A Symmetric Homotopy and Hybrid Polynomial System Solving Method for Mixed Trigonometric Polynomial Systems

Yu Bo, Dong Bo
School of Mathematics,
Dalian University of Technology,
Dalian, Liaoning 116024, China.
yubo@dlut.edu.cn dongbodlut@gmail.com

Abstract.

A mixed trigonometric polynomial system, which rather frequently occurs in applications, is a polynomial system where the monomials contain a mixture of some variables and sine and cosine functions applied to the other variables. Polynomial systems coming from the mixed trigonometric polynomial systems have a special structure. Applying this structure, we have presented a hybrid polynomial system solving method, which is more efficient than random product homotopy method and polyhedral homotopy method in solving this class of systems. In this paper, a symmetric homotopy is constructed and, combining homotopy method, decomposition and elimination techniques, a more efficient hybrid method for solving this class of polynomial systems is presented. Keeping the symmetric structure, only part of homotopy paths have to be traced, and more important, the computation work can be reduced due to the inconsistent subsystems, which need not to be solved at all. Exploiting the new hybrid method, some problems from the literature and a challenging practical problem, which cannot be solved by the existent polynomial system solving method, are solved. Numerical results show that our method is more efficient than the polyhedral homotopy method or hybrid solving method, which are the state-of-art sparse polynomial system solving methods for high dimensional and highly sparse polynomial systems coming from mixed trigonometric polynomial systems.

Key words. polynomial system, mixed trigonometric polynomial system, homotopy method, hybrid algorithm, symbolic-numeric computation.
AMS subject classifications. 65H10, 65H20, 68W30.

1 Introduction

A mixed trigonometric polynomial system (abbreviated by MTPS) is a polynomial system whose monomial is mixed by some variables and sine and cosine functions applied to the other variables. This class of systems occurs in many fields of science and engineering, such as neurophysiology [22], kinematics [3, 22], PUMA robot[17, 18], and etc.. Formally, A MTPS can be shown as follows:

$$F(y,\theta) = (f_1(y,\theta), \cdots, f_{n+m}(y,\theta))^T = 0, \tag{1}$$

where $y = (y_1, \cdots, y_n)$, $\theta = (\theta_1, \ldots, \theta_m)$, and for $1 \le i \le n+m$,

$$f_i(y, \theta) = \sum_{\alpha \in S_i} b_{i\alpha} y^\beta \sin^\mu \theta \cos^\nu \theta + c_i^1, \tag{2}$$

where $\alpha = (\beta,\ \mu,\ \nu)$, $S_i \subset \mathcal{N}^{n+2m}$ (here, for convenience, we write the constant term of $f_i(y, \theta)$ explicitly, i.e., $\forall\ i$, $0 \notin S_i$) and

$$\left\{ \begin{array}{rcl} y^\beta & = & y_1^{\beta_1} \cdots y_n^{\beta_n}, \\ \sin^\mu \theta & = & \sin^{\mu_1} \theta_1 \cdots \sin^{\mu_m} \theta_m, \\ \cos^\nu \theta & = & \cos^{\nu_1} \theta_1 \cdots \cos^{\nu_m} \theta_m, \end{array} \right. \tag{3}$$

Example 1.1. This is a challenging practical problem we meet in signal processing of sonar and radar, which is the original motivation of this research.

$$\left\{ \begin{array}{lcllcl} \sum_{i=1}^n y_i & = & c_1^1, & \sum_{i=1}^n y_i^2 & = & c_2^1, \\ \sum_{i=1}^n y_i \sin\theta_i & = & c_3^1, & \sum_{i=1}^n y_i \cos\theta_i & = & c_4^1, \\ \sum_{i=1}^n y_i \sin^2\theta_i & = & c_5^1, & \sum_{i=1}^n y_i \sin\theta_i \cos\theta_i & = & c_6^1, \\ \sum_{i=1}^n y_i^2 \sin\theta_i & = & c_7^1, & \sum_{i=1}^n y_i^2 \cos\theta_i & = & c_8^1, \\ \sum_{i=1}^n y_i^2 \sin^2\theta_i & = & c_9^1, & \sum_{i=1}^n y_i^2 \sin\theta_i \cos\theta_i & = & c_{10}^1, \\ \sum_{i=1}^n y_i^2 \sin^2\theta_i \cos\theta_i & = & c_{11}^1, & \sum_{i=1}^n y_i^2 \sin\theta_i \cos^2\theta_i & = & c_{12}^1, \\ \sum_{i=1}^n y_i^2 \sin^3\theta_i \cos\theta_i & = & c_{13}^1, & \sum_{i=1}^n y_i^2 \sin^4\theta_i & = & c_{14}^1. \end{array} \right. \tag{4}$$

where, $n = 2$ to 7 (if $n < 7$, then only the first $2n$ equations are needed).

Substituting occurrences of y_i, $\sin\theta_j$ and $\cos\theta_j$ with new variables x_i, x_{n+j} and x_{n+m+j} and adding quadratic equations of the form $x_{n+j}^2 + x_{n+m+j}^2 - 1 = 0$, where $i = 1, \ldots, n, j = 1, \ldots, m$, the MTPS $F(y, \theta) = 0$ in (1), (2) and (3) can be transformed to the following polynomial system:

$$P(x) = \begin{pmatrix} p_1(x) + c_1^1 \\ \cdots \\ p_{n+m}(x) + c_{n+m}^1 \\ x_{n+1}^2 + x_{n+m+1}^2 - 1 \\ \cdots \\ x_{n+m}^2 + x_{n+2m}^2 - 1 \end{pmatrix} = 0, \tag{5}$$

where

$$p_i(x) = \sum_{\alpha \in S_i} b_{i\alpha} x^\alpha. \tag{6}$$

Example 1.2 (Example 1.1 continued).

