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A Predator–Prey system with viral infection and anorexia response

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Abstract

The generalized Gause model of Predator–Prey system is proposed with an introduction of viral infection on prey population and anorexia response on predator population. By using the comparison theorem and constructing suitable Lyapunov function, we study such modified Predator–Prey system with almost periodic coefficients. Some sufficient conditions are obtained for the existence of a unique almost periodic solution. Numerical simulations of Predator–Prey system with anorexia response and the one without anorexia response are performed. Our observations suggest that anorexia response on predator population has a destabilizing effect on the persistence of such eco-epidemiological Predator–Prey system.

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1. Introduction

After the seminal models of Vito Volterra and Alfred James Lotka in the mid 1920s for Predator–Prey interactions, mutualist and competitive mechanisms have been studied extensively in the recent years by researchers. The Predator–Prey system becomes an important field of study in mathematical ecology and has been studied in [1]. Similarly, epidemiological models have also been received much attention from scientists. Relevant references are vast, we can see the book [2]. Both mathematical ecology and mathematical epidemiology are major fields of study in their own right. The study including ecology and epidemiology is now termed as eco-epidemiology. The study of eco-epidemiology has great ecological significance. Scientists have paid lots of attention to theoretical studies. Many good results can predict useful implications in both dynamics and control, we can refer to the book [3] and the references cited therein.

The importance of transmissible disease in Predator–Prey system arouses the interest of scientists. Theoretical studies have been carried out in eco-epidemiology where the effect of viral infection has been explored. Lots of good results have already been obtained (see [4–7] and the references therein). In [4], the authors studied the following eco-epidemiological system

$$\begin{cases} \dot{x}_1(t) = r(x_1+x_2)(1-\frac{x_1+x_2}{K}) - bx_1x_2 - \eta\delta_1(x_1)y, \\ \dot{x}_2(t) = bx_1x_2 - \gamma(x_2)y - cx_2, \\ \dot{y}(t) = y(\varepsilon\gamma(x_2) + \varepsilon\eta\delta_1(x_1) - d). \end{cases}$$

In the above eco-epidemiological system, the authors assume that the sound prey population grows according to a logistic law involving the whole prey population (sound and infected). The disease is spread among the prey population only and that disease is not genetically inherited. The infected population does not recover or becomes immune. The predator population predates mostly the infective prey and the functional response is of Holling-type II. Persistence and extinction conditions of the system are obtained.

In Predator–Prey theory and related topics in mathematical ecology, an important and ubiquitous problem concerns the long term coexistence of species. The Predator–Prey system has been studied extensively in [8–13] and the references therein. In their literature, the following Predator–Prey system with *linear growth* was considered,

$$\begin{cases} \dot{x}_1(t) = x_1(t)[b_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - c_{11}(t)y(t)], \\ \dot{x}_2(t) = x_2(t)[b_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - c_{21}(t)y(t)], \\ \dot{y}(t) = y(t)[-r_1(t) + d_{11}(t)x_1(t) + d_{12}(t)x_2(t) - e_{11}(t)y(t)]. \end{cases} \tag{1.1}$$

Some sufficient conditions were obtained for the uniform persistence and existence of a unique globally attractive periodic (almost periodic) solution for the Predator–Prey system (1.1).

According to the culture figure that obtained by Ayala on the studying of D. Psendoobscura and D. Serrata, the growth rate of species doesn't correspond with that of the Lotka–Volterra model (see Chen [1]). The reason is due to the linearize in mathematics and the linearize makes many important factors neglected, such as the effect of toxic (see [14,15]), the age-structure of a population (see [16]) and anorexia response (see [17]). And now lots of literature consider feedback controls and fuzzy control, we can refer to [18–24]. If these important factors are to be considered, then more complex models should be introduced. In 1975, Dubois introduced the following autonomous anorexia system,

$$\begin{cases} \dot{x} = k_1 x[1 - \alpha x] - k(x)y, \\ \dot{y} = y[-k_3 + \beta k(x)], \end{cases} \tag{1.2}$$

where

$$k(x) = \begin{cases} k_2 x, & x \leqslant \tau, \\ k_2 \tau, & x > \tau. \end{cases}$$

Later, Yang [17] considered the following anorexia system,

$$\begin{cases} \dot{x} = x\left[(a - bx) - \dfrac{\alpha y}{1 + \omega x + \beta x^2}\right] - G, \\ \dot{y} = y\left[-c + \dfrac{k\alpha x}{1 + \omega x + \beta x^2}\right], \end{cases}$$

where anorexia response function $\phi(x) = \frac{\alpha}{1+\omega x+\beta x^2}$. It then naturally leads one to incorporate the anorexia response into Predator–Prey system.

In Predator–Prey system, when a certain virus comes, infected prey may died, but it also can survive and give birth to next generation (see [4]). As predator, it eats all the prey, including sound prey(susceptible) and infected prey(infective). The infection weakens the prey, it also does harm to the predator and makes the predator feel uncomfortable with the infected prey. It means that the more infected prey, the less it wants to eat. When the density of infected prey amounts to a certain degree, the quantity of the preyed decreases. It is so-called anorexia response. We can take anorexia response function (see [25]) as

$$h(x) = \begin{cases} c(t)x(t), & x(t) \leqslant \tau, \\ \dfrac{c(t)\tau^2}{x(t)}, & x(t) > \tau. \end{cases}$$

For more background and biological adjustment, one can refer to [17,25] and the references cited therein.

However, most of works concern about Predator–Prey system with constant coefficients or periodic coefficients with a common period. If the various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be periodic or almost periodic since there is no a priori reason to expect the existence of constant circumstance. If we want to make the models suitable to the environment and reality, the assumption of periodicity or almost periodicity of parameters is realistic and important (e.g. seasonal effects of weather, food supplies, mating habits, harvesting etc.). In this paper, we consider a Predator–Prey system with almost periodic coefficients.

Stimulated by system (1.1) and (1.2) and works [4–7,17,25] and the references cited therein, We introduce the following modified Predator–Prey model.

1.1. The modified Predator–Prey model

Let us consider a generalized Gause Model (see [26]) for Predator–Prey interactions, e.g.,

$$\frac{dx}{dt} = xg(x) - yp(x),$$
$$\frac{dy}{dt} = y[-\gamma + q(x)],$$

where $g(x)$ is the specific growth rate of the prey in the absence of any predator and $p(x)$ is the predator response function for the predator with respect to that particular prey. But the role of disease in such systems cannot be ignored and we like to build the Gause type Predator–Prey model with viral infection on prey population only. We shall now modify the Gause Predator–Prey model by introducing infection on prey population (see [7]) and replacing $p(x)$ with anorexia response function $h(x)$.

The following basic assumptions are made:

(1) In the presence of viral infection, the prey population is divided into two classes, namely, susceptible prey, denoted by $x_1(t)$ and the infected prey, denoted by $x_2(t)$. The prey population grows according to a logistic law involving the whole prey population(susceptible and infected). But the disease is genetically inherited.
(2) The transmission rate among the sound prey population and the infected prey population follows the simple law of mass action, that is $b(t)x_1(t)x_2(t)$, where $b(t)$ is the transmission rate. The infected prey does not recover or becomes immune, the death rate of infected prey is $d_2(t)$.

And the virus weakens the prey, but it does not cause the predator to have any disease. Denote the predator by $y(t)$.

(3) The predator eats the whole prey, including sound prey and the infected prey. The predator population predates the sound prey according to Holling-type II (see [27]), predates the infected prey with Anorexia response (see [25]).

Based on the above assumptions, the modified Gause model can be written as:

$$\begin{cases} \dot{x}_1(t) = x_1(t)(a_{10}(t) - a_{11}(t)x_1(t)) - b(t)x_1(t)x_2(t) \\ \qquad\qquad -\dfrac{c_1(t)x_1(t)}{d_1(t)+x_1(t)}y(t) \triangleq F_1(t,X), \\ \dot{x}_2(t) = x_2(t)(a_{20}(t) - a_{21}(t)x_2(t)) + b(t)x_1(t)x_2(t) \\ \qquad\qquad -h(x_2)y(t) - d_2(t)x_2(t) \triangleq F_2(t,X), \\ \dot{y}(t) = y(t)\left(-a_{30}(t) - a_{31}(t)y(t) + k_1(t)\dfrac{c_1(t)x_1(t)}{d_1(t)+x_1(t)} + k_2(t)h(x_2)\right) \\ \qquad \triangleq G(t,X), \end{cases} \tag{E}$$

where

$$h(x_2) = \begin{cases} c_2(t)x_2(t), & x_2(t) \leqslant \tau, \\ \dfrac{c_2(t)\tau^2}{x_2(t)}, & x_2(t) > \tau. \end{cases}$$

$x_1(t)$ is the density of sound prey species, $x_2(t)$ is the density of infected prey species, $y(t)$ is the density of predator species, predator anorexia response function is $h(x_2)$ and index τ denotes the anorexia degree of the predator $y(t)$ to the infected prey $x_2(t)$, $X(t) = (x_1(t), x_2(t), y(t))$. If we consider the effects of the instinct factors, the assumption of anorexia response is more realistic, more important and more general. To the best of the author's knowledge, this is the first paper considering the almost periodic solutions of the Predator–Prey system with infection and anorexia response. By using the comparison theorem and constructing suitable Lyapunov function, some sufficient conditions are obtained for the existence of a unique almost periodic solution of system (E).

This paper is organized as follows. In next section, by using comparison theorem, we shall obtain that there exists a bounded solution of system (E) on R. In Section 3, by constructing a suitable Lyapunov function, some sufficient conditions are obtained for the existence of a unique almost periodic solution of system (E). In Section 4, to illustrate the generality of our results, we shall further our discussion for the value of anorexia index τ. Finally, a suitable example is given to perform numerical simulations of Predator–Prey system.

Our observations lead one to value the effect of anorexia response and viral infection on persistence of eco-epidemiological Predator–Prey system.

