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Dynamic effective properties of matrix composite materials with high volume concentrations of particles

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Abstract A new approach is proposed to investigate the propagation of a plane compressional wave in matrix composite materials with high volume concentrations of particles. The theory of quasicrystalline approximation and Waterman's T matrix formalism are employed to treat the multiple scattering resulting from the particles in composites. The addition theorem for spherical Bessel functions is used to accomplish the translation between different coordinate systems. The Percus–Yevick correlation function widely applied in the molecular theory of liquids is employed to analyze the interaction of the densely distributed particles. The analytical expression for the Percus–Yevick correlation function is also given. The closed form solution for the effective propagation constant is obtained in the low frequency limit. Only numerical solutions are obtained at higher frequencies. Numerical examples show that the phase velocities in the composite materials with low volume concentration are in good agreement with those in previous literatures. The effects of the incident wave number, the volume fraction and the material properties of the particles and matrix on the phase velocity are also examined.

Keywords Particle-reinforced composite materials · High volume concentration · Multiple scattering of elastic waves · Quasicrystalline approximation · Percus–Yevick correlation function

1 Introduction

The study of wave propagation in random medium is interesting both theoretically and from the experimental point of view due to its numerical practical applications [1–4]. With the advent of composite materials, the multiple scattering of waves in composites with randomly distributed inclusions has been studied extensively [5–11].

When the elastic waves propagate in composite materials, it is inevitable that the multiple scattering from discrete inhomogeneities and the dispersion and attenuation of elastic waves occur. Differing from the static dynamic effective properties under the static load, the effective properties of materials subjected to elastic waves are defined as the dynamic effective properties. Through analyzing the relation between the phase velocity and the microstructure of composite materials, one can provide the overall dynamic behavior of materials under various loading conditions and optimize the microstructure through quantitative nondestructive evaluation.

To the author's knowledge, up to present time most research has mainly focused on the multiple scattering of fibers and particles in materials with low volume concentration, and no direct interactions between inclusions are taken into consideration. Self-consistent theory was often applied to solve the problem of wave propagation

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in composite materials. Yang and Mal [5] implemented the multiple scattering theory of Waterman and Truell in the framework of self-consistent theory and obtained the formulae for the dynamic effective medium in a self-consistent form. Subsequently, Kim [6,7] employed the self-consistent theory to solve the effective properties of composite materials. Recently, Kanaun and Levin further developed the self-consistent theory and investigated the elastic wave propagation in fiber [8] and particle [9] reinforced composites, and the effective propagation wave number and attenuation of waves were also analyzed. In addition, Foldy's approximation theory was also applied to analyze the dynamic behavior of composite materials [10,11].

However, when elastic waves propagate in composite materials with high volume concentration, the interaction among inclusions makes the scattered field more complex. In such situations, the self-consistent theory and Foldy's theory noticeably break down. By using a micromechanics model, Zhou et al. [12] ever analyzed the mesofracture of metal matrix composites reinforced by particles of large volume fraction. In order to solve accurately the multiple scattering of elastic waves in composite materials with high volume concentration, it is necessary to introduce a pair distribution function to describe the direct interactions between inclusions. The study of pair distribution is a subject of interest in statistical mechanics and various integral equations have been proposed. The simplest and, on basis of comparisons, the most satisfactory of these is the Percus–Yevick (PY) equation. Tang and Kong [13] have successfully applied it to study the multiple scattering of electromagnetic waves by random distribution of dielectric scatterers with high volume concentration. Recently, Kanaun [14] calculated the dielectric properties of matrix composite materials with high volume concentration of inclusions by using the Percus–Yevick correction function together with the effective field approach.