$$\left\{\begin{array}{llllll}
\sum_{i=1}^{n} x_i & = & c_1^1, & \sum_{i=1}^{n} x_i^2 & = & c_2^1, \\
\sum_{i=1}^{n} x_i x_{n+i} & = & c_3^1, & \sum_{i=1}^{n} x_i x_{n+m+i} & = & c_4^1, \\
\sum_{i=1}^{n} x_i x_{n+i}^2 & = & c_5^1, & \sum_{i=1}^{n} x_i x_{n+i} x_{n+m+i} & = & c_6^1, \\
\sum_{i=1}^{n} x_i^2 x_{n+i} & = & c_7^1, & \sum_{i=1}^{n} x_i^2 x_{n+m+i} & = & c_8^1, \\
\sum_{i=1}^{n} x_i^2 x_{n+i}^2 & = & c_9^1, & \sum_{i=1}^{n} x_i^2 x_{n+i} x_{n+m+i} & = & c_{10}^1, \\
\sum_{i=1}^{n} x_i^2 x_{n+i}^2 x_{n+m+i} & = & c_{11}^1, & \sum_{i=1}^{n} x_i^2 x_{n+i} x_{n+m+i}^2 & = & c_{12}^1, \\
\sum_{i=1}^{n} x_i^2 x_{n+i}^3 x_{n+m+i} & = & c_{13}^1, & \sum_{i=1}^{n} x_i^2 x_{n+i}^4 & = & c_{14}^1, \\
x_{n+1}^2 + x_{n+m+1}^2 & = & 1, & x_{n+2}^2 + x_{n+m+2}^2 & = & 1, \\
\cdots & & & \cdots & & \\
x_{n+m-1}^2 + x_{n+2m-1}^2 & = & 1, & x_{n+m}^2 + x_{n+2m}^2 & = & 1.
\end{array}\right. \tag{7}$$

Polynomial systems coming from MTPSs have a special structure: its lower part consists of m equations of the form $x_{n+i}^2 + x_{n+m+i}^2 - 1, i = 1, \ldots, m$, which possesses an inherent symmetry: the permutation of x_{n+i} and x_{n+m+i} and that of (x_{n+i}, x_{n+m+i}) and (x_{n+j}, x_{n+m+j}) and that of (x_{n+i}, x_{n+m+i}) and (x_{n+m+j}, x_{n+j}) can not change the m equations except their orders. In this paper, exploiting the special structure and the symmetry, a homotopy, which is a combination of a special coefficient-parameter homotopy and the random product homotopy and keeps the symmetry of its lower part, is constructed. Combining homotopy method, decomposition and elimination techniques, an efficient method, named symmetric hybrid method, for solving this class of polynomial systems is presented.

The paper is organized as follows. Introductions to homotopy methods and symmetric homotopy methods for solving polynomial systems are given in the rest of this section; The symmetric hybrid method for solving polynomial systems coming from MTPSs is given in Section 2, and also a theorem to ensure applying our symmetric homotopy can find all isolated solutions of the target system is presented; The precise solving procedure and the advantage of our symmetric hybrid method are described in Section 3; Numerical examples and applications are given in Section 4.

1.1 Homotopy methods for solving polynomial systems

Homotopy method is an efficient numerical method for solving polynomial systems. The homotopy method is to define a polynomial system $Q(x) = 0$, known as the start system, and then follow the curves in the variable t consisting of the solutions of

$$H(x, t) = (1 - t)Q(x) + tP(x)$$

to get the solutions of the target system $P(x) = 0$. If the start system $Q(x) = 0$ is chosen correctly, the following three properties hold [7]:

Triviality: The solutions of $H(x, 0) = 0$ are known.

Smoothness: The solution set of $H(x, t) = 0$ for all $0 \le t < 1$ consists of a finite number of smooth paths, each parameterized by $t \in [0, 1)$.

Accessibility: Every isolated solutions of $H(x, 1) = 0$ can be reached by some paths originating at a solution of $H(x, 0) = 0$.

When these three properties hold, tracing numerically homotopy paths in the variable t from 0 to 1, we can find all isolated solutions of $P(x) = 0$.

Standard homotopies, which generate Bézout number of homotopy paths, are given in [6, 19, 28, 31]. Theoretically, standard homotopy can be used to find all isolated solutions of any polynomial system. However, the polynomial systems arising in practice are often deficient, that is, its actual number of solutions is less than Bézout number. For deficient systems, standard homotopies generate many extraneous paths that tend to infinity and hence make wasted computation. So studies on homotopy methods for solving polynomial systems are now focused on exploiting sparsity of the system, that is, constructing better homotopy such that the number of homotopy paths to be traced is proportional to the actual number of isolated solutions of the target system. Many homotopies for deficient polynomial systems have been constructed, the list is hardly exhaustive, see discussions in [4, 7, 8, 9, 11, 12, 13, 14, 15, 18, 20, 21, 23, 26, 27, 30], and the many references contained therein.

For its essential complexity, polynomial system solving problem is far from satisfactorily resolved. Above mentioned homotopy methods still can not be applied to efficiently solve practical polynomial systems of moderate size. As an example, for polynomial systems in (7), when $n = 5, 6, 7$, Bézout numbers are 331776, 16588800, 1194393600 and multi-homogeneous Bézout numbers are 73728, 3736576, 270811136 respectively. It is unsuitable to solve this class of polynomial systems by the standard homotopy methods, multi-homogeneous homotopy methods or other random product homotopy methods. And the polyhedral homotopy methods are also difficult to be applied because computation of the BKK-bound and the mixed subdivision needs too long time. Further effort needs to be made to solve polynomial systems more efficiently by exploiting their special structure.

1.2 Hybrid polynomial system solving method for MTPS

Polynomial systems coming from MTPSs have a special structure: the last m equations are

$$\begin{pmatrix} x_{n+1}^2 + x_{n+m+1}^2 - 1 \\ \cdots \\ x_{n+m}^2 + x_{n+2m}^2 - 1 \end{pmatrix} = 0.$$

In [29], by exploiting this special structure, we proposed a hybrid polynomial system solving method for MTPS. It hybridizes the homotopy method and some symbolic computation techniques, such as variable substitution, variable elimination, and some reduction techniques. The homotopy we constructed is also hybrid, it combines linear product homotopy and coefficient parameter homotopy or cheater's homotopy. The hybrid method is more efficient than the multi-homogeneous homotopy method or the polyhedral homotopy method, which is the state-of-art sparse polynomial system solving method for high dimensional and highly sparse problems.