Throughout this paper, we shall use the following notations, unless otherwise stated:

- We always use $i = 1, 2, 3$, $j = 1, 2$.
- If $f(t)$ is an almost periodic function defined on $(-\infty, +\infty)$, we set
$$f^l = \inf_{t\in(-\infty,+\infty)} f(t), \quad f^\mu = \sup_{t\in(-\infty,+\infty)} f(t).$$
- Moreover, we set
$$p_1 = \frac{a_{10}^\mu}{a_{11}^l}, \quad p_2 = \frac{a_{20}^\mu + b^\mu p_1}{a_{21}^l}, \quad q = \frac{k_2^\mu c_2^\mu p_2 + k_1^\mu c_1^\mu p_1 (d_1^l)^{-1} - a_{30}^l}{a_{31}^l},$$
$$\alpha_1 = \frac{a_{10}^l - b^\mu p_2 - c_1^\mu q (d_1^l)^{-1}}{a_{11}^\mu}, \quad \alpha_2 = \frac{a_{20}^l - d_2^\mu + b^l \alpha_1 - c_2^\mu q}{a_{21}^\mu},$$
$$\beta = (a_{31}^\mu)^{-1}\left(\frac{k_1^l c_1^l \alpha_1}{d_1^\mu + p_1} + \frac{k_2^l c_2^l \alpha_2^2}{p_2} - a_{30}^\mu\right).$$
- Denote mean value $m(f(t)) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t$. When $f(t)$ is a ω-periodic function, then $m(f(t)) = \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t$.

Throughout this paper, we suppose that the following conditions are satisfied:

- (H_1): $b(t)$, $d_j(t)$, $k_j(t)$, $a_{i0}(t)$, $a_{i1}(t)$, $c_j(t)$ are all nonnegative almost periodic functions defined on $t \in (-\infty, +\infty)$.
- (H_2): We always assume $a_{i1}^l > 0$, $d_1^l > 0$ and $m(a_{10}(t)) > 0$.

2. Existence of bounded solutions

First, we consider that anorexia index satisfies $\alpha_2 < \tau < p_2$. In the following, we will state some lemmas which will be used in the proving of Theorem 2.1.

For any given initial condition of system (E)
$$x_{j0} = x_j(0) > 0, \quad y_0 = y(0) > 0,$$
it is not difficult to see that the corresponding solution $X(t) = (x_1(t), x_2(t), y(t))$ exists for all $t \geqslant 0$ and satisfies
$$x_j(t) > 0, \quad y(t) > 0, \quad \text{for } t \geqslant 0.$$
Now we consider the almost periodic Logistic equation
$$\dot{x}(t) = x(t)[b(t) - a(t)x(t)], \tag{2.1}$$

where $b(t)$ and $a(t)$ are continuous almost periodic functions with $a^l > 0$ and $m(b(t)) > 0$.

We introduce Lemma 2.1 which improves Theorem 1 in [28].

Lemma 2.1. *If $a^l > 0$ and $m(b(t)) > 0$, then system (2.1) has a unique globally attractive positive almost periodic solution $\tilde{x}(t)$ with $\tilde{x}(t) \leqslant \frac{b^u}{a^l}$. Moreover, let $\tilde{x}_i(t), (i = 1, 2)$ be the unique positive almost periodic solution of (2.1) with $b(t) = b_i(t)$, $a(t) = a_i(t)$, $(i = 1, 2)$, respectively. If $b_2(t) > b_1(t)$ and $a_2(t) \leqslant a_1(t)$, then $\tilde{x}_2(t) > \tilde{x}_1(t)$.*

Lemma 2.2 (Fink [29], Theorem 6.2). *Let $\dot{x} = f(t, x)$ have a solution φ which is bounded on $[t_0, \infty)$. If $f(t, x)$ is almost periodic in t uniformly for $x \in K = \overline{\{\varphi(t) : t \geqslant t_0\}}$, then there is a solution of the equation on all of R with values in K.*

We shall make some preparations before stating our Theorem 2.1.

If (H_1) and (H_2) hold, from Lemma 2.1, we know

$$\dot{x}_1(t) = x_1(t)[a_{10}(t) - a_{11}(t)x_1(t)]$$

has a unique positive globally attractive almost periodic solution $x_1^*(t)$ with $0 < x_1^*(t) \leqslant p_1$.

If (H_1), (H_2) and

(H_3): $m(a_{20}(t) + b(t)x_1^*(t)) > 0$,

from Lemma 2.1, we know

$$\dot{x}_2(t) = x_2(t)\left[a_{20}(t) - a_{21}(t)x_2(t) + b(t)x_1^*(t)\right]$$

has a unique positive globally attractive almost periodic solution $x_2^*(t)$ with $0 < x_2^*(t) \leqslant p_2$.

If (H_1), (H_2) and

(H_4): $m\left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_1^*(t)}{d_1(t)} + k_2(t)c_2(t)x_2^*(t)\right) > 0$,

from Lemma 2.1, we know

$$\dot{y}(t) = y(t)\left[-a_{30}(t) - a_{31}(t)y(t) + k_1(t)\frac{c_1(t)x_1^*(t)}{d_1(t)} + k_2(t)c_2(t)x_2^*(t)\right]$$

has a unique positive globally attractive almost periodic solution $y^*(t)$ with $0 < y^*(t) \leqslant q$.

If (H_1), (H_2) and

(H_5): $m\left(a_{10}(t) - b(t)x_2^*(t) - \frac{c_1(t)}{d_1(t)}y^*(t)\right) > 0$, from Lemma 2.1, we know

$$\dot{x}_1(t) = x_1(t)\left[a_{10}(t) - a_{11}(t)x_1(t) - b(t)x_2^*(t) - \frac{c_1(t)}{d_1(t)}y^*(t)\right]$$

has a unique positive globally attractive almost periodic solution $x_{*1}(t)$ with $x_{*1}(t) \geqslant \inf_{t\in R} x_{*1}(t) \geqslant \alpha_1 > 0$.

If (H_1), (H_2) and
(H_6): $m(a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t)) > 0$,

from Lemma 2.1, we know

$$\dot{x}_2(t) = x_2(t)[a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t) - a_{21}(t)x_2(t)]$$

has a unique positive globally attractive almost periodic solution $x_{*2}(t)$ with $x_{*2}(t) \geqslant \inf_{t\in R} x_{*2}(t) \geqslant \alpha_2 > 0$.

If (H_1), (H_2) and
(H_7): $m\left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t)+x_1^*(t)} + k_2(t)\frac{c_2(t)\alpha_2^2}{p_2}\right) > 0$,

from Lemma 2.1, we know

$$\dot{y}(t) = y(t)\left[-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t)+x_1^*(t)} + k_2(t)\frac{c_2(t)\alpha_2^2}{p_2} - a_{31}(t)y(t)\right]$$

has a unique positive globally attractive almost periodic solution $y_*(t)$ with $y_*(t) \geqslant \inf_{t\in R} y_*(t) \geqslant \beta > 0$.

Let

$$m_j = \inf_{t\in R} x_{*j}(t), \quad \xi = \inf_{t\in R} y_*(t), \quad M_j = \sup_{t\in R} x_j^*(t), \quad \eta = \sup_{t\in R} y^*(t).$$

Clearly, $m_j > 0$, $\xi > 0$, $M_j > 0$, $\eta > 0$.

Denote

$$S = \left\{(x_1(t), x_2(t), y(t)) | m_j \leqslant x_j(t) \leqslant M_j, \quad \xi \leqslant y(t) \leqslant \eta\right\}.$$

Theorem 2.1. *If* (H_1)–(H_7) *hold, then system* (E) *has at least one positive* (*componentwise*) *solution defined on* R *with value in* S.

Proof. It is obviously that $h(x_2) \leqslant c_2(t)x_2(t)$. If $x_2(t) \leqslant \tau$, one obtains

$$h(x_2) = c_2(t)x_2(t) \geqslant c_2(t)\alpha_2 \geqslant c_2(t)\frac{\alpha_2^2}{p_2}.$$

If $x_2(t) > \tau$, one obtains

$$h(x_2) = \frac{c_2(t)\tau^2}{x_2(t)} \geqslant \frac{c_2(t)\tau^2}{x_2^*(t)} \geqslant c_2(t)\frac{\alpha_2^2}{p_2}.$$

From the above simple comparison, one obtains

$$c_2(t)\frac{\alpha_2^2}{p_2} \leqslant h(x_2) \leqslant c_2(t)x_2(t). \tag{2.2}$$

From the first equation of system (E), one obtains

$$\dot{x}_1(t) \leqslant x_1(t)[a_{10}(t) - a_{11}(t)x_1(t)]. \tag{2.3}$$

From (H_2), using comparison theorem on (2.3), we have

$$0 < x_1(t) \leqslant x_1^*(t), \quad \text{for } t \geqslant 0. \tag{2.4}$$

where $x_1(t)$ is a solution of system (E) which satisfies $0 < x_1(0) \leqslant x_1^*(0)$.

From the second equation of system (E) and (2.4), one obtains

$$\dot{x}_2(t) \leqslant x_2(t)\left[a_{20}(t) - a_{21}(t)x_2(t) + b(t)x_1^*(t)\right]. \tag{2.5}$$

From (H_3), using comparison theorem on (2.5), we have

$$0 < x_2(t) \leqslant x_2^*(t), \quad \text{for } t \geqslant 0. \tag{2.6}$$

where $x_2(t)$ is a solution of system (E) which satisfies $0 < x_2(0) \leqslant x_2^*(0)$.

From the third equation of system (E), (2.2), (2.4) and (2.6), one obtains

$$\dot{y}(t) \leqslant y(t)\left[-a_{30}(t) - a_{31}(t)y(t) + k_1(t)\frac{c_1(t)x_1^*(t)}{d_1(t)} + k_2(t)c_2(t)x_2^*(t)\right]. \tag{2.7}$$

From (H_4), using comparison theorem on (2.7), we have

$$0 < y(t) \leqslant y^*(t), \quad \text{for } t \geqslant 0, \tag{2.8}$$

where $y(t)$ is a solution of system (E) which satisfies $0 < y(0) \leqslant y^*(0)$.

The first equation of system (E), (2.6) and (2.8) imply that

$$\dot{x}_1(t) \geqslant x_1(t)\left[a_{10}(t) - a_{11}(t)x_1(t) - b(t)x_2^*(t) - \frac{c_1(t)}{d_1(t)}y^*(t)\right]. \tag{2.9}$$

From (H_5), using comparison theorem on (2.9), we have

$$x_1(t) \geqslant x_{*1}(t), \quad \text{for } t \geqslant 0, \tag{2.10}$$

where $x_1(t)$ is a solution of system (E) which satisfies $x_1(0) \geqslant x_{*1}(0) > 0$.

The second equation of system (E), (2.2), (2.8) and (2.10) imply that

$$\dot{x}_2(t) \geqslant x_2(t)[a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t) - a_{21}(t)x_2(t)]. \tag{2.11}$$

From (H_6), using comparison theorem on (2.11), we have

$$x_2(t) \geqslant x_{*2}(t), \quad \text{for } t \geqslant 0, \tag{2.12}$$

where $x_2(t)$ is a solution of system (E) which satisfies $x_2(0) \geqslant x_{*2}(0) > 0$.