The main objective of this paper is to investigate the propagation of a plane compressional wave in matrix composite materials with high volume concentrations of particles by using a combination of quasicrystalline approximation and Percus–Yevick correlation function. Modeling of this problem is presented in Sect. 2. In Sect. 3, the conditional probability density function for particle distribution is discussed, and the analytical expression of the Percus–Yevick correlation function is given. In Sect. 4, the quasicrystalline approximation and Waterman's transfer matrix formalism are applied to treat the multiple scattering problem. The addition theorem for spherical Bessel functions is used to accomplish the translation between different coordinate systems. In Sect. 5, according to the approximation theory in the low frequency limit, closed form solutions for the effective propagation constants and dynamic elastic modulus are obtained. The numerical solutions at higher frequencies are constructed in Sect. 6. Numerical examples of phase velocity and effective elastic modulus are presented in Sect. 7, and comparisons with other existing theories are also given. Finally, the conclusions are presented.

2 Formulation of this problem

Consider a semi-infinite matrix composite material with randomly distributed particles, as depicted in Fig. 1. Let E_0, υ_0 and ρ_0 be the elastic modulus, Poisson ratio and density of the matrix, and E, υ, ρ those of the

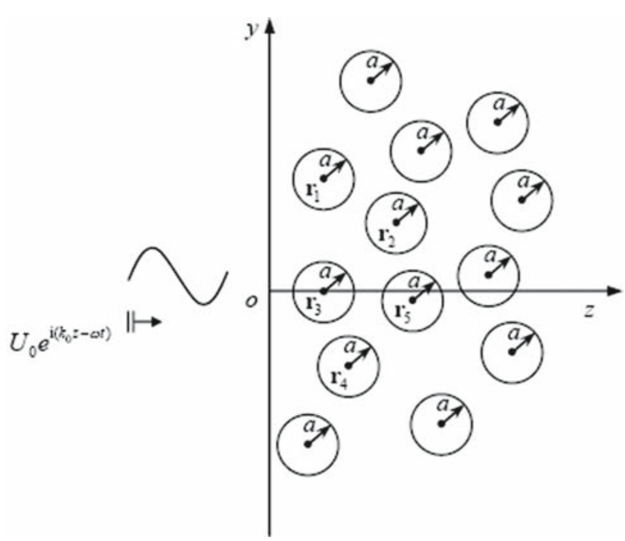

Fig. 1 Schematic of the incidence of elastic waves on a semi-infinite matrix composite material with randomly distributed particles

particles. The volume fraction of particles in materials is denoted by f. For simplicity, all particles are assumed to be identical ones fully bonded to the matrix. The particles of radius a are distributed discretely and randomly in the matrix. Suppose that a plane compressional wave of frequency ω is incident on the semi-infinite edge of the material structure. Due to the mutual interference of the particles and the multiple scattering from the edge, the propagation wave number will change. The propagation wave number is denoted as the effective wave number.

3 Conditional probability density function for particle distribution

To apply the quasicrystalline approximation to the multiple scattering of elastic waves by a random distribution of particles, the conditional probability density function for particle distribution must be specified. The position vector of the center of the ith particle is denoted by $\mathbf{r}_i$ and the probability density of the random variable $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ by $p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$, then due to the indistinguishability of the spherical particles, it is symmetric in its arguments and we have

$$\begin{aligned} p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) &= p(\mathbf{r}_i)p(\mathbf{r}_1, \mathbf{r}_2, \ldots,', \ldots, \mathbf{r}_N|\mathbf{r}_i) \\ &= p(\mathbf{r}_i)p(\mathbf{r}_j|\mathbf{r}_i)p(\mathbf{r}_1, \mathbf{r}_2, \ldots,', \ldots,' \ldots, \mathbf{r}_N|\mathbf{r}_j|\mathbf{r}_i), \\ p(\mathbf{r}_i) &= p(\mathbf{r}_1),\, p(\mathbf{r}_j|\mathbf{r}_i) = p(\mathbf{r}_2|\mathbf{r}_1), \quad i \neq j, \end{aligned} \tag{1}$$

where the probabilities with the vertical bar in their argument denote the customary conditional probabilities. A prime in the first part of Eq. (1) means $\mathbf{r}_i$ is absent, while two primes in the second part of Eq. (1) mean both $\mathbf{r}_i$ and $\mathbf{r}_j$ are absent. For a uniform composite material, the positions of a single spherical particle are equally probable within a large region V of the material and, hence, its distribution is uniform with density

$$p(\mathbf{r}_i) = 1/V, \quad \text{if} \quad \mathbf{r}_i \in V, \quad p(\mathbf{r}_i) = 0, \quad \text{if} \quad \mathbf{r}_i \notin V. \tag{2}$$

