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Physics Letters A 363 (2007) 356–360

PHYSICS LETTERS A

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A generalized new auxiliary equation method and its applications to nonlinear partial differential equations

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Received 30 July 2006; accepted 15 November 2006

Available online 27 November 2006

Communicated by R. Wu

Abstract

In this Letter, a generalized new auxiliary equation method is proposed for constructing more general exact solutions of nonlinear partial differential equations. With the aid of symbolic computation, we choose the combined KdV–mKdV equation and the $(2+1)$-dimensional asymmetric Nizhnik–Novikov–Vesselov equations to illustrate the validity and advantages of the method. As a result, many new and more general exact solutions are obtained.

PACS: 02.30.Jr; 04.20.Jb

Keywords: Exact solutions; Nonlinear partial differential equations; Auxiliary equation method; Combined KdV–mKdV equation; Asymmetric Nizhnik–Novikov–Vesselov equations

1. Introduction

It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better. With the development of soliton theory, various methods for obtaining exact solutions of NLPDEs have been presented, such as inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], tanh function method [5,6], sine–cosine method [7], homogeneous balance method [8], homotopy perturbation method [9], variational method [10], asymptotic methods [11], non-perturbative methods [12], exp-function method [13], Adomian Padé approximation [14], algebraic method [15,16], Jacobi elliptic function expansion method [17], F-expansion method [18], auxiliary equation method [19–22], Weierstrass semi-rational expansion method [23], unified rational expansion method [24] and so on. Recently, Sirendaoreji [25] proposed a new auxiliary equation method by introducing a new first-order nonlinear ordinary differential equation with six-degree nonlinear term and its solutions to construct exact traveling wave solutions of NLPDEs in a unified way. By using this method, Sirendaoreji [25] obtained many new exact traveling wave solutions of four NLPDEs under the constraint $c=0$.

The present Letter is motivated by the desire to generalize the work made in [25] by proposing a new and more general ansatz so that it can be used to construct more general exact solutions which contain not only the results obtained by using the method in [25] as special cases but also a series of new and more general exact solutions, in which the restrictions on ξ as merely a linear function of x and t, coefficients as constants and $c=0$ are removed. For illustration, we apply this method to the combined KdV–mKdV

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2006.11.035

equation and the (2 + 1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations and successfully obtain many new and more general exact solutions.

The rest of this Letter is organized as follows: in Section 2, we give the description of the generalized new auxiliary equation method; in Section 3, we apply this method to the combined KdV–mKdV equation and the (2 + 1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations; in Section 4, some conclusions are given.

2. A generalized new auxiliary equation method

For a given NLPDE with independent variables $x = (t, x_1, x_2, \ldots)$ and dependent variable u:

$$F(u, u_t, u_{x_1}, u_{x_2}, \ldots, u_{x_1 t}, u_{x_2 t} \ldots, u_{tt}, u_{x_1 x_1}, u_{x_2 x_2}, \ldots) = 0, \tag{1}$$

we seek its solutions in the new and more general form:

$$u = a_0 + \sum_{i=1}^{n} \left\{ a_i z^i(\xi) + b_i z^{i-1}(\xi) z'(\xi) \right\}, \tag{2}$$

with $z(\xi)$ satisfying the new auxiliary equation:

$$z'^2(\xi) = \left(\frac{dz}{d\xi}\right)^2 = a z^2(\xi) + b z^4(\xi) + c z^6(\xi), \tag{3}$$

where $a_0 = a_0(x)$, $a_i = a_i(x)$, $b_i = b_i(x)$ $(i = 1, 2, \ldots, n)$ and $\xi = \xi(x)$ are functions to be determined; a, b and c are real constants. Eq. (3) has some special solutions which are listed in Table 1. To determine u explicitly, we take the following four steps.

Step 1. Determine the integer n. Substituting (2) along with (3) into Eq. (1) and balancing the highest order partial derivative with the nonlinear terms in Eq. (1), we can obtain the value of n. For example, in the case of KdV equation:

$$u_t + 6 u u_x + u_{xxx} = 0, \tag{4}$$

we have $n = 4$.

Step 2. Derive the system of equations. Taking account of the value of n obtained in Step 2 and substituting (2) along with (3) into Eq. (1) and collecting coefficients of $z^j(\xi) z'^l(\xi)$ $(l = 0, 1;\ j = 0, 1, 2, \ldots)$, then setting each coefficient to zero, we can derive a set of over-determined partial differential equations for a_0, a_i, b_i and ξ.