However, for the polynomial systems in (7) in the case that $n = 7$, when applying the polyhedral homotopy method to solve the start system, there are 6705152 homotopy paths have to be traced, and finding the subdivision is also time-consuming; when applying hybrid homotopy method to solve the target system, there are 40320 homotopy paths need to be traced. The computational amount is still very large.

Further discoveries of the structure of the transformed polynomial system have to be explored. Besides the above special structure, the last m equations of polynomial systems coming from MTPSs have an inherent symmetric structure: permutations

$$x_{n+i} \leftrightarrow x_{n+m+i},$$
$$(x_{n+i}, x_{n+m+i}) \leftrightarrow (x_{n+j}, x_{n+m+j}),\ (x_{n+i}, x_{n+m+i}) \leftrightarrow (x_{n+i}, x_{n+m+i})$$

do not change the last m equations except their orders. For a symmetric polynomial system, its solutions can be divided to some equivalent classes, and all solutions in a class can be generated in some rules by any solution, named "*representative solution*", in it. If a homotopy is constructed to keep the symmetry of the target polynomial system, then only the representative solution path in each equivalent class should be traced, and hence the computational work can be decreased a lot.

1.3 Homotopy methods for symmetric polynomial systems

In many cases, polynomial systems arising in practical problems are symmetric, and some symmetric homotopy methods have been published.

Some symmetric homotopy methods have been published. In [16], Meravý provided a sign-symmetric homotopy for solving polynomial systems which have a symmetric solution set and gave conditions upon start systems in the general cases. However, this homotopy was constructed according to the classic theorem of Bézout which is not suitable for the deficient polynomial systems. In [10], Li and Sauer applied the random product homotopy to what they called a self-symmetric polynomial system, which belongs to polynomial systems that suitable to apply the random product homotopy to solve. In [24], for the polynomial systems with a (G, V, W)-symmetric structure, Verschelde and Cools gave an algorithm to verify whether a given system is (G, V, W)-symmetric and presented how to construct a (G, V, W)-symmetric homotopy. Applying this homotopy, only the generating solution paths have to be traced. In [25], symmetric polyhedral homotopies were constructed to deal with (G, V, W)-symmetric polynomial systems.

The existent methods for symmetric polynomial systems are mainly focused on the specially symmetric structure, for the generally symmetric structure, these methods are hard to be applied or generalized. Symmetric homotopy methods mentioned above are all unsuitable to be applied to solve polynomial systems with the specially symmetric structure mentioned in Section 1.2.

2 Symmetric hybrid methods for solving polynomial systems coming from MTPSs

Motivated by [29] and the existent symmetric homotopies, we present a symmetric homotopy and hybrid polynomial system solving method to solve the MTPSs more efficiently.

2.1 Symmetric hybrid homotopy

To keep the symmetric structure of polynomial systems in (5) and (6), we construct the following symmetric homotopy:

$$H(x,t)=\gamma(1-t)Q(x)+tP(x), \tag{8}$$

where

$$Q(x)=\begin{pmatrix} p_1(x)+c_1^0 \\ \cdots \\ p_{n+m}(x)+c_{n+m}^0 \\ (s_1x_{n+1}+s_2x_{n+m+1}+s_0)(s_2x_{n+1}+s_1x_{n+m+1}+s_0) \\ \cdots \\ (s_1x_{n+m}+s_2x_{n+2m}+s_0)(s_2x_{n+m}+s_1x_{n+2m}+s_0) \end{pmatrix}, \tag{9}$$

and $\gamma, c_1^0, \dots, c_{n+m}^0, s_0, s_1, s_2$ are all random nonzero complex numbers.

The start system $Q(x)=0$ can be decomposed into 2^m subsystems

$$Q_\sigma(x)=\begin{pmatrix} p_1(x)+c_1^0 \\ \cdots \\ p_{n+m}(x)+c_{n+m}^0 \\ s_{\sigma_1}x_{n+1}+s_{3-\sigma_1}x_{n+m+1}+s_0 \\ \cdots \\ s_{\sigma_m}x_{n+m}+s_{3-\sigma_m}x_{n+2m}+s_0 \end{pmatrix}=0, \tag{10}$$

where $\sigma\in\Sigma\triangleq\{(\sigma_1,\dots,\sigma_m)|\sigma_i\in\{1,2\}\}$.

Remark 2.1. The last m equations of the start system (9) have the same symmetric structure as those of the target system. If the target system is symmetric, then in the process of tracing homotopy paths, only the representative solution paths have to be traced, and thus the computational work can be reduced.

Specially, we can fix some coefficients in the start system, such as

$$Q(x)=\begin{pmatrix} p_1(x)+c_1^0 \\ \cdots \\ p_{n+m}(x)+c_{n+m}^0 \\ (x_{n+1}+x_{n+m+1}\mathbf{i}+s)(x_{n+1}\mathbf{i}+x_{n+m+1}+s) \\ \cdots \\ (x_{n+m}+x_{n+2m}\mathbf{i}+s)(x_{n+m}\mathbf{i}+x_{n+2m}+s) \end{pmatrix}, \tag{11}$$

where s must be a random complex number instead of a fixed complex number, otherwise the homotopy will not satisfy accessibility.

2.2 Homotopy theorem and its proof

In order to find all the solutions of the target system through numerically tracing the solution paths of the homotopy map, we need to prove there exists a smooth path connecting every solution of the target system and one solution of the start system.