If the center of the ith particle, well within V, is held fixed, the distribution of the spherical particles around it will be spherically symmetrical. Thus, the conditional probability density function $p(\mathbf{r}_j|\mathbf{r}_i)$ is usually expressed in term of the pair correlation function $g(\bar{r})$, i.e.,

$$p\big(\mathbf{r}_j|\mathbf{r}_i\big) = \frac{1}{V}g\big(|\mathbf{r}_j - \mathbf{r}_i|\big). \tag{3}$$

Function $g(\bar{r})$ satisfies the following conditions

$$g(\bar{r}) = 0 \quad \text{if} \quad \bar{r} < b; \quad \lim_{\bar{r}\to\infty} g(\bar{r}) = 1. \tag{4}$$

where $b = 2a$ is the diameter of particles. The first of these conditions holds for non-overlapping sets of spherical particles. The second condition is correct if the correlation in spatial positions of the particles disappears when the distance between their centers tends to infinity.

For a random set of non-overlapping spherical particles, the most reliable two-point correlation function is the solution of the so-called Percus–Yevick equation proposed in the molecular theory of liquids [15]. In the three-dimensional case, the closed form solution of the PY equation has been found, and is expressed in term of inverse Laplace transforms [16]

$$g(x) = \sum_{n=1}^{\infty} g_n(x), \tag{5}$$

where

$$\begin{aligned} g_n(x) &= \frac{(-1)^{n+1}}{2\pi \mathrm{i} f x} \int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty} t e^{t(x-n)} \left(\frac{L(t)}{S(t)}\right)^n \mathrm{d}t, \quad x = r/b,\, L(t) \\ &= 12f\left[\left(1 + \frac{f}{2}\right)t + \big(1 + 2f\big)\right], \\ S(t) &= (1-f)^2 t^3 + 6f(1-f)t^2 + 18f^2 t - 12f(1+2f). \end{aligned}$$

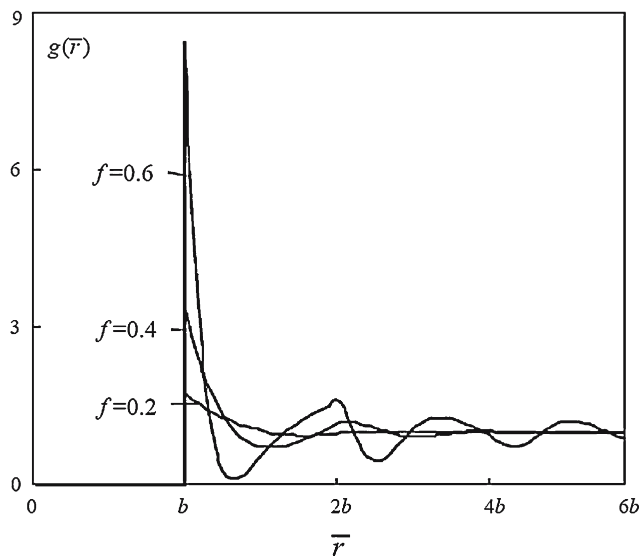

Fig. 2 Two-point Percus–Yevick correlation function as a function of $\bar{r}$

Note that $f = 4\pi a^3 n_0/3$ is the volume fraction of particles in materials, and n_0 is the number of particles per unit volume.

In Fig. 2, the PY correlation function has been evaluated for f = 0.2,0.4 and 0.6. It can be seen that the maximum value of the correlation function is at the position of $\bar{r} = b$. The value of the correlation function oscillates as a function of $\bar{r}$ and the extent of oscillation increases with f.