The third equation of system (E), (2.2), (2.4) and (2.10) imply that

$$\dot{y}(t) \geqslant y(t)\left[-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t) + x_1^*(t)} + k_2(t)\frac{c_2(t)\alpha_2^2}{p_2} - a_{31}(t)y(t)\right]. \tag{2.13}$$

From (H_7), using comparison theorem on (2.13), we have

$$y(t) \geqslant y_*(t), \quad \text{for } t \geqslant 0, \tag{2.14}$$

where $y(t)$ is a solution of system (E) which satisfies $y(0) \geqslant y_*(0) > 0$.

Therefore system (E) has a bounded solution $X(t) = (x_1(t), x_2(t), y(t)) \subset S$ for $t \geqslant 0$. Since $F_j(t, X)$, $G(t, X)$ (defined in (E)) are almost periodic in t uniformly for $X(t) = (x_1(t), x_2(t), y(t)) \subset S$. Hence, by Lemma 2.2, system (E) has at least one bounded solution $Y(t) = (u_1(t), u_2(t), v(t)) \subset S$ for all $t \in R$. This completes the proof of Theorem 2.1. □

3. Existence of a unique almost periodic solution

Now we state a definition and a lemma which will be used in the proving of our main results.

Definition 3.1. A bounded positive solution $Y(t) = (u_1(t), u_2(t), v(t))$ of system (E) with $Y(0) > 0$ is said to be globally attractive, if for any other solution $X(t) = (x_1(t), x_2(t), y(t))$ of system (E) with $X(0) > 0$, we have

$$\lim_{t\to+\infty} |x_j(t) - u_j(t)| = 0, \quad \lim_{t\to+\infty} |y(t) - v(t)| = 0.$$

A consequence of such a global attractivity of a bounded positive (componentwise) solution of system (E) on R is that there cannot be another positive (componentwise) bounded solution of system (E) on R.

Lemma 3.1 (Fink [29], Theorem 10.1). *Consider system $\dot{x} = f(t, x)$, suppose $f(t, x)$ is almost periodic in t uniformly for x in K, K compact in E^n. If each equation $\dot{x} = g(t, x)$, $g \in H(f)$ (where $H(f)$ is the hull of f) has a unique solution on R with values in K, then these solutions are almost periodic with module contained in mod(f).*

Theorem 3.1. *If system* (E) *satisfies*

(H_1)–(H_7) *and*

(H_8): *There exist positive constants s_j, ω and ε_j, δ such that*

$$\begin{cases} s_1 a_{11}(t) \geqslant \dfrac{s_1 c_1(t) y^*(t)}{(d_1(t) + x_{*1}(t)) d_1(t)} + s_2 b(t) + \dfrac{\omega k_1(t) c_1(t)}{d_1(t) + x_{*1}(t)} + \varepsilon_1, \\ s_2 a_{21}(t) \geqslant s_1 b(t) + 2\omega k_2(t) c_2(t) + 2\dfrac{s_2 c_2(t) y^*(t)}{x_{*2}(t)} + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1 c_1(t)}{d_1(t)} + s_2 c_2(t) + \delta. \end{cases}$$

Then system (E) has a unique positive bounded solution $X(t) = (x_1(t), x_2(t), y(t)) \subset S$ on R, which is globally attractive.

Proof. From Theorem 2.1, system (E) has at least one solution $X(t) = (x_1(t), x_2(t), y(t)) \subset S$ for all $t \in R$. Let $Y(t) = (u_1(t), u_2(t), v(t))$ be any other solution of system (E) with $Y(0) > 0$.

Consider the following Lyapunov function

$$V(t) = \sum_{k=1}^{2} s_k |\ln x_k(t) - \ln u_k(t)| + \omega |\ln y(t) - \ln v(t)|, \quad t \in R.$$

Denote

$$\lambda = \min_j \{s_j, \omega\}, \quad \Lambda = \max_j \{s_j, \omega\}.$$

Calculating the upper right derivative $D^+V(t)$ of $V(t)$ along the solution of (E), one obtains

$$\begin{aligned} D^+V(t) &= \sum_{k=1}^{2} s_k D^+ |\ln x_k(t) - \ln u_k(t)| + \omega D^+ |\ln y(t) - \ln v(t)| \\ &= s_1 \operatorname{sgn}\{x_1(t) - u_1(t)\} \Big[-a_{11}(t)(x_1(t) - u_1(t)) - b(t)(x_2(t) - u_2(t)) \\ &\quad - \left(\frac{c_1(t) y(t)}{d_1(t) + x_1(t)} - \frac{c_1(t) v(t)}{d_1(t) + u_1(t)} \right) \Big] + s_2 \operatorname{sgn}\{x_2(t) - u_2(t)\} \\ &\quad \times \Big[-a_{21}(t)(x_2(t) - u_2(t)) + b(t)(x_1(t) - u_1(t)) \\ &\quad - \frac{h(x_2) y(t)}{x_2(t)} + \frac{h(u_2) v(t)}{u_2(t)} \Big] + \omega \operatorname{sgn}\{y(t) - v(t)\} \\ &\quad \times \Big[-a_{31}(t)(y(t) - v(t)) + k_2(t)(h(x_2) - h(u_2)) \\ &\quad + \frac{k_1(t) c_1(t) x_1(t)}{d_1(t) + x_1(t)} - \frac{k_1(t) c_1(t) u_1(t)}{d_1(t) + u_1(t)} \Big]. \end{aligned}$$

From simple calculation,we have

$$v(t)\frac{h(u_2)}{u_2(t)} - y(t)\frac{h(x_2)}{x_2(t)} = \frac{h(u_2)}{u_2(t)}(v(t) - y(t)) + y(t)\left(\frac{h(u_2)}{u_2(t)} - \frac{h(x_2)}{x_2(t)}\right),$$

$$\frac{c_1(t)y(t)}{d_1(t) + x_1(t)} - \frac{c_1(t)v(t)}{d_1(t) + u_1(t)} = c_1(t)\frac{d_1(t)(y(t) - v(t)) + x_1(t)(y(t) - v(t)) + y(t)(u_1(t) - x_1(t))}{(d_1(t) + x_1(t))(d_1(t) + u_1(t))},$$

$$\frac{k_1(t)c_1(t)x_1(t)}{d_1(t) + x_1(t)} - \frac{k_1(t)c_1(t)u_1(t)}{d_1(t) + u_1(t)} = k_1(t)c_1(t)d_1(t)\frac{x_1(t) - u_1(t)}{(d_1(t) + u_1(t))(d_1(t) + x_1(t))}.$$

Denote

$$\Delta_1 = h(x_2) - h(u_2), \quad \Delta_2 = \frac{h(u_2)}{u_2(t)} - \frac{h(x_2)}{x_2(t)}.$$

Then there are four cases for us to compare $x_2(t)$ and $u_2(t)$ with anorexia response index τ.

(1) If $x_2(t) \leqslant \tau$ and $u_2(t) \leqslant \tau$, one obtains

$$|\Delta_1| = c_2(t)|x_2(t) - u_2(t)|, \quad |\Delta_2| = 0.$$

(2) If $x_2(t) \leqslant \tau$ and $u_2(t) > \tau$, one obtains

$$\begin{aligned}
|\Delta_1| &= \left|c_2(t)x_2(t) - \frac{c_2(t)\tau^2}{u_2(t)}\right| \\
&\leqslant \frac{c_2(t)}{u_2(t)}|\tau(u_2(t) - \tau) + u_2(t)(x_2 - \tau)| \\
&\leqslant c_2(t)\left|(x_2(t) - \tau) + \frac{\tau(u_2(t) - \tau)}{u_2(t)}\right| \\
&\leqslant 2c_2(t)|x_2(t) - u_2(t)|, \\
|\Delta_2| &= \left|\frac{c_2(t)\tau^2}{u_2^2(t)} - c_2(t)\right| \\
&\leqslant c_2(t)\left|\frac{(\tau + u_2(t))(\tau - u_2(t))}{u_2^2(t)}\right| \\
&\leqslant 2\frac{c_2(t)}{u_2(t)}|x_2(t) - u_2(t)| \\
&\leqslant 2\frac{c_2(t)}{x_2(t)}|x_2(t) - u_2(t)|.
\end{aligned}$$

(3) If $x_2(t) > \tau$ and $u_2(t) \leqslant \tau$, one obtains

$$\begin{aligned}
|\Delta_1| &= \left|\frac{c_2(t)\tau^2}{x_2(t)} - c_2(t)u_2(t)\right| \\
&= c_2(t)\left|\frac{\tau(\tau - u_2(t)) + u_2(t)(\tau - x_2(t))}{x_2(t)}\right| \\
&\leqslant c_2(t)\left|\frac{\tau(\tau - u_2(t))}{x_2(t)} + (\tau - x_2(t))\right| \\
&\leqslant 2c_2(t)|x_2(t) - u_2(t)|, \\
|\Delta_2| &= \left|c_2(t)\frac{\tau^2}{x_2^2(t)} - c_2(t)\right| \\
&= c_2(t)\left|\frac{\tau(\tau - x_2(t))}{x_2^2(t)} + \frac{\tau - x_2(t)}{x_2(t)}\right| \\
&\leqslant c_2(t)\left(\frac{\tau}{x_2^2(t)} + \frac{1}{x_2(t)}\right)|x_2(t) - u_2(t)| \\
&\leqslant 2\frac{c_2(t)}{x_2(t)}|x_2(t) - u_2(t)|.
\end{aligned}$$

(4) If $x_2(t) > \tau$ and $u_2(t) > \tau$, one obtains

$$\begin{aligned}
|\Delta_1| &= c_2(t)\left|\frac{\tau^2}{x_2(t)} - \frac{\tau^2}{u_2(t)}\right| \\
&= \frac{c_2(t)\tau^2}{u_2(t)x_2(t)}|x_2(t) - u_2(t)| \\
&\leqslant c_2(t)|x_2(t) - u_2(t)|, \\
|\Delta_2| &= c_2(t)\tau^2\left|\frac{1}{u_2^2(t)} - \frac{1}{x_2^2(t)}\right| \\
&= c_2(t)\tau\left|\left(\frac{\tau}{u_2(t)} + \frac{\tau}{x_2(t)}\right)\left(\frac{1}{u_2(t)} - \frac{1}{x_2(t)}\right)\right| \\
&\leqslant c_2(t)\frac{2\tau}{x_2(t)u_2(t)}|x_2(t) - u_2(t)| \\
&\leqslant 2\frac{c_2(t)}{x_2(t)}|x_2(t) - u_2(t)|.
\end{aligned}$$

