t) = (a + i\alpha)\Delta w - w_t \tag{1.11}$$

and

$$v(x, 0) = h(x) - w(x, 0), \quad x \in \Omega, \tag{1.12}$$

$$\frac{\partial v}{\partial n} = 0, \quad t > 0,\ x \in \partial\Omega. \tag{1.13}$$

Clearly $f(x, t) \in L^2(Q_T)$ based on (1.7). By the embedding theorem in [15], we know that $H^{2,1}(Q_T) \hookrightarrow C([0, T], H^1(\Omega))$.

Lemma 1.2. *Let v be a smooth solution to the initial-boundary value problem for the Ginzburg–Landau Eqs. (1.9), (1.12) and (1.13). Then the following identities are available.*

$$\partial_t \int_\Omega |v|^2 \mathrm{d}x = -2a\|\nabla v\|_2^2 - 2b\|v\|_4^4 - 2\mathrm{Re}(b + \mathrm{i}\beta)\int_\Omega G\bar{v}\mathrm{d}x + 2\mathrm{Re}\int_\Omega f\bar{v}\mathrm{d}x, \tag{1.14}$$

$$\partial_t \int_\Omega |\nabla v|^2 \mathrm{d}x = -2a\|\Delta v\|_2^2 + 2\mathrm{Re}(b - \mathrm{i}\beta)\int_\Omega |v|^2\bar{v}\Delta v\mathrm{d}x + 2\mathrm{Re}(b - \mathrm{i}\beta)\int_\Omega \overline{G}\Delta v\mathrm{d}x + 2\mathrm{Re}\int_\Omega \bar{f}\Delta v\mathrm{d}x \tag{1.15}$$

and

$$\frac{1}{4a}\partial_t \int_\Omega |v|^4 \mathrm{d}x = \mathrm{Re}\left(1 + \mathrm{i}\frac{\alpha}{a}\right)\int_\Omega \Delta v|v|^2\bar{v}\mathrm{d}x - \frac{b}{a}\|v\|_6^6 - \frac{1}{a}\mathrm{Re}(b + \mathrm{i}\beta)\int_\Omega |v|^2\bar{v}G\mathrm{d}x + \frac{1}{a}\mathrm{Re}\int_\Omega |v|^2\bar{v}f\mathrm{d}x. \tag{1.16}$$

We note that $\partial v/\partial n = 0$ on $\partial\Omega$. The proof is straight forward after integration by parts and substitution of Eq. (1.9).

2. Global solution in H^1

With the above identities (1.14)–(1.16) we are able to prove the following global existence theorem.

Theorem 2.1. *If $h \in H^1 \cap L^4(\Omega)$, $a > 0$, $b > 0$, $K \in H^{\frac{1}{2},\frac{1}{4}}(\partial\Omega \times (0, T))$ for any given $T > 0$, then 1.4,1.5,1.6 has an unique solution in*

$$u \in (C([0, T^*), H^1) \cap (L^2([0, T^*), H^2), \quad \textit{for some } T^* > 0,$$

and this solution is global (i.e., $T^* = \infty$) provided that $-1 - \frac{\alpha\beta}{ab} < \sqrt{3}|\frac{\alpha}{a} - \frac{\beta}{b}|$.

In order to obtain the local existence of (1.4)–(1.6), we should homogenized the boundary condition by $v = u - w$. Using the theory of [13,14], local existence is proved by verifying that both $|v|^2 v$ and $G(v, w)$ are locally Lipshitz continuous from $H^1(\Omega) \to L^2(\Omega)$.

In fact, for any v_1 and v_2 in $H^1(\Omega)$, we note that $w \in C([0, T], H^1(\Omega))$. In addition, we note that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ for $n \leqslant 3$ by Gagliardo–Nirenberg estimates [13]. Therefore,

$$\||v_1|^2 v_1 - |v_2|^2 v_2\| \leqslant C\left(\|v_1\|_{H^1}^2 + \|v_2\|_{H^1}^2\right)\|v_1 - v_2\|_{H^1}. \tag{2.1}$$

Similarly, we see that

$$\|G(v_1, w) - G(v_2, w)\| \leqslant C\left(\|v_1\|_{H^1}^2 + \|v_2\|_{H^1}^2 + \|w\|_{H^1}^2\right)\|v_1 - v_2\|_{H^1}. \tag{2.2}$$

Therefore, the local existence is obtained.