For low frequency scattering, one is only required to know the expression $n_0 \int [g(\bar{r}) - 1]\mathrm{d}\bar{r}$ rather than the detailed solution of $g(\bar{r})$ as a function of $\bar{r}$. Making use of the Ornstein–Zernike equation [17], it can be shown that

$$n_0 \int \left[g(\bar{r}) - 1\right]\mathrm{d}\bar{r} = \frac{(1-f)^4}{(1+2f)^2} - 1. \tag{6}$$

4 Solution of the integral equation: quasicrystalline approximation and Percus–Yevick correlation function (QA-PY)

Under the quasicrystalline approximation and Waterman's T matrix formalism, the integral equation for the configurational average of exciting field $\left\langle U^E(\mathbf{r}/\mathbf{r}_1)\right\rangle$ at field point r acting on a scatterer at $\mathbf{r}_1$ is expressed as

$$\left\langle U^E(\mathbf{r}/\mathbf{r}_1)\right\rangle = U^i(\mathbf{r}) + n_0 \int_\tau g(\mathbf{r}_2 - \mathbf{r}_1) T(\mathbf{r}_2) \left\langle U^E(\mathbf{r}/\mathbf{r}_2)\right\rangle d\tau_2, \tag{7}$$

where $U^i(\mathbf{r})$ is the incident field in the composite materials, and $T(\mathbf{r}_2)\left\langle U^E(\mathbf{r}/\mathbf{r}_2)\right\rangle$ is the field scattered by a single scatterer at $\mathbf{r}_2$ when excited by $\left\langle U^E(\mathbf{r}/\mathbf{r}_2)\right\rangle$. It should be noted that the integral is taken over the whole volume τ accessible to particles.

Consider a plane compressional wave being normally incident on the edge of the semi-infinite composite material, as shown in Fig. 1. Then, the incident field can be expanded using spherical wave function as

$$\begin{aligned} U^i &= U_0 e^{\mathrm{i}k_0 z} = U_0 e^{\mathrm{i}k_0 z_1} e^{\mathrm{i}k_0 (z - z_1)} \\ &= U_0 e^{\mathrm{i}k_0 z_1} \sum_{n=0}^{\infty} (2n+1)\mathrm{i}^n j_n(k_0 |r - r_1|) P_n(\cos\theta_{rr_1}), \end{aligned} \tag{8}$$

where $k_0 = \omega/c_0$ with $c_0 = \sqrt{[(1-v_0)E_0]/(1+v_0)(1-2v_0)\rho_0}$, U_0 is the amplitude of the incident waves, $j_n(\cdot)$ is the nth spherical Bessel function of the first kind, and $P_n(\cdot)$ is the associate Legendre polynomial. It should be noted that all field quantities have the same time variation $e^{\mathrm{i}\omega t}$ which is suppressed in all representations for notational convenience.

To solve Eq. (7), the scattered field is expanded as

$$T(\mathbf{r}_2)\left\langle U^E(\mathbf{r}/\mathbf{r}_2)\right\rangle = \sum_{n=0}^{\infty} \mathrm{i}^n(2n+1)A_n(z_2)T_n h_n\left(k_0|\mathbf{r}-\mathbf{r}_2|\right)P_n\left(\cos\theta_{rr_2}\right), \tag{9}$$

where $A_n(z_2)$ is the expanded coefficients, $h_n(\cdot)$ is the nth spherical Hankel function of the first kind, and T_n is the scattering coefficient. For spherical particles, the scattering coefficient can be obtained by satisfying the continuous boundary conditions of particles, and is written as

$$T_n = \frac{\rho_0 c_0 j_n(k_0 a) j_n'(ka) - \rho c j_n'(k_0 a) j_n(ka)}{\rho_0 c_0 h_n(k_0 a) j_n'(ka) - \rho c h_n'(k_0 a) j_n(ka)}. \tag{10}$$

Here the prime ($'$) denotes the differentiation of the spherical Bessel or Hankel functions with respect to their arguments, k denotes the wave numbers in materials, and $c = \sqrt{[(1-v)E]/(1+v)(1-2v)\rho}$.