From (1)–(4), one obtains

$$\begin{aligned}
|\Delta_1| &= |h(x_2) - h(u_2)| \leqslant 2c_2(t)|x_2(t) - u_2(t)|, \\
|\Delta_2| &= \left|\frac{h(u_2)}{u_2(t)} - \frac{h(x_2)}{x_2(t)}\right| \leqslant 2\frac{c_2(t)}{x_2(t)}|x_2(t) - u_2(t)|.
\end{aligned}$$

By simplifying, we have

$$
\begin{aligned}
D^+V(t) &\leqslant -s_1a_{11}(t)|x_1(t)-u_1(t)| + s_1b(t)|x_2(t)-u_2(t)| + s_1c_1(t) \\
&\quad \times \frac{(d_1(t)+x_1(t))|y(t)-v(t)| + y(t)|x_1(t)-u_1(t)|}{(d_1(t)+x_1(t))(d_1(t)+u_1(t))} - s_2a_{21}(t) \\
&\quad \times |x_2(t)-u_2(t)| + s_2b(t)|x_1(t)-u_1(t)| + s_2c_2(t)|y(t)-v(t)| \\
&\quad + s_2y(t)\frac{2c_2(t)}{x_{*2}(t)}|x_2(t)-u_2(t)| - \omega a_{31}(t)|y(t)-v(t)| \\
&\quad + \frac{\omega k_1(t)c_1(t)d_1(t)|x_1(t)-u_1(t)|}{(d_1(t)+x_1(t))(d_1(t)+u_1(t))} + 2\omega k_2(t)c_2(t)|x_2(t)-u_2(t)| \\
&\leqslant \left[-s_1a_{11}(t) + \frac{s_1c_1(t)y^*(t)}{d_1(t)(d_1(t)+x_{*1}(t))} + s_2b(t) + \frac{\omega k_1(t)c_1(t)}{d_1(t)+x_{*1}(t)}\right] \\
&\quad \times |x_1(t)-u_1(t)| + \left[s_1b(t) - s_2a_{21}(t) + 2\omega k_2(t)c_2(t) + 2\frac{s_2c_2(t)y^*(t)}{x_{*2}(t)}\right] \\
&\quad \times |x_2(t)-u_2(t)| + \left[\frac{s_1c_1(t)}{d_1(t)} + s_2c_2(t) - \omega a_{31}(t)\right]|y(t)-v(t)| \\
&\leqslant -\varepsilon_1|x_1(t)-u_1(t)| - \varepsilon_2|x_2(t)-u_2(t)| - \delta|y(t)-v(t)| \\
&\leqslant -\gamma\left(\sum_{i=1}^{2}|x_i(t)-u_i(t)| + |y(t)-v(t)|\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_1 &= \inf_{t\in R}\left\{s_1a_{11}(t) - \frac{s_1c_1(t)y^*(t)}{d_1(t)(d_1(t)+x_{*1}(t))} - s_2b(t) - \frac{\omega k_1(t)c_1(t)}{d_1(t)+x_{*1}(t)}\right\}, \\
\varepsilon_2 &= \inf_{t\in R}\left\{s_2a_{21}(t) - s_1b(t) - 2\omega k_2(t)c_2(t) - 2\frac{s_2c_2(t)y^*(t)}{x_{*2}(t)}\right\}, \\
\delta &= \inf_{t\in R}\left\{\omega a_{31}(t) - \frac{s_1c_1(t)}{d_1(t)} - s_2c_2(t)\right\}, \quad \gamma = \min\{\varepsilon_1, \varepsilon_2, \delta\}.
\end{aligned}
$$

Then $V(t)$ is decreasing on $[0,\infty)$, thus $0 \leqslant V(t) \leqslant V(0)$, and $\lim_{t\to+\infty}V(t) = V^* \geqslant 0$.

Now we prove that $V^* = 0$ and $X(t)$ is globally attractive. Since $X(t) \subset S$, then $\ln x_j(t)$, $\ln y(t)$ are bounded. As

$$
\begin{aligned}
|\ln u_j(t)| &\leqslant |\ln u_j(t) - \ln x_j(t)| + |\ln x_j(t)| \leqslant \frac{1}{s_j}V(t) + |\ln x_j(t)|, \\
|\ln v(t)| &\leqslant |\ln v(t) - \ln y(t)| + |\ln y(t)| \leqslant \frac{1}{\omega}V(t) + |\ln y(t)|.
\end{aligned}
$$

So, $u_j(t)$, $v(t)$ are bounded. Hence, $Y(t) = (u_1(t), u_2(t), v(t))$ is bounded. From the mean-value theorem, there are positive constants ρ_1, ρ_2 such that

$$\sum_{k=1}^{2}\frac{\lambda}{\rho_1}|x_k(t)-u_k(t)|+\frac{\lambda}{\rho_2}|y(t)-v(t)|$$
$$\leqslant V(t)\leqslant \sum_{k=1}^{2}\frac{\Lambda}{\rho_1}|x_k(t)-u_k(t)|+\frac{\Lambda}{\rho_2}|y(t)-v(t)|.$$

Take $\rho_3=\max\{\frac{\rho_1}{\lambda},\frac{\rho_2}{\lambda}\},\rho_4=\min\{\frac{\rho_1}{\Lambda},\frac{\rho_2}{\Lambda}\}$, we have

$$\frac{1}{\rho_3}\left[\sum_{k=1}^{2}|x_k(t)-u_k(t)|+|y(t)-v(t)|\right]$$
$$\leqslant V(t)\leqslant \frac{1}{\rho_4}\left[\sum_{k=1}^{2}|x_k(t)-u_k(t)|+|y(t)-v(t)|\right], \tag{3.1}$$

thus $D^+V(t)\leqslant -\gamma\rho_4V(t)$. We claim $V^*=0$. Otherwise $V^*>0$, and we have $V(t)\geqslant V^*>0$, it follows $D^+V(t)\leqslant -\gamma\rho_4V^*$, which implies

$$V(t)\leqslant V(0)-\gamma\rho_4V^*t\to-\infty,\quad (t\to\infty).$$

This contradicts with the positivity of $V(t)$, so $V^*=0$. From (3.1), we have

$$0\leqslant \lim_{t\to+\infty}\left[\sum_{k=1}^{2}|x_k(t)-u_k(t)|+|y(t)-v(t)|\right]\leqslant \lim_{t\to+\infty}\rho_3V(t)=0,$$

which implies

$$\lim_{t\to+\infty}|x_j(t)-u_j(t)|=0,\quad \lim_{t\to+\infty}|y(t)-v(t)|=0.$$

Thus $X(t)=(x_1(t),x_2(t),y(t))$ is the unique positive bounded solution of system (E) contained in S for all $t\in R$, which is globally attractive. This completes the proof of Theorem 3.1. □

Now we consider the system

$$\begin{cases}\dot{x}_1(t)=x_1(t)\big(a_{10}^*(t)-a_{11}^*(t)x_1(t)\big)-b^*(t)x_1(t)x_2(t)-\dfrac{c_1^*(t)x_1(t)}{d_1^*(t)+x_1(t)}y,\\ \dot{x}_2(t)=x_2(t)\big(a_{20}^*(t)-a_{21}^*(t)x_2(t)\big)+b^*(t)x_1(t)x_2(t)-h^*(x_2)y-d_2^*(t)x_2(t),\\ \dot{y}(t)=y(t)\left(-a_{30}^*(t)-a_{31}^*(t)y(t)+k_1^*(t)\dfrac{c_1^*(t)x_1(t)}{d_1^*(t)+x_1(t)}+k_2^*(t)h^*(x_2)\right),\end{cases} \tag{E*}$$

where

$$h^*(x_2)=\begin{cases}c_2^*(t)x_2(t), & x_2(t)\leqslant\tau,\\ \dfrac{c_2^*(t)\tau^2}{x_2(t)}, & x_2(t)>\tau.\end{cases}$$

For some sequence $\{t_v\}$ with $t_v \to \infty$ as $v \to \infty$, one has

$$\begin{aligned}
&d_j(t+t_v) \to d_j^*(t), \quad a_{i0}(t+t_v) \to a_{i0}^*(t), \quad a_{i1}(t+t_v) \to a_{i1}^*(t),\\
&k_j(t+t_v) \to k_j^*(t), \quad c_j(t+t_v) \to c_j^*(t), \quad x_j^*(t+t_v) \to \varphi_j^*(t),\\
&y^*(t+t_v) \to \psi^*(t), \quad x_{*j}(t+t_v) \to \varphi_{*j}(t), \quad b^*(t+t_v) \to b^*(t),\\
&\quad \text{uniformly for all} \quad t \in R, \quad \text{as } v \to \infty,
\end{aligned}$$

where $x_j^*(t),\ x_{*j}(t),\ y^*(t)$ are defined in Section 2. It is not difficult to obtain that

$$\lim_{v\to\infty} \{a_{20}(t+t_v) + b(t+t_v)x_1^*(t+t_v)\} = a_{20}^*(t) + b^*(t)\varphi_1^*(t), \tag{3.2}$$

$$\begin{aligned}
&\lim_{v\to\infty} \left\{ -a_{30}(t+t_v) + k_1(t+t_v)\frac{c_1(t+t_v)x_1^*(t+t_v)}{d_1(t+t_v)} + k_2(t+t_v)c_2(t+t_v)x_2^*(t+t_v) \right\}\\
&\quad = -a_{30}^*(t) + k_1^*(t)\frac{c_1^*(t)\varphi_1^*(t)}{d_1^*(t)} + k_2^*(t)c_2^*(t)\varphi_2^*(t),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\lim_{v\to\infty} \left\{ a_{10}(t+t_v) - b(t+t_v)x_2^*(t+t_v) - \frac{c_1(t+t_v)}{d_1(t+t_v)}y^*(t+t_v) \right\}\\
&\quad = a_{10}^*(t) - b^*(t)\varphi_2^*(t) - \frac{c_1^*(t)}{d_1^*(t)}\psi^*(t),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\lim_{v\to\infty} \{a_{20}(t+t_v) - d_2(t+t_v) - c_2(t+t_v)y^*(t+t_v) + b(t+t_v)x_{*1}(t+t_v)\}\\
&\quad = a_{20}^*(t) - d_2^*(t) - c_2^*(t)\psi^*(t) + b^*(t)\varphi_{*1}(t),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\lim_{v\to\infty} \left\{ -a_{30}(t+t_v) + k_1(t+t_v)\frac{c_1(t+t_v)x_{*1}(t+t_v)}{d_1(t+t_v)+x_1^*(t+t_v)} + k_2(t+t_v)\frac{c_2(t+t_v)\alpha_2}{p_2} \right\}\\
&\quad = -a_{30}^*(t) + k_1^*(t)\frac{c_1^*(t)\varphi_{*1}(t)}{d_1^*(t)+\varphi_1^*(t)} + k_2^*(t)\frac{c_2^*(t)\alpha_2}{p_2}.
\end{aligned} \tag{3.6}$$

Note that $k_j^*(t),\ a_{i0}^*(t),\ a_{i1}^*(t),\ \varphi_j^*(t),\ \varphi_{*j}(t),\ d_j^*(t),\ \psi^*(t),\ c_j^*(t),\ b^*(t)$ are also almost periodic in t.