Step 3. Solve the system of equations. Solving the system of over-determined partial differential equations obtained in Step 2 by use of *Mathematica*, we can obtain the explicit expressions for a_0, a_i, b_i and ξ.

Step 4. Obtain exact solutions. By using the results obtained in the above steps, we can derive a series of fundamental solutions of Eq. (1) depending on the solution $z(\xi)$ of Eq. (3). Selecting appropriate $z_i(\xi)$ from Table 1 and substituting it into the fundamental solutions, we can obtain exact solutions of Eq. (1).

Remark 1. By using the method in [25], we have $n = 2$ for Eq. (4). From (2.6) in [15], we get $m = 4$ which leads to $c = 0$ in Eq. (3). By our method, however, we get $m = 6$ which removes the constraint $c = 0$.

Table 1
Solutions of Eq. (3) with $\Delta = b^2 - 4ac$, $\varepsilon = \pm 1$

No.	$z_i(\xi)$	No.	$z_i(\xi)$
1	$\operatorname{sech}(\sqrt{a}\xi)\left(\frac{-ab}{b^2 - ac(1+\varepsilon\tanh(\sqrt{a}\xi))^2}\right)^{1/2}$, $a > 0$	8	$\sec(\sqrt{-a}\xi)\left(\frac{-a}{b + 2\varepsilon\sqrt{-ac}\tan(\sqrt{-a}\xi)}\right)^{1/2}$, $a < 0$, $c > 0$
2	$\operatorname{csch}(\sqrt{a}\xi)\left(\frac{ab}{b^2 - ac(1+\varepsilon\coth(\sqrt{a}\xi))^2}\right)^{1/2}$, $a > 0$	9	$\operatorname{csch}(\sqrt{a}\xi)\left(\frac{a}{b + 2\varepsilon\sqrt{ac}\coth(\sqrt{a}\xi)}\right)^{1/2}$, $a > 0$, $c > 0$
3	$\left(\frac{2a}{\varepsilon\sqrt{\Delta}\cosh(2\sqrt{a}\xi) - b}\right)^{1/2}$, $a > 0$, $\Delta > 0$	10	$\csc(\sqrt{-a}\xi)\left(\frac{-a}{b + 2\varepsilon\sqrt{-ac}\cot(\sqrt{-a}\xi)}\right)^{1/2}$, $a < 0$, $c > 0$
4	$\left(\frac{2a}{\varepsilon\sqrt{\Delta}\cos(2\sqrt{-a}\xi) - b}\right)^{1/2}$, $a < 0$, $\Delta > 0$	11	$\left(-\frac{a}{b}(1 + \varepsilon\tanh(\sqrt{a}\xi))\right)^{1/2}$, $a > 0$, $\Delta = 0$
5	$\left(\frac{2a}{\varepsilon\sqrt{-\Delta}\sinh(2\sqrt{a}\xi) - b}\right)^{1/2}$, $a > 0$, $\Delta < 0$	12	$\left(-\frac{a}{b}(1 + \varepsilon\coth(\sqrt{a}\xi))\right)^{1/2}$, $a > 0$, $\Delta = 0$
6	$\left(\frac{2a}{\varepsilon\sqrt{\Delta}\sin(2\sqrt{-a}\xi) - b}\right)^{1/2}$, $a < 0$, $\Delta > 0$	13	$4\left(\frac{a e^{2\varepsilon\sqrt{a}\xi}}{(e^{2\varepsilon\sqrt{a}\xi} - 4b)^2 - 64ac}\right)^{1/2}$, $a > 0$
7	$\operatorname{sech}(\sqrt{a}\xi)\left(\frac{-a}{b + 2\varepsilon\sqrt{ac}\tanh(\sqrt{a}\xi)}\right)^{1/2}$, $a > 0$, $c > 0$	14	$4\left(\frac{a\varepsilon e^{2\varepsilon\sqrt{a}\xi}}{(1 - 64ac e^{4\varepsilon\sqrt{a}\xi})}\right)^{1/2}$, $a > 0$, $b = 0$

3. Applications of the method

In this section, we would like to apply our method to obtain new and more general exact solutions of two important NLPDEs.