Likewise, the exciting field $\left\langle U^E(\mathbf{r}/\mathbf{r}_1)\right\rangle$ is expressed as

$$\left\langle U^E(\mathbf{r}/\mathbf{r}_1)\right\rangle = \sum_{n=0}^{\infty}(2n+1)\mathrm{i}^n A_n(z_1) j_n\left(k_0\mathbf{r}_q\right)P_n\left(\cos\theta_q\right). \tag{11}$$

To make computation tractable, the expression of a scattered field is translated to the coordinate system with origin at $\mathbf{r}_1$. According to the addition theorem for spherical wave function [18,19], the following can be obtained

$$\begin{aligned} h_j\left(k_0|\mathbf{r}-\mathbf{r}_2|\right)P_j\left(\cos\theta_{rr_2}\right) = &\sum_{n=0}^{\infty}\mathrm{i}^n(2n+1)j_n\left(k_0|\mathbf{r}-\mathbf{r}_1|\right)P_n(\cos\theta_{rr_1}) \\ &\times\sum_p(\text{-}\,\mathrm{i})^p a(0,j,0,n,p)h_p(k_0|\mathbf{r}_2-\mathbf{r}_1|) \\ &\times P_p(\cos\theta_{r_2r_1})e^{\mathrm{i}(j-p)\phi_{r_1r_2}}, \end{aligned} \tag{12}$$

where $p = |j-n|, j+n+2, \ldots, j+n$, and

$$\begin{aligned} a(0,j,0,n,p) = {} &(2p+1)\frac{(j+n-p)!(j+p-n)!(n+p-j)!}{(j+n+p-1)!} \\ &\times\left[\frac{\left[\frac{1}{2}(j+n+p)\right]!}{\left[\frac{1}{2}(j+n-p)\right]!\left[\frac{1}{2}(j+p-n)\right]!\left[\frac{1}{2}(n+p-j)\right]!}\right]^2. \end{aligned} \tag{13}$$

Eqs. (8)–(11) can be substituted into Eq. (7). Owing to the axial symmetry of the volume of the integration, only terms independent of the azimuthal angle $\phi_{r_1r_2}$ contribute. Then the wave function $(2n+1)\mathrm{i}^n \times j_n(k_0|r-r_1|)P_n(\cos\theta_{rr_1})$ is factored out of the three terms of Eq. (7) and each coefficient in this sum is set to be equal to zero, because of the orthogonality of these functions on a spherical surface with center at $\mathbf{r}_1$. As a result, the infinite set of coupled integral equation are obtained as

$$\begin{aligned} A_n(z_1) = {} &e^{\mathrm{i}k_0z_1} + n_0\sum_{j=0}^{\infty}(2j+1)T_j\sum_p(-\mathrm{i})^p a(0,j,0,n,p) \\ &\times\int_\tau g\left(\mathbf{r}_2-\mathbf{r}_1\right)A_j(z_2)h_p\left(k_0|\mathbf{r}_2-\mathbf{r}_1|\right)P_p(\cos\theta_{r_2r_1})\mathrm{d}\tau_2, \quad n=0,1,2\ldots \end{aligned} \tag{14}$$

Note that the volume of integration of Eq. (14) consists of the semi-infinite space $z > 0$ less a sphere of radius b centered at the point $\mathbf{r}_1$.

In order to solve the set of equations (14), it is assumed that all coefficients $A_n(z_1)$ may be expressible in term of the same function $G(z_1)$, which satisfies the wave motion equation

$$\left(\nabla^2 + K^2\right)G(z_1) = 0, \tag{15}$$

where K is a constant to be determined. It can be seen that the exciting field $\left\langle U^E(\bar{r}/\bar{r}_1)\right\rangle$ consists of two functions satisfying the wave motion equation for K and k_0.

From Eq. (15), one can obtain

$$A_n(z_1) = A_n^0 e^{\mathrm{i}Kz_1}. \tag{16}$$

Here A_n^0 are the constants to be determined.