Theorem 1 *$P(x), Q(x)$ are respectively as the systems in (5), (6) and (9). There exists an open dense full-measure subset U of $(n+m+4)$-dimensional complex space $\mathcal{C}^{n+m+4}$, such that for all $\{c_1^0, \ldots, c_{n+m}^0, s_0, s_1, s_2, \gamma\} \in U$ with $s_1^2 \neq s_2^2$,*

1. *solutions of $Q(x) = 0$ are isolated and regular,*
2. *the smoothness and accessibility properties hold for the homotopy map $H(x,t)$ defined in (8) and (9).*

To ensure the process of path following can be started, we must show that from every solution of the start system, a homotopy path can be generated, that is, the solutions of the start system should be nonsingular.

Lemma 1 *There exists an open dense full-measure subset W_1 of $\mathcal{C}^{n+m}$, such that for $(c_1^0, \ldots, c_{n+m}^0) \in W_1$ and any nonzero complex numbers s_0, s_1 and s_2 with $s_1^2 \neq s_2^2$,*

$$s_1 x_{n+i} + s_2 x_{n+m+i} + s_0, \;\; s_2 x_{n+i} + s_1 x_{n+m+i} + s_0$$

are not both zero at any zero point of $Q(x)$ in (9).

Proof: Without loss of generality, we prove the result for $i = m$ under the assumption that

$$s_1 x_{n+j} + s_2 x_{n+m+j} + s_0, \;\; s_2 x_{n+j} + s_1 x_{n+m+j} + s_0$$

$(1 \leq j \leq m-1)$ are not both zero at any zero point of $Q(x)$.

Consider the following polynomial system

$$R(x) = \begin{pmatrix} p_1(x) + c_1^0 \\ \cdots \\ p_{n+m}(x) + c_{n+m}^0 \\ (s_1 x_{n+1} + s_2 x_{n+m+1} + s_0)(s_2 x_{n+1} + s_1 x_{n+m+1} + s_0) \\ \cdots \\ (s_1 x_{n+m-1} + s_2 x_{n+2m-1} + s_0)(s_2 x_{n+m-1} + s_1 x_{n+2m-1} + s_0) \\ s_1 x_{n+m} + s_2 x_{n+2m} + s_0 \\ s_1 x_{n+2m} + s_2 x_{n+m} + s_0 \end{pmatrix}.$$

Let $\mathcal{Z}_0$ be the zero set of

$$\begin{pmatrix} (s_1x_{n+1}+s_2x_{n+m+1}+s_0)(s_2x_{n+1}+s_1x_{n+m+1}+s_0) \\ \cdots \\ (s_1x_{n+m-1}+s_2x_{n+2m-1}+s_0)(s_2x_{n+m-1}+s_1x_{n+2m-1}+s_0) \\ s_1x_{n+m}+s_2x_{n+2m}+s_0 \\ s_1x_{n+2m}+s_2x_{n+m}+s_0 \end{pmatrix}. \tag{12}$$

The Jacobian of (12) with respect to $x_{n+1},\dots,n_{n+2m}$ is

$$A=\begin{pmatrix} a_1 & & & 0 & a_{m+1} & & & \\ & \ddots & & 0 & & \ddots & & \\ & & a_{m-1} & 0 & & & a_{2m-1} & \\ 0 & \cdots & 0 & s_1 & 0 & \cdots & 0 & s_2 \\ 0 & \cdots & 0 & s_2 & 0 & \cdots & 0 & s_1 \end{pmatrix},$$

where

$$\begin{cases} a_i=s_1(s_2x_{n+i}+s_1x_{n+m+i}+s_0)+s_2(s_1x_{n+i}+s_2x_{n+m+i}+s_0), \\ a_{m+i}=s_2(s_2x_{n+i}+s_1x_{n+m+i}+s_0)+s_1(s_1x_{n+i}+s_2x_{n+m+i}+s_0), \end{cases} \tag{13}$$

for $1\le i\le m-1$. Because $s_1^2\ne s_2^2$, and one polynomial of

$$s_1x_{n+i}+s_2x_{n+m+i}+s_0,\ \ s_2x_{n+i}+s_1x_{n+m+i}+s_0$$

is nonzero for any solution of $Q(x)=0$, A is of full rank and hence $\mathcal{Z}_0$ is a nonsingular $(n+m-1)$-dimensional variety. Applying Bertini's theorem ([2]) $n+m-1$ times, there exist $n+m-1$ subsets $M_1,\dots,M_{n+m-1}$ of $\mathcal{C}$, such that for $c_i^0\notin M_i$, the solution set $\mathfrak{N}$ to

$$\begin{pmatrix} p_1(x)+c_1^0 \\ \cdots \\ p_{n+m-1}(x)+c_{n+m-1}^0 \\ (s_1x_{n+1}+s_2x_{n+m+1}+s_0)(s_2x_{n+1}+s_1x_{n+m+1}+s_0) \\ \cdots \\ (s_1x_{n+m-1}+s_2x_{n+2m-1}+s_0)(s_2x_{n+2m-1}+s_1x_{n+m-1}+s_0) \\ s_1x_{n+m}+s_2x_{n+2m}+s_0 \\ s_2x_{n+2m}+s_1x_{n+m}+s_0 \end{pmatrix}=0$$

is a 0 dimensional set and the set $M_{n+m}=\{c_{n+m}^0=-p_{n+m}(x):x\in\mathfrak{N}\}$ is a finite set in $\mathcal{C}$, and for $(c_1^0,\dots,c_{n+m}^0)$ chosen from $W_1=\mathcal{C}^{n+m}\setminus(M_1\times\cdots\times M_{n+m})$, an open dense full-measure subset of $\mathcal{C}^{n+m}$, $R(x)=0$ has no solution. Thus the lemma follows. □

Lemma 2 *There exists an open dense full-measure subset W_2 of $\mathcal{C}^{n+m}$, such that for $(c_1^0,\dots,c_{n+m}^0)\in W_2$ and any nonzero complex numbers s_0,s_1,s_2 with $s_1^2\ne s_2^2$, the solution set of $Q(x)=0$ in (9) consists of isolated nonsingular points.*

Proof: Let $\mathcal{Y}$ be the solution set to

$$\begin{pmatrix} (s_1x_{n+1}+s_2x_{n+m+1}+s_0)(s_2x_{n+1}+s_1x_{n+m+1}+s_0) \\ \cdots \\ (s_1x_{n+m}+s_2x_{n+2m}+s_0)(s_2x_{n+m}+s_1x_{n+2m}+s_0) \end{pmatrix} = 0. \tag{14}$$

The Jacobian of (14) with respect to $x_{n+1},\dots,x_{n+2m}$ is

$$B = \begin{pmatrix} a_1 & & & a_{m+1} & & \\ & \ddots & & & \ddots & \\ & & a_m & & & a_{2m} \end{pmatrix},$$

where a_i's are as in (13) for $i = 1,\cdots,m$.