Example 1. Consider the combined KdV–mKdV equation:

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + \gamma u_{xxx} = 0, \tag{5}$$

where α, β and γ are constants. This equation is widely used in various physics and quantum fields such as solid-state physics, plasma physics, fluid physics and quantum field theory. Lou and Chen [26] found solitary wave solutions and cnoidal wave solutions of Eq. (5) by using a mapping approach. Zhao et al. [27] got soliton-like solutions of Eq. (5) by applying the extended tanh method. Very recently, Zhao et al. [28] and Sirendaoreji [25] obtained traveling wave solutions by means of a new Riccati equation expansion method and a new auxiliary equation method.

According to Step 1, we get $n = 2$ for u. In order to search for explicit solutions, we assume that Eq. (5) has the following formal solution:

$$u = a_0 + a_1 z(\xi) + a_2 z^2(\xi) + b_1 z'(\xi) + b_2 z(\xi) z'(\xi), \tag{6}$$

where $a_0 = a_0(t)$, $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$, $b_2 = b_2(t)$, $\eta = \eta(t)$, $\xi = kx + \eta$, k is a non-zero constant.

With the aid of *Mathematica*, substituting (6) along with Eq. (3) into Eq. (5) and setting each coefficient of $z^j(\xi) z'^l(\xi)$ ($l = 0, 1$; $j = 1, 2, \ldots$) to zero, we get a set of over-determined partial differential equations for a_0, a_1, a_2, b_1, b_2 and η. Solving the system of over-determined partial differential equations by use of *Mathematica* and choosing c_0 as an arbitrary constant, we obtain the following results:

Case 1.1.

$$a_0 = \frac{-c\alpha \pm kb\beta\sqrt{-\frac{c\gamma}{\beta}}}{2c\beta}, \qquad a_1 = 0, \qquad a_2 = \pm 2k\sqrt{-\frac{c\gamma}{\beta}}, \tag{7}$$

$$b_1 = 0, \qquad b_2 = 0, \qquad \eta = \frac{k(3c\alpha^2 + k^2\beta\gamma(3b^2 - 8ac))}{2c\beta} t + c_0. \tag{8}$$

Case 1.2.

$$a_0 = -\frac{\alpha}{2\beta}, \qquad a_1 = \pm k\sqrt{-\frac{b\gamma}{\beta}}, \qquad a_2 = 0, \tag{9}$$

$$b_1 = 0, \qquad b_2 = 0, \qquad \eta = \frac{k(3\alpha^2 - 2k^2 a\beta\gamma)}{2\beta} t + c_0, \qquad c = 0. \tag{10}$$

Substituting Case 1.1 along with $z_1(\xi)$ in Table 1 into (6), we obtain exact traveling wave solution of Eq. (5):

$$u = \frac{-c\alpha + kb\beta\sqrt{-\frac{c\gamma}{\beta}}}{2c\beta} \mp \frac{2kab\sqrt{-\frac{c\gamma}{\beta}}\operatorname{sech}^2(\sqrt{a}\xi)}{b^2 - ac(1 + \varepsilon\tanh(\sqrt{a}\xi))^2},$$

where $\xi = kx + \frac{k(3c\alpha^2 + k^2\beta\gamma(3b^2 - 8ac))}{2c\beta} t + c_0$.

From (6), Cases 1.1–1.2 and Table 1, we can obtain other exact solutions of Eq. (5), here we omit them for simplicity. It is noted that the special case $\omega = \frac{k(3\alpha^2 - 2k^2 a\beta\gamma)}{2\beta}$, $c_0 = 0$ and $k = 1$ in Case 1.2 will recover (29) given in [25]. All the solutions obtained from Case 1.1 cannot be obtained by the method in [25].

Example 2. Consider the $(2+1)$-dimensional asymmetric Nizhnik–Novikov–Vesselov equations:

$$u_t - u_{xxx} + \alpha(uv)_x = 0, \tag{11}$$

$$u_x + \beta v_y = 0, \tag{12}$$

where α and β are all nonzero constants. Jiao et al. [29] obtained new traveling wave solutions of Eqs. (11) and (12) by using an extended method.