Lemma 3.2. *Suppose* (H_1)–(H_8) *hold, then system* (E^*) *has a unique bounded solution* $\varphi(t) = (\varphi_1(t), \varphi_2(t), \psi(t)) \in S$ *on* R, *which is globally attractive.*

Proof. By the definition of mean value, the assumptions (H_1)–(H_7) and (3.2)–(3.6), it follows that $m(a_{10}(t)) = m(a_{10}^*(t))$,

$$\begin{aligned}
&m(a_{20}(t) + b(t)x_1^*(t)) = m(a_{20}^*(t) + b^*(t)\varphi_1^*(t)),\\
&m\left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_1^*(t)}{d_1(t)} + k_2(t)c_2(t)x_2^*(t) \right)\\
&\quad = m\left(-a_{30}^*(t) + k_1^*(t)\frac{c_1^*(t)\varphi_1^*(t)}{d_1^*(t)} + k_2^*(t)c_2^*(t)\varphi_2^*(t) \right),
\end{aligned}$$

$$
\begin{aligned}
& m\left(a_{10}(t)-b(t)x_2^*(t)-y^*(t)\frac{c_1(t)}{d_1(t)}\right)\\
&\quad = m\left(a_{10}^*(t)-b^*(t)\varphi_2^*(t)-\psi^*(t)\frac{c_1^*(t)}{d_1^*(t)}\right),\\
& m(a_{20}(t)-d_2(t)-c_2(t)y^*(t)+b(t)x_{*1}(t))\\
&\quad = m(a_{20}^*(t)-d_2^*(t)-c_2^*(t)\psi^*(t)+b^*(t)\varphi_{*1}(t)),\\
& m\left(-a_{30}(t)+k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t)+x_1^*(t)}+k_2(t)\frac{c_2(t)\alpha_2^2}{p_2}\right)\\
&\quad = m\left(-a_{30}^*(t)+k_1^*(t)\frac{c_1^*(t)\varphi_{*1}(t)}{d_1^*(t)+\varphi_1^*(t)}+k_2^*(t)\frac{c_2^*(t)\alpha_2^2}{p_2}\right).
\end{aligned}
$$

And (H_8) leads to

(H_8^*) There exist positive constants s_j, ω and ε_j, δ such that

$$
\begin{cases}
s_1a_{11}^*(t) \geqslant \dfrac{s_1c_1^*(t)\psi^*(t)}{d_1^*(t)\left(d_1^*(t)+\varphi_{*1}(t)\right)}+s_2b^*(t)+\dfrac{\omega k_1^*(t)c_1^*(t)}{d_1^*(t)+\varphi_{*1}(t)}+\varepsilon_1,\\[2ex]
s_2a_{21}^*(t) \geqslant s_1b^*(t)+2\omega k_2^*(t)c_2^*(t)+2\dfrac{s_2c_2^*(t)\psi^*(t)}{\varphi_{*2}(t)}+\varepsilon_2,\\[2ex]
\omega a_{31}^*(t) \geqslant \dfrac{s_1c_1^*(t)}{d_1^*(t)}+s_2c_2^*(t)+\delta.
\end{cases}
$$

From Theorem 3.1, $\varphi(t) = (\varphi_1(t), \varphi_2(t), \psi(t)) \subset S$ is the unique bounded solution of system (E*) on R, which is globally attractive. This completes the proof Lemma 3.2. $\square$

By Lemma 3.2, it follows that for each $h(t, X) \in H(f(t, X))$, the hull equation

$$\dot{x} = h(t,X)$$

has a unique bounded solution on R with value in S. Hence, from Lemma 3.1, these unique solutions are all almost periodic. Therefore, by the global attractivity, $X(t)$ is the unique almost periodic solution of system (E) contained in S. Thus our main results follows:

Theorem 3.2. *Suppose (H_1)–(H_8) hold, then system* (E) *has a unique positive (componentwise) almost periodic solution $X(t) = (x_1(t), x_2(t), y(t)) \subset S$ on R, which is globally attractive.*

Corollary 3.1. *In addition to (H_1), if system* (E) *satisfies*

(H_9): $a_{10}^l > 0$, $d_1^l > 0$, $\beta > 0$ *and* $\alpha_j > 0$.

(H_{10}): *There exist positive constants s_j, ω and ε_j, δ such that*

$$\begin{cases} s_1a_{11}(t) \geqslant \dfrac{s_1c_1(t)q}{d_1(t)(d_1(t)+\alpha_1)} + s_2b(t) + \dfrac{\omega k_1(t)c_1(t)}{d_1(t)+\alpha_1} + \varepsilon_1, \\ s_2a_{21}(t) \geqslant s_1b(t) + 2\omega k_2(t)c_2(t) + 2\frac{s_2c_2(t)q}{\alpha_2} + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1c_1(t)}{d_1(t)} + s_2c_2(t) + \delta. \end{cases}$$

Then system (E) has a unique almost periodic solution $Y(t) = (u_1(t), u_2(t), v(t)) \subset S$ *on* R, *which is globally attractive.*

Proof. From (H_{10}), we have $a^l_{i1} > 0$. From the discussion in Section 2, we have $\alpha_j \leqslant x^*_j(t) \leqslant p_j$ and $\beta \leqslant y^*(t) \leqslant q$, which imply

$$m\left(a_{10}(t) - b(t)x^*_2(t) - \frac{c_1(t)y^*(t)}{d_1(t)}\right) \geqslant a^l_{10} - b^\mu p_2 - \frac{c^\mu_1 q}{d^l_1} = a^\mu_{11}\alpha_1 > 0,$$

$$m(a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t)) \geqslant a^l_{20} - d^\mu_2 - c^\mu_2 q + b^l\alpha_1 = a^\mu_{21}\alpha_2 > 0,$$

$$m\left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t)} + k_2(t)c_2(t)x^*_2(t)\right) \geqslant -a^\mu_{30} + k^l_1\frac{c^l_1\alpha_1}{d^\mu_1} + k^l_2c^l_2\alpha_2$$
$$\geqslant -a^\mu_{30} + k^l_1\frac{c^l_1\alpha_1}{d^\mu_1+p_1} + k^l_2\frac{c^l_2\alpha_2}{p_2} = a^\mu_{31}\beta > 0,$$

$$m\left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t)+x^*_1(t)} + k_2(t)\frac{c_2(t)\alpha^2_2}{p_2}\right)$$
$$\geqslant -a^\mu_{30} + k^l_1\frac{c^l_1\alpha_1}{d^\mu_1+p_1} + k^l_2\frac{c^l_2\alpha^2_2}{p_2} = a^\mu_{31}\beta > 0.$$

Since $m(a_{10}(t)) \geqslant a^l_{10} > 0$ and $m\big(a_{20}(t) + b(t)x^*_1(t)\big) \geqslant a^l_{20} + b^l\alpha_1 > 0$, it is obviously that (H_9) and (H_{10}) lead to (H_2)–(H_8), by Theorem 3.2, Corollary 3.1 holds. □

Now consider system (E) with periodic coefficients, i.e, $b(t)$, $k_j(t)$, $d_j(t)$, $a_{i0}(t)$, $c_j(t)$, $a_{i1}(t)$ are nonnegative ω-periodic functions defined on R with $a^l_{i1} > 0, d^l_1 > 0$.

Theorem 3.3. *If system (E) satisfies*

(H_{11}): $\int_0^\omega a_{10}(t)\,\mathrm{d}t > 0, \quad \int_0^\omega (a_{20}(t) + b(t)x^*_1(t))\,\mathrm{d}t > 0,$

(H_{12}): $\int_0^\omega \left(-a_{30}(t) + k_1(t)\frac{c_1(t)x^*_1(t)}{d_1(t)} + k_2(t)c_2(t)x^*_2(t)\right)\mathrm{d}t > 0,$

(H_{13}): $\int_0^\omega \left(a_{10}(t) - b(t)x^*_2(t) - \frac{c_1(t)}{d_1(t)}y^*(t)\right)\mathrm{d}t > 0,$

(H_{14}): $\int_0^\omega (a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t))\,\mathrm{d}t > 0,$

(H_{15}): $\int_0^\omega \left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t)+x_1^*(t)} + k_2(t)\frac{c_2(t)\alpha_2^2}{p_2}\right)\mathrm{d}t > 0$,

(H_{16}): *There exist positive constants* s_j, ω *and* ε_j, δ *such that*

$$\begin{cases} s_1a_{11}(t) \geqslant \dfrac{s_1c_1(t)y^*(t)}{d_1(t)(d_1(t)+x_{*1}(t))} + s_2b(t) + \dfrac{\omega k_1(t)c_1(t)}{d_1(t)+x_{*1}(t)} + \varepsilon_1, \\ s_2a_{21}(t) \geqslant s_1b(t) + 2\omega k_2(t)c_2(t) + 2\dfrac{s_2c_2(t)y^*(t)}{x_{*2}(t)} + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1c_1(t)}{d_1(t)} + s_2c_2(t) + \delta. \end{cases}$$

Then system (E) *has a unique* ω*-periodic solution in* S, *which is globally attractive.*