By Lemma 1, there exists a full-measure subset W_1 of $\mathcal{C}^{n+m}$ such that for $(c_1^0,\dots,c_{n+m}^0) \in W_1$ and any nonzero complex numbers s_0, s_1, s_2 with $s_1^2 \neq s_2^2$, only one equation of

$$s_1x_{n+k}+s_2x_{n+m+k}+s_0 = 0,\ \ s_2x_{n+k}+s_1x_{n+m+k}+s_0 = 0$$

holds for any solution of $Q(x) = 0$, thus $\mathrm{rank}_{\mathcal{C}}B = m$ and $\mathcal{Y}$ is a nonsingular $(n+m)$-dimensional variety.

After repeatedly utilizing Bertini's Theorem $n+m$ times, there exists a full-measure subset $W_2 \subset W_1$ of $\mathcal{C}^{n+m}$ such that for $(c_1^0,\dots,c_{n+m}^0) \in W_2$ and any nonzero complex numbers s_0, s_1, s_2 with $s_1^2 \neq s_2^2$, all solutions to $Q(x) = 0$ are isolated and nonsingular. □

Lemma 3 **([1])** *If $F_0,\dots,F_n$ are homogeneous polynomials in $x_0,\dots,x_n$ of degrees $d_0,\dots,d_n$, then $F_0 = \cdots = F_n = 0$ has a nontrivial solution if and only if $\mathrm{Res}(F_0,\dots,F_n) = 0$, here, $\mathrm{Res}(F_0,\dots,F_n)$ is the resultant of $F_0,\dots,F_n$, a homogeneous polynomial in all possible coefficients of F_i $(i = 0,\dots,n)$.*

Proof of Theorem 1: **Proof:** (1). Rewrite $H(x,t) = 0$ as $H(x,\tau) = Q(x)+\tau P(x) = 0$, where $\tau = \dfrac{t}{\gamma(1-t)}$, and let $\widetilde{H}(\widetilde{x},\tau) = \widetilde{Q}(\widetilde{x})+\tau\widetilde{P}(\widetilde{x})$ be the homogenization of $H(x,\tau)$, where $\widetilde{Q}(\widetilde{x})$ and $\widetilde{P}(\widetilde{x})$ are the homogenization of $Q(x)$ and $P(x)$.

Denote $\left(\dfrac{\partial\widetilde{H}(\widetilde{x},\tau)}{\partial\widetilde{x}}\right)$ by $\widetilde{H}'(\widetilde{x},\tau)$. If the solutions to $\widetilde{H}(\widetilde{x},\tau) = 0$ are singular, then the following system

$$\begin{pmatrix} \det(\widetilde{H}'(\widetilde{x},\tau)) \\ \widetilde{H}(\widetilde{x},\tau) \end{pmatrix} = 0 \tag{15}$$

has nontrivial solutions, thus by Lemma 3, the resultant $\mathrm{Res}(\widetilde{H},\det(\widetilde{H}'))$, denoted by $r(\tau)$, of $\widetilde{H},\det(\widetilde{H}')$ equals 0. Since solutions to $Q(x) = 0$ are regular, $r(0) \neq 0$ and $r(\tau)$ is not identically zero, the solution set S_1 to $r(\tau) = 0$ is finite, and the complementary set S_2 of $\{\gamma \in \mathcal{C} : \frac{t}{\gamma(1-t)} \in S_1\}$ is an open dense full-measure subset of $\mathcal{C}$. For $\gamma \in S_2$, $r(\tau) \neq 0$, and thus (15) has no solutions, that is, there exists an open dense full-measure subset U_1 of $\mathcal{C}^{n+m+1}$ such that the solutions to $H(x,t) = 0$ are nonsingular for $(c_1^0, \ldots, c_{n+m}^0, s_0, s_1, s_2, \gamma) \in U_1$ with $s_1^2 \neq s_2^2$.

(2) Refer to [29] for the proof of accessibility. $\square$

3 The solution procedure

Applying the symmetric homotopy (8) and (9), and combining the homotopy method with symbolic method, such as decomposition and variable substitution and some reduction techniques, we present our symmetric hybrid method as follows:

Algorithm 3.1 (Symmetric Hybrid Method for MTPS):

Step 1: Construct the symmetric hybrid homotopy as in (8) and (9).

Step 2: Decompose the start system in (9) into 2^m subsystems $Q_\sigma(x) = 0$ as (10).

Step 3: By variable substitution

$$x_{n+m+i} = l_i(x_{n+i}) \triangleq -\frac{s_{\sigma_i}}{s_{3-\sigma_i}} x_{n+i} - \frac{s_0}{s_{3-\sigma_i}} \tag{16}$$

for $i = 1, \ldots, m$, $Q_\sigma(x) = 0$ can be transformed to a new simpler system $\widehat{Q}_\sigma(\widehat{x}) = 0$ with less variables and equations:

$$\widehat{Q}_\sigma(\widehat{x}) = \begin{pmatrix} q_1(\widehat{x}) + c_1^0 \\ \cdots \\ q_{n+m}(\widehat{x}) + c_{n+m}^0 \end{pmatrix},$$

where $\widehat{x} = (x_1, \ldots, x_{n+m})$ and for all $1 \leq i \leq n+m$,

$$q_i(\widehat{x}) = \sum_{\alpha \in S_i} b_{i\alpha} (x_1, \ldots, x_{n+m}, l_1(x_{n+1}), \ldots, l_m(x_{n+m}))^\alpha.$$

Hence solving $Q(x) = 0$ is equivalent to solving 2^m systems $\widehat{Q}_\sigma(\widehat{x}) = 0$ of dimension $n+m$.