According to Step 1, we get $n = 4$ for u and v. We assume that Eqs. (11) and (12) have the following formal solutions:

$$u = a_0 + a_1 z(\xi) + a_2 z^2(\xi) + a_3 z^3(\xi) + a_4 z^4(\xi) + b_1 z'(\xi) + b_2 z(\xi) z'(\xi) + b_3 z^2(\xi) z'(\xi) + b_4 z^3(\xi) z'(\xi), \tag{13}$$

$$v = A_0 + A_1 z(\xi) + A_2 z^2(\xi) + A_3 z^3(\xi) + A_4 z^4(\xi) + B_1 z'(\xi) + B_2 z(\xi) z'(\xi) + B_3 z^2(\xi) z'(\xi) + B_4 z^3(\xi) z'(\xi), \tag{14}$$

where $a_0 = a_0(y,t)$, $a_1 = a_1(y,t)$, $a_2 = a_2(y,t)$, $a_3 = a_3(y,t)$, $a_4 = a_4(y,t)$, $b_1 = b_1(y,t)$, $b_2 = b_2(y,t)$, $b_3 = b_3(y,t)$, $b_4 = b_4(y,t)$, $A_0 = A_0(y,t)$, $A_1 = A_1(y,t)$, $A_2 = A_2(y,t)$, $A_3 = A_3(y,t)$, $A_4 = A_4(y,t)$, $B_1 = B_1(y,t)$, $B_2 = B_2(y,t)$, $B_3 = B_3(y,t)$, $B_4 = B_4(y,t)$, $\eta = \eta(y,t)$, $\xi = kx + \eta$, k is a nonzero constant.

With the aid of *Mathematica*, substituting (13) and (14) along with Eq. (3) into Eqs. (11) and (12), then setting each coefficient of $z^j(\xi)z'^l(\xi)$ $(l = 0, 1;\ j = 1, 2, \ldots)$ to zero, we get a set of over-determined partial differential equations for a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, A_0, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 and η. Solving the system of over-determined partial differential equations by use of *Mathematica*, we obtain the following results:

Case 2.1.

$$a_0 = -\frac{\beta(4k^2a + \alpha C)f_1(y)}{k\alpha}, \qquad a_1 = 0, \qquad a_2 = -\frac{6kb\beta f_1(y)}{\alpha}, \qquad a_3 = 0, \qquad a_4 = -\frac{12kc\beta f_1(y)}{\alpha}, \tag{15}$$

$$b_1 = 0, \qquad b_2 = \mp\frac{12k\beta\sqrt{c}f_1(y)}{\alpha}, \qquad b_3 = 0, \qquad b_4 = 0, \qquad A_0 = -\frac{k\alpha C + f_2'(t)}{k\alpha}, \qquad A_1 = 0, \qquad A_2 = \frac{6k^2b}{\alpha}, \tag{16}$$

$$A_3 = 0, \qquad A_4 = \frac{12k^2c}{\alpha}, \qquad B_1 = 0, \qquad B_2 = \pm\frac{12k^2\sqrt{c}}{\alpha}, \qquad B_3 = 0, \qquad B_4 = 0, \qquad \eta = \int f_1(y)\,dy + f_2(t), \tag{17}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t, respectively, C is an arbitrary constant, and $f_2'(t) = df_2(t)/dt$.

Case 2.2.

$$a_0 = -\frac{\beta(4k^2a + \alpha C)f_1(y)}{k\alpha}, \qquad a_1 = 0, \qquad a_2 = -\frac{12kb\beta f_1(y)}{\alpha}, \qquad a_3 = 0, \qquad a_4 = -\frac{24kc\beta f_1(y)}{\alpha}, \tag{18}$$

$$b_1 = 0, \qquad b_2 = 0, \qquad b_3 = 0, \qquad b_4 = 0, \qquad A_0 = -\frac{k\alpha C + f_2'(t)}{k\alpha}, \qquad A_1 = 0, \qquad A_2 = \frac{12k^2b}{\alpha}, \qquad A_3 = 0, \tag{19}$$

$$A_4 = \frac{24k^2c}{\alpha}, \qquad B_1 = 0, \qquad B_2 = 0, \qquad B_3 = 0, \qquad B_4 = 0, \qquad \eta = \int f_1(y)\,dy + f_2(t), \qquad c = \frac{b^2}{4a}, \tag{20}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t, respectively, C is an arbitrary constant, and $f_2'(t) = df_2(t)/dt$.