Proof. From the proof of Theorem 3.2, since $b(t)$, $k_j(t)$, $d_j(t)$, $a_{i0}(t)$, $c_j(t)$,$a_{i1}(t)$ are all ω-periodic, we can take

$$\varepsilon_1 = \inf_{t\in[0,\omega]}\left\{s_1a_{11}(t) - \frac{s_1c_1(t)y^*(t)}{d_1(t)(d_1(t)+x_{*1}(t))} - s_2b(t) - \frac{\omega k_1(t)c_1(t)}{d_1(t)+x_{*1}(t)}\right\},$$
$$\varepsilon_2 = \inf_{t\in[0,\omega]}\left\{s_2a_{21}(t) - s_1b(t) - 2\omega k_2(t)c_2(t) - 2\frac{s_2c_2(t)y^*(t)}{x_{*2}(t)}\right\},$$
$$\delta = \inf_{t\in[0,\omega]}\left\{\omega a_{31}(t) - \frac{s_1c_1(t)}{d_1(t)} - s_2c_2(t)\right\}.$$

Let $X(t)$ be the unique positive almost periodic solution of system (E), but in the periodic case, $X(t+\omega)$ is also an almost periodic solution of system (E). By the uniqueness of almost periodic solution, it follows that $X(t) = X(t+\omega)$ for all $t \in R$. This completes the proof of Theorem 3.3. □

Take similar proof as Corollary 3.1, from Theorem 3.3, one obtains

Corollary 3.2. *In addition to* (H_{11})–(H_{15}), *if system* (E) *satisfies*
(H_{17}): *There exist positive constants* s_j, ω *such that*

$$\begin{cases} s_1a_{11}(t) \geqslant \dfrac{s_1c_1(t)q}{d_1(t)(d_1(t)+\alpha_1)} + s_2b(t) + \dfrac{\omega k_1(t)c_1(t)}{d_1(t)+\alpha_1}, \\ s_2a_{21}(t) \geqslant s_1b(t) + 2\omega k_2(t)c_2(t) + 2\dfrac{s_2c_2(t)q}{\alpha_2}, \\ \omega a_{31}(t) \geqslant \dfrac{s_1c_1(t)}{d_1(t)} + s_2c_2(t). \end{cases}$$

Then system (E) *has a unique* ω*-periodic solution in* S, *which is globally attractive.*

4. Further discussions

The modified Predator–Prey system is constructed by an introduction of viral infection on the prey population and anorexia response on the predator population. Stability behavior of such modified system is carried out to observe the dynamics of the system. Under the force of viral infection and instinct anorexia response, the existence of almost periodic solutions supports the coexistence of the complex system. But there are three cases for us to further discussion.

Discussion 4.1. In Sections 2 and 3, we only consider anorexia index satisfies $\alpha_2 < \tau < p_2$. If anorexia index $\tau > p_2$, From the proof of Theorems 2.1, 3.1 and 3.2, it is easy for us to have the same results.

Theorem 4.1. *If* (H_1)–(H_7) *hold, then system* (E) *has at least one positive* (*componentwise*) *solution defined on R with value in S.*

Theorem 4.2. *If system* (E) *satisfies* (H_1)–(H_7) *and*

(H_8): *There exist positive constants* s_j, ω *and* ε_j, δ *such that*

$$\begin{cases} s_1 a_{11}(t) \geqslant \dfrac{s_1 c_1(t) y^*(t)}{d_1(t)(d_1(t) + x_{*1}(t))} + s_2 b(t) + \dfrac{\omega k_1(t) c_1(t)}{d_1(t) + x_{*1}(t)} + \varepsilon_1, \\ s_2 a_{21}(t) \geqslant s_1 b(t) + 2\omega k_2(t) c_2(t) + 2\dfrac{s_2 c_2(t) y^*(t)}{x_{*2}(t)} + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1 c_1(t)}{d_1(t)} + s_2 c_2(t) + \delta. \end{cases}$$

Then system (E) *has a unique almost periodic solution* $X(t) = (x_1(t), x_2(t), y(t)) \subset S$ *on R, which is globally attractive.*

Note 1. To prove the Theorem 4.2, similarity to Theorem 3.1, If $\tau > p_2$, we only consider $x_2(t) \leqslant \tau$ and $u_2(t) \leqslant \tau$, or $x_2(t) \leqslant \tau$ and $u_2(t) > \tau$. Thus the proof is the same to that of Theorem 3.1.

Discussion 4.2. If anorexia index $\tau < \alpha_2$, we have $\frac{c_2(t)\tau^2}{p_2} \leqslant h(x_2) \leqslant c_2(t) x_2(t)$. We take

$$(H_7):\ m\left(-a_{30}(t) + k_1(t)\frac{c_1(t) x_{*1}(t)}{d_1(t) + x_1^*(t)} + k_2(t)\frac{c_2(t)\tau^2}{p_2}\right) > 0,$$

$$\beta = (a_{31}^{\mu})^{-1}\left(\frac{k_1^l c_1^l \alpha_1}{d_1^{\mu} + p_1} + \frac{k_2^l c_2^l \tau^2}{p_2} - a_{30}^{\mu}\right).$$

From the proof of Theorems 2.1, 3.1 and 3.2, it is easy for us to have the same results.

Theorem 4.3. *If* (H_1)–(H_7) *hold, then system* (E) *has at least one positive* (*componentwise*) *solution defined on* R *with value in* S.

Theorem 4.4. *If system* (E) *satisfies* (H_1)–(H_7) *and*

(H_8): *There exist positive constants* s_j, ω *and* ε_j, δ *such that*

$$\begin{cases} s_1a_{11}(t) \geqslant \dfrac{s_1c_1(t)y^*(t)}{d_1(t)(d_1(t)+x_{*1}(t))} + s_2b(t) + \dfrac{\omega k_1(t)c_1(t)}{d_1(t)+x_{*1}(t)} + \varepsilon_1, \\ s_2a_{21}(t) \geqslant s_1b(t) + 2\omega k_2(t)c_2(t) + 2\dfrac{s_2c_2(t)y^*(t)}{x_{*2}(t)} + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1c_1(t)}{d_1(t)} + s_2c_2(t) + \delta. \end{cases}$$

Then system (E) *has a unique almost periodic solution* $X(t) = (x_1(t), x_2(t), y(t)) \subset S$ *on* R, *which is globally attractive.*

Note 2. To prove the Theorem 4.4, similarity to Theorem 3.1, if $\tau < \alpha_2$, we only consider $x_2(t) > \tau$ and $u_2(t) \leqslant \tau$ or $x_2(t) > \tau$ and $u_2(t) > \tau$. Thus the proof is the same to that of Theorem 3.1.

Discussion 4.3. Now we consider anorexia index $\tau = \infty$, that is, the predator has no anorexia response, $h(x_2) = c_2(t)x_2(t)$, system (E) is reduce to

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)(a_{10}(t) - a_{11}(t)x_1(t)) - b(t)x_1(t)x_2(t) - \frac{c_1(t)x_1(t)}{d_1(t)+x_1(t)}y(t), \\ \dot{x}_2(t) &= x_2(t)(a_{20}(t) - a_{21}(t)x_2(t)) + b(t)x_1(t)x_2(t) - c_2(t)x_2(t)y(t) - d_2(t)x_2(t), \\ \dot{y}(t) &= y(t)\left(-a_{30}(t) - a_{31}(t)y(t) + k_1(t)\frac{c_1(t)x_1(t)}{d_1(t)+x_1(t)} + k_2(t)c_2(t)x_2(t)\right). \end{aligned} \tag{4.1}$$

We shall make some preparations before stating our Theorem 4.3.

In the following, we suppose that (H_1)–(H_2) hold, Unless otherwise stated, and we use the following notations:

$$p_1 = \frac{a_{10}^{\mu}}{a_{11}^{l}}, \quad p_2 = \frac{a_{20}^{\mu} + b^{\mu}p_1}{a_{21}^{l}}, \quad q = \frac{k_2^{\mu}c_2^{\mu}p_2 + k_1^{\mu}c_1^{\mu}p_1(d_1^{l})^{-1} - a_{30}^{l}}{a_{31}^{l}},$$

$$\alpha_1 = \frac{a_{10}^{l} - b^{\mu}p_2 - c_1^{\mu}q(d_1^{l})^{-1}}{a_{11}^{\mu}}, \quad \alpha_2 = \frac{a_{20}^{l} - d_2^{\mu} + b^{l}\alpha_1 - c_2^{\mu}q}{a_{21}^{\mu}},$$

$$\beta = (a_{31}^{\mu})^{-1}\left(\frac{k_1^{l}c_1^{l}\alpha_1}{d_1^{\mu} + p_1} + k_2^{l}c_2^{l}\alpha_2 - a_{30}^{\mu}\right).$$

From Lemma 2.1, we know

$$\dot{x}_1(t) = x_1(t)[a_{10}(t) - a_{11}(t)x_1(t)]$$

has a unique positive globally attractive almost periodic solution $x_1^*(t)$ with $0 < x_1^*(t) \leqslant p_1$.

If (A_1): $m\big(a_{20}(t) + b(t)x_1^*(t)\big) > 0$,

from Lemma 2.1, we know

$$\dot{x}_2(t) = x_2(t)\big[a_{20}(t) - a_{21}(t)x_2(t) + b(t)x_1^*(t)\big]$$

has a unique positive globally attractive almost periodic solution $x_2^*(t)$ with $0 < x_2^*(t) \leqslant p_2$.

If (A_2): $m\left(-a_{30}(t) + k_1(t)\dfrac{c_1(t)x_1^*(t)}{d_1(t)} + k_2(t)c_2(t)x_2^*(t)\right) > 0$,

from Lemma 2.1, we know

$$\dot{y}(t) = y(t)\left[-a_{30}(t) - a_{31}(t)y(t) + k_1(t)\frac{c_1(t)x_1^*(t)}{d_1(t)} + k_2(t)c_2(t)x_2^*(t)\right]$$

has a unique positive globally attractive almost periodic solution $y^*(t)$ with $0 < y^*(t) \leqslant q$.

If (A_3): $m\left(a_{10}(t) - b(t)x_2^*(t) - \dfrac{c_1(t)}{d_1(t)}y^*(t)\right) > 0$,

from Lemma 2.1, we know

$$\dot{x}_1(t) = x_1(t)\left[a_{10}(t) - a_{11}(t)x_1(t) - b(t)x_2^*(t) - \frac{c_1(t)}{d_1(t)}y^*(t)\right]$$

has a unique positive globally attractive almost periodic solution $x_{*1}(t)$ with $x_{*1}(t) \geqslant \inf_{t\in R} x_{*1}(t) \geqslant \alpha_1 > 0$.