Step 4: Reduce subsystems. Applying some symbolic reduction techniques to reduce the subsystems so that they will have less upper bounds of solution number, and even assert some of them have no solutions indeed.

Step 5: Solve the reduced subsystems by some existent homotopy method. If the target system, and hence the start system, is symmetric, it is not necessarily to solve all the reduced subsystems. Only the subsystems with *representative solutions* should be solved.

Step 6: Expand the solutions of all subsystems by (16) to get all solutions to $Q(x) = 0$. These solutions serve as the start points of the homotopy in (8).

Step 7: Trace homotopy paths. Applying predict-correct method to follow solution paths of (8) to get all isolated solutions of the target system $P(x) = 0$.

Similar to the hybrid method in [29], the start system can be decomposed to some subsystems in $n + m$ variables, so it also possesses the advantage of the hybrid method in [29], which can be stated briefly as follows:

- Tracing homotopy paths in $(n + m)$-dimensional space will be considerably faster than that in $(n + 2m)$-dimensional space;
- Finding the best multi-homogeneous structure or the mixed subdivision of the reduced subsystems of dimension $n+m$ is much easier than that of the target system of dimension $n + 2m$.
- Sometimes the subsystems $Q_\sigma(\widehat{x})$ can be considerably reduced in Bézout number even when the target system $P(x) = 0$ can hardly be reduced.

On the other hand, due to the application of the symmetry, the symmetric hybrid method has two additional advantages as following:

- In Step 5, if the reduced subsystems are symmetric, exploiting homotopy methods for symmetric polynomial systems, only the representative paths should be traced, and hence decreases the computational work a lot. In Step 7, the same advantage holds if the target system is symmetric.
- The symmetric hybrid method can keep the sparse structure of the target system better than the hybrid method in [29] due to the symmetry of the last m equations in the start system. The subsystems can be considerably reduced in multi-Bézout number and mixed volume even when the target system can hardly be reduced. More important, at the extreme case, we can find that part of the reduced subsystems $\widehat{Q}_\sigma(\widehat{x}) = 0$ need not be solved because some equations in them are obviously inconsistent. We name this class of subsystems "*inconsistent system*" and other subsystems "*consistent system*". Thus the computational work can be reduced a lot, as stated in the following example.

Example 3.1 (Example 1.2 continued). Consider the transformed polynomial system in (7) for $n = 3$. We construct the start system as that in the hybrid method in [29]:

$$Q(x) = \begin{pmatrix} x_1 + x_2 + x_3 - c_1^0 \\ x_1^2 + x_2^2 + x_3^2 - c_2^0 \\ x_1x_4 + x_2x_5 + x_3x_6 - c_3^0 \\ x_1x_7 + x_2x_8 + x_3x_9 - c_4^0 \\ x_1x_4^2 + x_2x_5^2 + x_3x_6^2 - c_5^0 \\ x_1x_4x_7 + x_2x_5x_8 + x_3x_6x_9 - c_6^0 \\ (a_{11}^1x_4 + a_{12}^1x_7 + a_{10}^1)(a_{11}^2x_4 + a_{12}^2x_7 + a_{10}^2) \\ (a_{21}^1x_5 + a_{22}^1x_8 + a_{20}^1)(a_{21}^2x_5 + a_{22}^2x_8 + a_{20}^2) \\ (a_{31}^1x_6 + a_{32}^1x_9 + a_{30}^1)(a_{31}^2x_6 + a_{32}^2x_9 + a_{30}^2) \end{pmatrix} = 0.$$

$Q(x) = 0$ can be decomposed into 8 subsystems. For $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_i \in \{1, 2\}$, eliminating x_7, x_8, x_9 and by some simple symbolic computation, $Q_\sigma(x) = 0$ is equivalently transformed to system of form

$$\widehat{Q}_\sigma(\widehat{x}) = \begin{pmatrix} x_1 + x_2 + x_3 - c_1^0 \\ x_1^2 + x_2^2 + x_3^2 - c_2^0 \\ x_1x_4 + x_2x_5 + x_3x_6 - c_3^0 \\ b_1^1x_2x_5 + b_2^1x_3x_6 + b_3^1x_1 + b_4^1x_2 + b_5^1x_3 + b_6^1 \\ x_1x_4^2 + x_2x_5^2 + x_3x_6^2 - c_5^0 \\ b_1^2x_2x_5^2 + b_2^2x_3x_6^2 + b_3^2x_1x_4 + b_4^2x_2x_5 + b_5^2x_3x_6 + b_6^2 \end{pmatrix} = 0,$$

whose mixed volume is 12.

We construct the start system as that in the symmetric hybrid method:

$$Q^s(x) = \begin{pmatrix} x_1 + x_2 + x_3 - c_1^0 \\ x_1^2 + x_2^2 + x_3^2 - c_2^0 \\ x_1x_4 + x_2x_5 + x_3x_6 - c_3^0 \\ x_1x_7 + x_2x_8 + x_3x_9 - c_4^0 \\ x_1x_4^2 + x_2x_5^2 + x_3x_6^2 - c_5^0 \\ x_1x_4x_7 + x_2x_5x_8 + x_3x_6x_9 - c_6^0 \\ (x_4 + x_7\mathbf{i} + s)(x_4\mathbf{i} + x_7 + s) \\ (x_5 + x_8\mathbf{i} + s)(x_5\mathbf{i} + x_8 + s) \\ (x_6 + x_9\mathbf{i} + s)(x_6\mathbf{i} + x_9 + s) \end{pmatrix} = 0.$$

$Q^s(x) = 0$ can be decomposed into 8 subsystems. For $\sigma = (1, 2, 2)$, eliminating x_7, x_8, x_9 and by some simple symbolic computation, e.g., polynomial addition and variable substitution,