Case 2.3.

$$a_0 = -\frac{\beta(k^2a + \alpha C)f_1(y)}{k\alpha}, \qquad a_1 = 0, \qquad a_2 = -\frac{3kb\beta f_1(y)}{\alpha}, \qquad a_3 = 0, \qquad a_4 = 0, \tag{21}$$

$$b_1 = \mp\frac{3k\beta\sqrt{b}f_1(y)}{\alpha}, \qquad b_2 = 0, \qquad b_3 = 0, \qquad b_4 = 0, \qquad A_0 = -\frac{k\alpha C + f_2'(t)}{k\alpha}, \qquad A_1 = 0, \qquad A_2 = \frac{3k^2b}{\alpha}, \tag{22}$$

$$A_3 = 0, \qquad A_4 = 0, \qquad B_1 = \pm\frac{3k^2\sqrt{b}}{\alpha}, \qquad B_2 = 0, \qquad B_3 = 0, \qquad B_4 = 0, \qquad \eta = \int f_1(y)\,dy + f_2(t), \qquad c = 0, \tag{23}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t, respectively, C is an arbitrary constant, and $f_2'(t) = df_2(t)/dt$.

Substituting Case 2.1 along with $z_2(\xi)$ in Table 1 into (13) and (14), we obtain exact nontraveling wave solutions of Eqs. (11) and (12):

$$\begin{aligned}
u &= -\frac{\beta(4k^2a + \alpha C)f_1(y)}{k\alpha} - \frac{6kab^2\beta f_1(y)\operatorname{csch}^2(\sqrt{a}\xi)}{\alpha(b^2 - ac(1 + \varepsilon\coth(\sqrt{a}\xi))^2)} - \frac{12ka^2b^2c\beta f_1(y)\operatorname{csch}^4(\sqrt{a}\xi)}{\alpha[b^2 - ac(1 + \varepsilon\coth(\sqrt{a}\xi))^2]^2} \\
&\quad \mp \frac{6kab\sqrt{ac}\beta f_1(y)\operatorname{csch}^4(\sqrt{a}\xi)[(2ac - b^2)\sinh(2\sqrt{a}\xi) + 2ac\varepsilon\cosh(2\sqrt{a}\xi)]}{\alpha[b^2 - ac(1 + \varepsilon\coth(\sqrt{a}\xi))^2]^2}, \\
v &= -\frac{k\alpha C + f_2'(t))}{k\alpha} + \frac{6k^2ab^2\operatorname{csch}^2(\sqrt{a}\xi)}{\alpha(b^2 - ac(1 + \varepsilon\coth(\sqrt{a}\xi))^2)} + \frac{12k^2a^2b^2c\operatorname{csch}^4(\sqrt{a}\xi)}{\alpha[b^2 - ac(1 + \varepsilon\coth(\sqrt{a}\xi))^2]^2} \\
&\quad \pm \frac{6k^2ab\sqrt{ac}\operatorname{csch}^4(\sqrt{a}\xi)[(2ac - b^2)\sinh(2\sqrt{a}\xi) + 2ac\varepsilon\cosh(2\sqrt{a}\xi)]}{\alpha[b^2 - ac(1 + \varepsilon\coth(\sqrt{a}\xi))^2]^2},
\end{aligned}$$

where $\xi = kx + \int f_1(y)\,dy + f_2(t)$.

From (13) and (14), Cases 2.1–2.3 and Table 1, we can obtain other exact solutions of Eqs. (11) and (12), here we also omit them for simplicity. The Cases 2.1 and 2.2 cannot be obtained by the method in [25]. All the solutions obtained from Cases 2.1–2.3 are new and have not been reported yet.

Remark 2. All the results reported in this Letter have been checked with *Mathematica*. By using our method, we can obtain many new and more general exact solutions of the other NLPDEs in [25] including all the solutions given there as special cases. It shows that our method is more powerful than the method in [25] in constructing exact solutions of NLPDEs.

4. Conclusion

In short, we have presented a generalized new auxiliary equation method to construct more general exact solutions of NLPDEs. With the help of *Mathematica*, the method provides a powerful mathematical tool to obtain more general exact solutions of a great many NLPDEs in mathematical physics. Compared with the method in [25], our method is more powerful. It can be used to construct more general exact solutions which contain not only the results obtained by using the method in [25] as special cases but also a series of new and more general exact solutions. In order to illustrate the validity and advantages of our method, we have applied it to the combined KdV–mKdV equation and the $(2+1)$-dimensional asymmetric Nizhnik–Novikov–Vesselov equations. As a result, many new and more general exact solutions have been obtained. To our knowledge, these solutions have not been reported. It may be important to explain some physical phenomena.

Acknowledgements

This work was supported by the Natural Science Foundation of Educational Committee of Liaoning Province of China.

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