If (A_4): $m(a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t)) > 0$,

from Lemma 2.1, we know

$$\dot{x}_2(t) = x_2(t)[a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t) - a_{21}(t)x_2(t)]$$

has a unique positive globally attractive almost periodic solution $x_{*2}(t)$ with $x_{*2}(t) \geqslant \inf_{t\in R} x_{*2}(t) \geqslant \alpha_2 > 0$.

If (A_5): $m\left(-a_{30}(t) + k_1(t)\dfrac{c_1(t)x_{*1}(t)}{d_1(t) + x_1^*(t)} + k_2(t)c_2(t)x_{*2}(t)\right) > 0$,

from Lemma 2.1, we know

$$\dot{y}(t) = y(t)\left[-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t) + x_1^*(t)} + k_2(t)c_2(t)x_{*2}(t) - a_{31}(t)y(t)\right]$$

has a unique positive globally attractive almost periodic solution $y_*(t)$ with $y_*(t) \geqslant \inf_{t\in R} y_*(t) \geqslant \beta > 0$.

Let

$$m_j^* = \inf_{t\in R} x_{*j}(t), \quad \xi^* = \inf_{t\in R} y_*(t), \quad M_j^* = \sup_{t\in R} x_j^*(t), \quad \eta^* = \sup_{t\in R} y^*(t).$$

Clearly, $m_j^* > 0,\ \xi^* > 0,\ M_j^* > 0,\ \eta^* > 0$. Denote

$$S^* = \left\{(x_1(t), x_2(t), y(t)) | m_j^* \leqslant x_j(t) \leqslant M_j^*, \xi^* \leqslant y(t) \leqslant \eta^*\right\}.$$

Similar proof as Theorem 2.1, by using comparison theorem and Lemma 2.2, we have the following results.

Theorem 4.5. *If* $(A_1)-(A_5)$ *hold, then system* (4.1) *has at least one positive* (*componentwise*) *solution defined on* R *with value in* S^*.

Construct the following Lyapunov function:

$$V(t) = \sum_{k=1}^{2} s_k |\ln x_k(t) - \ln u_k(t)| + \omega |\ln y(t) - \ln v(t)|, \quad t \in R.$$

Similar proof as Theorem 3.1, by calculating the upper right derivative $D^+V(t)$ of $V(t)$ along the solution of (4.1), one obtains

Theorem 4.6. *If system* (4.1) *satisfies* (A_1)–(A_5) *and* (A_6): *There exist positive constants* s_j, ω *and* ε_j, δ *such that*

$$\begin{cases} s_1 a_{11}(t) \geqslant \dfrac{s_1 c_1(t) y^*(t)}{d_1(t)(d_1(t) + x_{*1}(t))} + s_2 b(t) + \dfrac{\omega k_1(t) c_1(t)}{d_1(t) + x_{*1}(t)} + \varepsilon_1, \\ s_2 a_{21}(t) \geqslant s_1 b(t) + \omega k_2(t) c_2(t) + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1 c_1(t)}{d_1(t)} + s_2 c_2(t) + \delta. \end{cases}$$

Then system (4.1) *has a unique positive bounded solution* $Y(t) = (u_1(t), u_2(t), v(t)) \subset S^*$ *on* R, *which is globally attractive.*

By Lemma 3.2 and Theorem 4.6, we have

Theorem 4.7. *If system* (4.1) *satisfies* $(A_1)-(A_6)$, *then system* (4.1) *has a unique almost periodic solution* $Y(t) = (u_1(t), u_2(t), v(t)) \subset S^*$ *on* R, *which is globally attractive.*

Similarity to Corollary 3.1, we have the following theorem.

Theorem 4.8. *If system* (4.1) *satisfies* (A_7): $a_{10}^l > 0$, $d_1^l > 0$, $\beta > 0$ *and* $\alpha_j > 0$.

And there exist positive constants s_j, ω *and* ε_j, δ *such that*

$$\begin{cases} s_1a_{11}(t) \geqslant \dfrac{s_1c_1(t)q}{d_1(t)(d_1(t)+\alpha_1)} + s_2b(t) + \dfrac{\omega k_1(t)c_1(t)}{d_1(t)+\alpha_1} + \varepsilon_1, \\ s_2a_{21}(t) \geqslant s_1b(t) + \omega k_2(t)c_2(t) + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1c_1(t)}{d_1(t)} + s_2c_2(t) + \delta. \end{cases}$$

Then system (4.1) has a unique almost periodic solution $Y(t) = (u_1(t), u_2(t), v(t)) \subset S^*$ *on* R, *which is globally attractive.*

Now we consider system (4.1) with periodic coefficients, i.e., $b(t)$, $c_j(t)$, $a_{i0}(t)$, $d_j(t)$, $k_j(t)$, $a_{il}(t)$ are nonnegative ω-periodic functions defined on R with $a_{i1}^l > 0,\ d_1^l > 0$. Similar proof as Theorem 3.3, we have the following results.

Theorem 4.9. *If system (4.1) satisfies*

(i) $\int_0^\omega a_{10}(t)\,\mathrm{d}t > 0, \quad \int_0^\omega \left(a_{20}(t) + b(t)x_1^*(t)\right)\mathrm{d}t > 0.$

(ii) $\int_0^\omega \left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_1^*(t)}{d_1(t)} + k_2(t)c_2(t)x_2^*(t)\right)\mathrm{d}t > 0.$

(iii) $\int_0^\omega \left(a_{10}(t) - b(t)x_2^*(t) - \frac{c_1(t)}{d_1(t)}y^*(t)\right)\mathrm{d}t > 0.$

(iv) $\int_0^\omega (a_{20}(t) - d_2(t) - c_2(t)y^*(t) + b(t)x_{*1}(t))\,\mathrm{d}t > 0.$

(v) $\int_0^\omega \left(-a_{30}(t) + k_1(t)\frac{c_1(t)x_{*1}(t)}{d_1(t)+x_1^*(t)} + k_2(t)c_2(t)x_{*2}(t)\right)\mathrm{d}t > 0.$

(vi) *There exist positive constants* s_j, ω *and* ε_j, δ *such that*

$$\begin{cases} s_1a_{11}(t) \geqslant \dfrac{s_1c_1(t)q}{d_1(t)(d_1(t)+\alpha_1)} + s_2b(t) + \dfrac{\omega k_1(t)c_1(t)}{d_1(t)+\alpha_1} + \varepsilon_1, \\ s_2a_{21}(t) \geqslant s_1b(t) + \omega k_2(t)c_2(t) + \varepsilon_2, \\ \omega a_{31}(t) \geqslant \dfrac{s_1c_1(t)}{d_1(t)} + s_2c_2(t) + \delta. \end{cases}$$

Then system (4.1) has a unique ω*-periodic solution in* S^*, *which is globally attractive.*

5. Example and remark

From the results obtained in Sections 3 and 4, we can see that conditions of existence of almost periodic solutions in Corollary 3.1 are less easy to satisfy than that of Theorem 4.8. Under circumstance of viral infection and anorexia response, the persistence of Predator–Prey system is relatively difficult to obtain. The following example will reveal the truth.

Example. Consider the following Predator–Prey system with viral infection and anorexia response.

$$\begin{cases} \dot{x}_1(t) = x_1(t)[11+\cos 2t - 2x_1(t)] - x_1(t)x_2(t) - \dfrac{x_1(t)}{4+x_1(t)}y(t), \\ \dot{x}_2(t) = x_2(t)[11+\sin 5t - 2x_2(t)] + x_1(t)x_2(t) - h(x_2)y(t) - x_2(t), \\ \dot{y}(t) = y(t)\left[-\dfrac{3+\sin 3t}{8} + \dfrac{\frac{1}{3}x_1(t)}{4+x_1(t)} + \dfrac{1}{3}h(x_2) - 2y(t)\right], \end{cases} \tag{5.1}$$

where anorexia response function is

$$h(x_2) = \begin{cases} \dfrac{1}{3}x_2(t), & x_2(t) \leqslant \tau, \\ \dfrac{\frac{1}{3}\tau^2}{x_2(t)}, & x_2(t) > \tau. \end{cases}$$

Corresponding to system (E), we have $a_{10}(t) = 11+\cos 2t$, $a_{20}(t) = 11+\sin 5t$, $d_1(t) = 4$, $a_{11}(t) = a_{21}(t) = 2$, $a_{30}(t) = \frac{3+\sin 3t}{8}$, $a_{31}(t) = 2$, $k_1(t) = \frac{1}{3}$, $k_2(t) = c_2(t) = \frac{1}{3}$, $b(t) = c_1(t) = d_2(t) = 1$, $\tau = 0.1$. It is easy to verify that system (5.1) does not satisfy the results obtained in Section 3.

Without anorexia response, we consider the following reductive Predator–Prey system.

$$\begin{cases} \dot{x}_1(t) = x_1(t)[11+\cos 2t - 2x_1(t)] - x_1(t)x_2(t) - \dfrac{x_1(t)}{4+x_1(t)}y(t), \\ \dot{x}_2(t) = x_2(t)[11+\sin 5t - 2x_2(t)] + x_1(t)x_2(t) - \dfrac{1}{3}x_2(t)y(t) - x_2(t), \\ \dot{y}(t) = y(t)\left[-\dfrac{3+\sin 3t}{8} + \dfrac{\frac{1}{3}x_1(t)}{4+x_1(t)} + \dfrac{1}{9}x_2(t) - 2y(t)\right]. \end{cases} \tag{5.2}$$

In fact,

(I) $p_1 = 6$, $p_2 = 9$, $q = \frac{5}{8}$, $\alpha_1 = \frac{27}{64}$, $\alpha_2 = \frac{1769}{384}$.