$Q_\sigma(x) = 0$ is equivalently transformed to

$$\widehat{Q}_\sigma^s(\widehat{x}) = \begin{pmatrix} x_1 + x_2 + x_3 - c_1^0 \\ x_1^2 + x_2^2 + x_3^2 - c_2^0 \\ 2x_1x_4 + (s - s\mathbf{i})x_1 + sc_1^0\mathbf{i} + c_4^0\mathbf{i} - c_3^0 \\ 2x_2x_5 + 2x_3x_6 + (s - s\mathbf{i})(x_2 + x_3) - sc_1^0 - c_4^0\mathbf{i} - c_3^0 \\ 2x_1x_4^2 + (s - s\mathbf{i})x_1x_4 + c_3^0 s\mathbf{i} + c_6^0\mathbf{i} - c_5^0 \\ 2x_2x_5^2 + 2x_3x_6^2 + (s - s\mathbf{i})(x_2x_5 + x_3x_6) - sc_3^0 - c_6^0\mathbf{i} - c_5^0 \end{pmatrix} = 0.$$

whose mixed volume is 8.

For $\sigma = (1, 1, 1)$, eliminating x_7, x_8, x_9, and by some simple symbolic computation, (**??**) is equivalently transformed to

$$\widehat{Q}_\sigma^s(\widehat{x}) = \begin{pmatrix} x_1 + x_2 + x_3 - c_1^0 \\ x_1^2 + x_2^2 + x_3^2 - c_2^0 \\ x_1x_4 + x_2x_5 + x_3x_6 - c_3^0 \\ c_3^0\mathbf{i} + s\mathbf{i}c_1^0 - c_4^0 \\ x_1x_4^2 + x_2x_5^2 + x_3x_6^2 - c_5^0 \\ c_5^0\mathbf{i} + s\mathbf{i}c_3^0 - c_6^0 \end{pmatrix} = 0.$$

For the generic choice of $c_1^0, c_3^0, c_4^0, c_5^0, c_6^0, s$,

$$c_3^0\mathbf{i} + s\mathbf{i}c_1^0 - c_4^0 \neq 0, \;\; c_5^0\mathbf{i} + s\mathbf{i}c_3^0 - c_6^0 \neq 0.$$

Thus the subsystem is “inconsistent system” and need not be solved.

4 Numerical experiments and applications

In this section, symmetric hybrid method and hybrid method are respectively applied to solve several deficient polynomial systems coming from MTPSs which appeared in the literature [3, 22] and the practical problem in (7) for $n = 3, 4, 5, 6, 7$.

The elapsed time for solving start system were obtained by HOM4PS2 [5] running on a machine with Windows XP Professional operation system, Intel(R) Core(TM)2 2.0GHz processor and 1.99GB of memory.

Throughout the remainder of this section, it will be convenient to use the following notations.

1. “BN”, “MBN” and “MV” respectively denote the Bézout number, multi-homogeneous Bézout number and mixed volume of the target system.

2. “SN” denotes the total solution number of the target systems, and “RSN” dnotes the representative solution number of the target system.

3. "Hybrid-PH" is the elapsed time by hybrid method. "T_1" is the elapsed time for solving the start system by polyhedral homotopy method, and "T_2" is the elapsed time for tracing homotopy paths from $t = 0$ to $t = 1$, and "T", the sum of "T_1" and "T_2", is the total time for solving the target system, and "Hybrid-SPH" is the elapsed time by symmetric hybrid method.

4. "N_T" is the total number of subsystems, and "N_C" is the number of consistent subsystems.

Example 4.1 (d1 system in [3, 22]). For the target polynomial system, we construct the start system $Q(x) = 0$ as (11). $Q(x) = 0$ can be decomposed into 2^6 6-dimensional polynomial subsystems $Q_\sigma(x) = 0$. For $\sigma_2 = \sigma_3 = \sigma_4 = 1$,

$$Q_\sigma(x) = \begin{pmatrix} 3x_2 + 2x_3 + x_4 - c_1^0 \\ 3x_1x_8 + 2x_1x_9 + x_1x_{10} - c_2^0 \\ 3x_7x_8 + 2x_7x_9 + x_7x_{10} - c_3^0 \\ x_2x_5 + x_3x_5 + x_4x_5 - c_4^0 \\ x_7x_8x_5 + x_7x_9x_5 + x_7x_{10}x_5 + x_1x_{11} - c_5^0 \\ -x_2x_{11}x_6 - x_3x_{11}x_6 - x_4x_{11}x_6 + x_8x_{12} + x_9x_{12} + x_{10}x_{12} - c_6^0 \\ x_1\mathbf{i}^{\sigma_1+3} + x_7\mathbf{i}^{3\sigma_1+2} + s \\ x_8 + x_2\mathbf{i} + s \\ x_9 + x_3\mathbf{i} + s \\ x_{10} + x_4\mathbf{i} + s \\ x_5\mathbf{i}^{\sigma_5+3} + x_{11}\mathbf{i}^{3\sigma_5+2} + s \\ x_6\mathbf{i}^{\sigma_6+3} + x_{12}\mathbf{i}^{3\sigma_6+2} + s \end{pmatrix}. \tag{17}$$

where $c_1^0, c_2^0, c_3^0, c_4^0, c_5^0, c_6^0, s$ are random nonzero complex numbers and $\sigma_i \in \{1, 2\}$.