In order to prove the global existence, we work on estimates based on (1.14)–(1.16). By Young's inequality, we have

$$-2\mathrm{Re}(b + \mathrm{i}\beta)\int_\Omega G\bar{v}\mathrm{d}x \leqslant c\int_\Omega \left(|v|^3|w| + |v|^2|w|^2 + |v||w|^3\right)\mathrm{d}x \leqslant b\int_\Omega |v|^4\mathrm{d}x + C\int_\Omega |w|^4\mathrm{d}x \tag{2.3}$$

and

$$2\mathrm{Re}\int_\Omega f\bar{v}\mathrm{d}x \leqslant 2\|f\|_2 \cdot \|v\|_2 \leqslant \|f\|_2^2 + \|v\|_2^2. \tag{2.4}$$

Now we substitute (2.3) and (2.4) into (1.16) to get

$$\partial_t \int_\Omega |v|^2 \mathrm{d}x = -2a\|\nabla v\|_2^2 - b\|v\|_4^4 + C\int_\Omega |w|^4 \mathrm{d}x + \|f\|_2^2 + \|v\|_2^2$$
$$\leqslant -2a\|\nabla v\|_2^2 + C\int_\Omega |w|^4 \mathrm{d}x + \|f\|_2^2 + \|v\|_2^2. \tag{2.5}$$

Since $f \in L^2, w \in C([0,T], H^1(\Omega))(\Omega \subset \mathbb{R}^n, n \leqslant 3)$, Gronwall's inequality implies that the L^2 norm of v is bounded.

For H^1 estimate, we need to define

$$E_\delta(v(t)) = \frac{1}{2b}\|\nabla v\|_2^2 + \frac{\delta}{4a}\|v\|_4^4, \tag{2.6}$$

where δ is a positive constant. Under the condition that $-1 - \frac{\alpha\beta}{ab} < \sqrt{3}|\frac{\alpha}{a} - \frac{\beta}{b}|$, based on similar approach as in [11] (pp. 203–205), (1.15) and (1.16) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\delta(v(t)) \leqslant CE_\delta(v(t)) - \mu\left(\|\Delta v\|_2^2 + \|v\|_6^6\right)$$
$$+ C\left(\int_\Omega |G||\Delta v|\mathrm{d}x + \int_\Omega |f||\Delta v|\mathrm{d}x + \int_\Omega |G||v|^3 + \int_\Omega |f||v|^3 \mathrm{d}x\right), \tag{2.7}$$

for some $\mu > 0$. The following estimates are easily verified:

$$\int_\Omega |G||\Delta v|\mathrm{d}x \leqslant \frac{\mu}{2}\|\Delta v\|_2^2 + \frac{1}{2\mu}\int_\Omega |G|^2 \mathrm{d}x, \tag{2.8}$$

$$\int_\Omega |G|^2 \mathrm{d}x \leqslant \epsilon_1 \int_\Omega |v|^6 \mathrm{d}x + C\int_\Omega |w|^6 \mathrm{d}x, \tag{2.9}$$

$$\int_\Omega |f||\Delta v|\mathrm{d}x \leqslant \frac{\mu}{2}\|\Delta v\|_2^2 + \frac{1}{2\mu}\int_\Omega |f|^2 \mathrm{d}x, \tag{2.10}$$

$$\int_\Omega |G||v|^3 \mathrm{d}x \leqslant \epsilon_2 \int_\Omega |v|^6 \mathrm{d}x + \frac{1}{4\epsilon_2}\int_\Omega |G|^2 \mathrm{d}x, \tag{2.11}$$

$$\int_\Omega |f||v|^3 \mathrm{d}x \leqslant \epsilon_3 \int_\Omega |v|^6 + \frac{1}{4\epsilon_3}\int_\Omega |f|^2 \mathrm{d}x \tag{2.12}$$

for sufficiently small ϵ_1, ϵ_2, ϵ_3.

From these estimates along with (2.6) we obtain the following estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\delta(v(t)) \leqslant CE_\delta(v(t)) + C\left(\|f\|_2^2 + \|w\|_6^6\right). \tag{2.13}$$

Here we treat C as a generic constant. Since $f \in L^2(Q_T)$ and $w \in C([0,T], H^1(\Omega))$ (which implies that $w \in L^6$ for $n \leqslant 3$). Gronwall's inequality yields that

$$E_\delta(v(t)) \leqslant C. \tag{2.14}$$

This shows that v is a global solution. Consequently u is also a global solution and Theorem 2.1 is proved.

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