(II) Take $s_1 = s_2 = 1$, $\omega = 1$, $\varepsilon_1 = \frac{2}{849}$, $\varepsilon_2 = \frac{2}{9}$, $\delta_1 = \frac{1}{12}$, we have

$$\begin{cases} s_1a_{11}(t) = 2 > s_1\dfrac{c_1(t)q}{d_1(t)(d_1(t)+\alpha_1)} + s_2b(t) + \omega\dfrac{k_1(t)c_1(t)}{d_1(t)+\alpha_1} + \varepsilon_1 = \dfrac{315}{683}, \\ s_2a_{21}(t) = 2 > s_1b(t) + \omega k_2(t)c_2(t) + \varepsilon_2 = \dfrac{4}{3}, \\ \omega a_{31}(t) = 2 > s_1\dfrac{c_1(t)}{d_1(t)} + s_2c_2(t) + \delta = \dfrac{2}{3}. \end{cases}$$

Thus by Theorem 4.8, there exists a unique positive almost periodic solution of system (5.2). In fact, if we take $x_1(0) = x_2(0) = 1$ and $y(0) = 3$, predator population of system (5.1) is extinct, predator population of system (5.2)

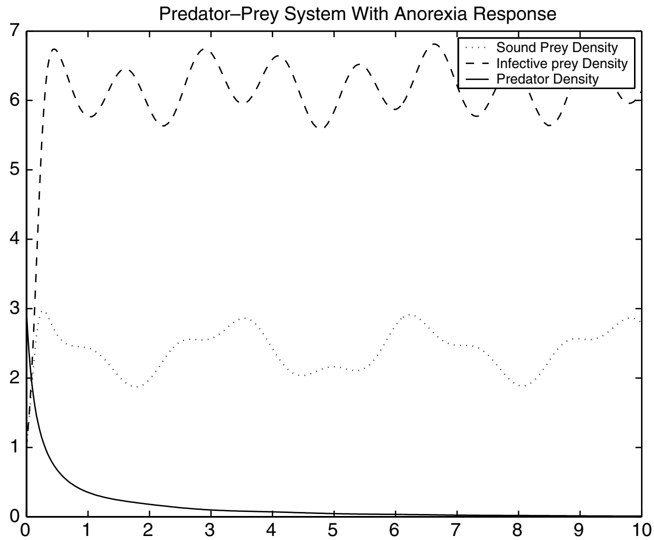

Fig. 1. The extinction of predator population.

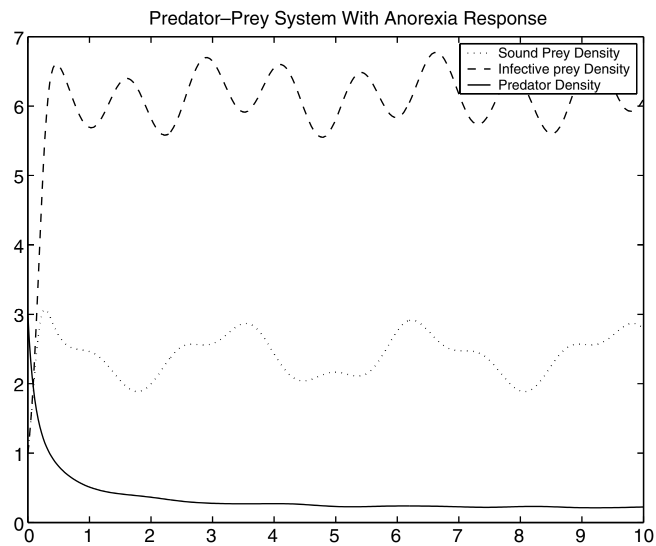

Fig. 2. The persistence of predator population.

persists. we can see Figs. 1 and 2. If we take $x_1(0) = x_2(0) = 1$ and $y(0) = 0.5$, dynamics of infective prey population and predator population are showed by Figs. 3 and 4.

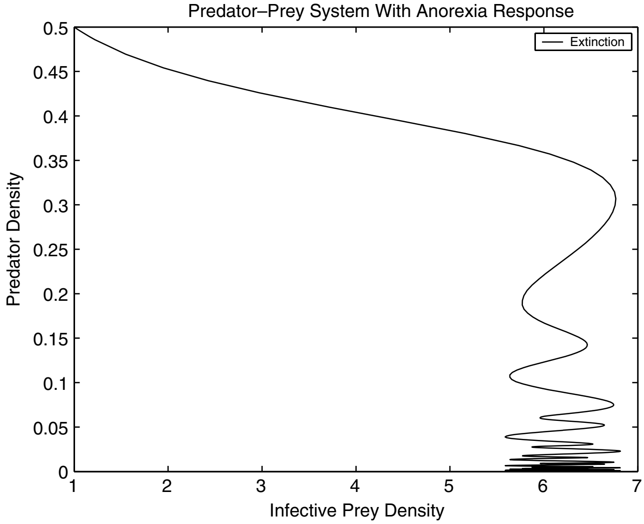

Fig. 3. The dynamics of infective prey and predator population.

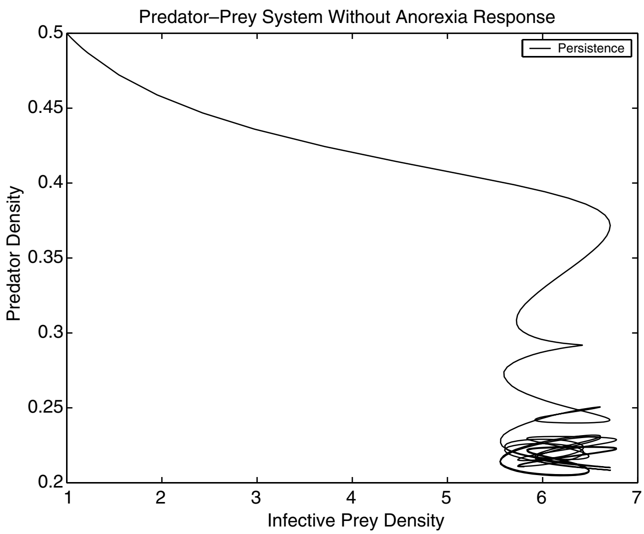

Fig. 4. The dynamics of infective prey and predator population.

Remark. In this paper, we investigate an eco-epidemiological Predator–Prey system in which an important factor such as anorexia response of predator is considered simultaneously. By constructing a suitable V-function, the conditions which guarantee the existence of a unique almost periodic solution (periodic solution) can be obtained and easy testified. To the same eco-epidemiological Predator–Prey system, persistence does not always exist under the effect of anorexia response on predator population. So our results about the anorexia response are new, realistic, important. Hence the study with Predator–Prey system with anorexia response may be useful to have some insight on realistic ecology.

References

[1] L. Chen, Mathematical Models and Methods in Ecology, Science Press, Beijing, 1988 (in Chinese).
[2] R.M. Anderson, R.M. May, Infectious Diseases of Humans, Dynamics and Control, Oxford University Press, Oxford, 1991.
[3] Z. Ma, Y. Zhou, et al., Mathematical Models and Studies of Infection Dynamics, Science Press, Beijing, 2004 (in Chinese).
[4] J. Chattopadhyay, O. Arino, A predator–prey model with disease in the prey, Nonlinear Dynamics 36 (1999) 747–766.
[5] K.P. Hadeler, H.I. Freedman, Predator–Prey population with parasite infection, J. Math. Biol. 27 (1989) 609–631.
[6] Y. Xiao, L. Chen, Modelling and analysis of a predator–prey model with disease in the prey, Mathematical Biosciences 171 (2001) 59–82.
[7] J. Chattopadhyay, A. Mukhopadhyay, P.K. Roy, Effect of viral infection on the generalized Gause model of predator–prey system, J. Biol. Syst. 11 (1) (2003) 19–26.
[8] Z. Ma, W. Wang, Asymptotic behavior of predator–prey system with time dependent coefficients, Appl. Anal. 34 (1989) 79–90.
[9] Z. Lu, L. Chen, Global Asymptotic stability of the periodic Lotka–Volterra system with two-predator and one prey, Appl. Math. J. Chin. Univ. Ser. B 10 (1995) 267–274.
[10] Z. Lu, L. Chen, Analysis of the periodic Lotka–Volterra system three species mixed model, Pure Appl. Math. 11 (2) (1995) 81–85.
[11] P. Yang, R. Xu, Global attractivity of the periodic Lotka–Volterra system, J. Math. Anal. Appl. 233 (1999) 221–232.
[12] J. Zhao, W. Chen, Global asymptotic stability of a periodic ecological model, Appl. Math. Comput. 147 (3) (2004) 881–892.
[13] Y. Xia, F. Chen, J. Cao, A. Chen, Existence and global attractivity of an almost periodic ecological model, Appl. Math. Comput. 157 (2) (2004) 449–475.
[14] J. Chattopadhyay, Effect of toxic substance on a two-species competitive system, Ecol. Model 84 (1996) 287–289.
[15] F.D. Chen et al., Periodicity in a food-limited population model with toxicants and state dependent delays, J. Math. Anal. Appl. 288 (1) (2003) 132–142.
[16] J. Cui, L. Chen, Asymptotic behavior of the solution for a class of time-dependent competitive system with feedback controls, Ann. Diff. Eqs. 9 (1) (1993) 11–17.
[17] Y. Yang, Qualitative analysis of anorexia predator–prey system under constant stocking rate of prey, J. Baotou Univ. Iroa Steel Technol. 12 (2) (1993) 1–7.

[18] Z. Huang, F. Chen, Stability of two-species almost periodic competition system with feedback control, J. Biol. Math. 20 (1) (2005) 28–32.
[19] Z. Huang, F. Chen, Almost periodic solution in two species competitive system with feedback controls, J. Eng. Math. 21 (1) (2004) 33–40.
[20] K. Gopalsamy, P. Weng, Feedback regulation of Logistic growth, Int. J. Math. Math. Sci. 16 (1) (1993) 177–192.
[21] F.D. Chen, F.X. Lin, X.X. Chen, Sufficient conditions for the existence of positive periodic solutions of a class of neutral delay models with feedback control, Appl. Math. Comput. 158 (1) (2004) 45–68.
[22] F.D. Chen, Positive periodic solutions of neutral Lotka–Volterra system with feedback control, Appl. Math. Comput. 162 (3) (2005) 1279–1302.
[23] Y. Xia, J. Cao, H. Zhang, F. Chen, Almost periodic solutions of n-species competitive system with feedback controls, J. Math. Anal. Appl. 294 (2) (2004) 503–522.
[24] Z. Huang, S. Chen, Optimal fuzzy control of a poisoning-pest model, Appl. Math. Comput., in press, doi:10.1016/j.amc.2005.01.082.
[25] S. Ma, A predator system affected by collectively defensive capability of the predator population, J. China Agri. Univ. 9 (2) (2004) 89–92.
[26] G.F. Gause, N.P. Smaragdova, A.A. Witt, Further studies of interaction between predator and prey, J. Anim. Ecol. 5 (1936) 1–18.
[27] Z. Huang, F. Chen, Global attractivity of three-species almost periodic system with II functional response, J. Math. Study 36 (2) (2003) 124–132.
[28] C. He, On almost periodic solutions of Lotka–Volterra almost periodic competition systems, Ann. Diff. Eqs. 9 (1) (1993) 26–36.
[29] A.M. Fink, Almost Periodic Differential Equations, Springer-Verlag, Berlin, Heldelberg, New York, 1974.