The eighth to tenth equations are equivalent to

$$x_{k+6} = -x_k\mathbf{i} - s, k = 2, 3, 4, \tag{18}$$

and by variable substitution (18), the second equation and the third equation in $Q_\sigma(x) = 0$ are transformed to

$$\begin{cases} -x_1(3x_2 + 2x_3 + x_4)\mathbf{i} - 6sx_1 = c_2^0, \\ -x_7(3x_2 + 2x_3 + x_4)\mathbf{i} - 6sx_7 = c_3^0. \end{cases}$$

Because $3x_2 + 2x_3 + x_4 = c_1^0$, x_1, x_7 should satisfy

$$\begin{cases} -c_1^0x_1\mathbf{i} - 6sx_1 - c_2^0 = 0, \\ -c_1^0x_7\mathbf{i} - 6sx_7 - c_3^0 = 0, \\ x_1\mathbf{i}^{\sigma_1+3} + x_7\mathbf{i}^{3\sigma_1+2} + s = 0, \end{cases} \tag{19}$$

and the system (19) is inconsistent due to the randomization of c_1^0, c_2^0, c_3^0. Through the simple analysis, we can find $Q_\sigma(x) = 0$ has no solutions for all $\sigma \in \{(\sigma_1, \ldots, \sigma_6) | \sigma_2 = \sigma_3 = \sigma_4\}$. Thus the actual number of the subsystems need to be solved is $2^6 - 2 \cdot 2^3 = 48$, and hence the number of subsystems have to be solved is $\frac{3}{4}$ times of the number of the total subsystems decomposed from the start system.

Results are given in TABLE 4.1a and TABLE 4.1b.

TABLE 4.1a: Sizes of Example 4.1.

n	m	BN	MBN	MV	SN	RSN	N_T	N_C
0	6	4608	320	192	48	48	64	48

TABLE 4.1b: Elapsed time for Example 4.1.

Hybrid-PH			Hybrid-SPH		
T1	T2	T	T1	T2	T
2.125	1.536	3.661	1.501	1.536	3.037

Example 4.2 (Polynomial system in (7) for $n = 3$). The solving procedure can see Example 3.1, and results are listed in TABLE 4.2a and TABLE 4.2b.

TABLE 4.2a: Sizes of Example 4.2.

n	m	BN	MBN	MV	SN	RSN	N_T	N_C
3	3	576	192	96	24	4	8	2

TABLE 4.2b: Elapsed time for Example 4.2.

Hybrid-PH			Hybrid-SPH		
T1	T2	T	T1	T2	T
0.750	0.375	1.125	0.031	0.063	0.094

Example 4.3 (Polynomial system in (7) for $n = 4$).

TABLE 4.3a: Sizes of Example 4.3.

n	m	BN	MBN	MV	SN	RSN	N_T	N_C
4	4	10368	2464	992	192	8	16	1

TABLE 4.3b: Elapsed time for Example 4.3.

Hybrid-PH			Hybrid-SPH		
T1	T2	T	T1	T2	T
9.750	3.774	13.524	0.266	0.472	0.738

Example 4.4 (Polynomial system in (7) for $n = 5$).

TABLE 4.4a: Sizes of Example 4.4.

n	m	BN	MBN	MV	SN	RSN	N_T	N_C
5	5	331776	73728	14592	960	8	32	2

Table 4.4b: Elapsed time for Example 4.4.

Hybrid-PH			Hybrid-SPH		
T1	T2	T	T1	T2	T
253.042	118.080	371.122	6.156	0.984	7.140

Example 4.5 (Polynomial system in (7) for $n = 6$).

Table 4.5a: Sizes of Example 4.5.

n	m	BN	MBN	MV	SN	RSN	N_T	N_C
6	6	16588800	3736576	301568	5760	8	64	1

Table 4.5b: Elapsed time for Example 4.5.

Hybrid-PH			Hybrid-SPH		
T1	T2	T	T1	T2	T
1.018e+004	3.059e+003	1.324e+004	116.734	4.248	120.982

Example 4.6 (Polynomial system in (7) for $n = 7$).

Table 4.6a: Sizes of Example 4.6.

n	m	BN	MBN	MV	SN	RSN	N_T	N_C
7	7	1194393600	270811136	6705152	40320	8	128	2

Table 4.6b: Elapsed time for Example 4.6.

Hybrid-PH			Hybrid-SPH		
T1	T2	T	T1	T2	T
>50 hours	/	>50 hours	3036.375	19.456	3.056e+003

Remark 4.1. In the procedure of solving polynomial systems, some symbolic computation, e.g., decomposition, variable substitution and reduction should be proceeded, as stated in Step 2, Step 3, and Step 4 in Algorithm 3.1. The elapsed time for these operations are much less than that of solving the start system and the target system, so we omit these time and only record the elapsed time for solving start system and target system.
Remark 4.2. In [29], we can see that the hybrid method is more efficient than multi-homogeneous homotopy method and polyhedral homotopy method, so in this paper we only compare the elapsed time by symmetric hybrid method and hybrid method.
Remark 4.3. In Example 4.2 to Example 4.6, as showed in Example 3.1, on the one hand, the subsystems can be considerably reduced in mixed volume, on the other hand, some subsystems are inconsistent systems and need not be solved, and moreover, due to the symmetry, only one or two consistent systems need to be solved, the computational work are reduced a lot.

5 Conclusion

Mixed trigonometric polynomial systems rather frequently occur in many applications, such as kinematics, mechanical system design and signal processing. A mixed trigonometric polynomial system is a polynomial system whose monomial is a mixture of some variables and sine or cosine functions applied to other variables. Through a simple variable substitution and adding m quadratic equations of the form $x^2 + y^2 - 1 = 0$, a mixed trigonometric polynomial system can be naturally transformed to a polynomial system. The appended m quadratic equations are inherently symmetric.

In this paper, to keep the symmetry of the m appended equations and further the (partial) symmetry of the target system, a symmetric hybrid homotopy and hybrid polynomial system solving method are presented. The main work of our new algorithm is focused on solving the start system, due to the special product structure of the last m polynomials, this work is equivalent to find all isolated solutions of some subsystems of lower dimension. On the other hand, by simple symbolic computation techniques, some subsystems can be reduced in Bézout number or mixed volume, and furthermore, due to the symmetry, only partial subsystems need to be solved, and in the process of path following, only the representative solution paths should be followed. Numerical results show our new algorithm performs really well and is more efficient than the hybrid method, and also the random product homotopy method or the polyhedral homotopy method.

6 Acknowledgment

This research was supported in part by the National Nature Science Foundation of China (10671029), the Research Fund for the Doctoral Programme of Higher Education (20060141029), DLUT Grants 3004-842321 and 3004-893327.

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