Substitution of Eq. (16) into (14) yields the following equation

$$A_n^0 e^{\mathrm{i}Kz_1} = e^{\mathrm{i}k_0 z_1} + n_0 e^{\mathrm{i}Kz_1} \sum_{j=0}^{\infty} (2j+1) T_j A_j^0 \sum_p (-\mathrm{i})^p a(0, j, 0, n, p)$$
$$\times \int_\tau g(\mathbf{r}_2 - \mathbf{r}_1) e^{\mathrm{i}Kz} h_p(k_0|\mathbf{r}_2 - \mathbf{r}_1|) P_p(\cos\theta_{r_2 r_1}) \mathrm{d}\tau_2. \tag{17}$$

Let us write $g(\mathbf{r}_2 - \mathbf{r}_1)$ as follows:

$$g(\mathbf{r}_2 - \mathbf{r}_1) = 1 + \left[g(\mathbf{r}_2 - \mathbf{r}_1) - 1\right]. \tag{18}$$

The first term in Eq. (18), when substituted into Eq. (17), produces two waves with propagation constant K and k_0. Then, Eq. (17) is rewritten as

$$A_n^0 e^{\mathrm{i}Kz_1} = e^{\mathrm{i}k_0 z_1} + n_0 e^{\mathrm{i}Kz_1} \sum_{j=0}^{\infty} (2j+1) T_j A_j^0 \sum_p (-\mathrm{i})^p a(0, j, 0, n, p)$$
$$\times \int_\tau e^{\mathrm{i}Kz} h_p\left(k_0|\mathbf{r}_2 - \mathbf{r}_1|\right) P_p(\cos\theta_{r_2 r_1}) \mathrm{d}\tau_2. \tag{19}$$

Note that the derivation of $\int_\tau e^{\mathrm{i}Kz} h_p(k_0|\mathbf{r}_2 - \mathbf{r}_1|) P_p(\cos\theta_{r_2 r_1}) \mathrm{d}\tau_2$ is shown in Appendix 1.

By making use of the derivation in Appendix 1, and balancing terms with wave number k_0, the following can be obtained

$$K = k_0 + \frac{2\pi n_0}{\mathrm{i}k_0^2} \sum_{j=0}^{\infty} (2j+1) T_j A_j^0. \tag{20}$$

Note that the convergence condition of the series in Eq. (20) is that the wave number is not too large.

In view of the results in Fig. 2, it can be seen that the second term of Eq. (18) vanishes for $|r_2 r_1|$ larger than a few b's. Thus, if the point r_1 is several diameters deep in the semi-infinite composite material, the volume of integration in Eq. (19) can be extended to the infinite space giving rise to a wave with propagation constant K. Balancing terms with propagation constant K in Eq. (19) leads to the following equations:

$$A_n^0 = n_0 \sum_{j=0}^{\infty} (2j+1) T_j A_j^0 \sum_p (-\mathrm{i})^p a(0, j, 0, n, p) \left[L_p(k_0, K, b) + M_p(k_0, K, b)\right], \tag{21}$$

where

$$L_p(k_0, K, b) = D_p(k_0, K, b)\mathrm{i}^{-p}/4\pi. \tag{22}$$

$$M_p(k_0, K, b) = \int_b^\infty r^2[g(r) - 1] h_p(k_0 r) j_p(Kr) \mathrm{d}r. \tag{23}$$

Note that $D_p(k_0, K, b)$ is shown in Appendix 1.

The set of equations (21) consists of an infinite number of homogeneous linear equations determining the coefficients A_n^0. For a nontrivial solution of A_n^0, the determinant must vanish, and this leads to the equation for the effective wave number K. Then A_n^0 are determined by using Eq. (20).

5 The phase velocity in the low frequency limit

A closed form solution can be obtained for the effective propagation wave number K in the low frequency limit. In such a case, because $|k_0 a| \ll 1.0$, only the first and second terms contribute in Eq. (21).

When $|x| \ll 1.0$, one can obtain

$$j_n(x) \approx \frac{2^n n! x^n}{(2n+1)!}, \quad h_n(x) \approx -\frac{\mathrm{i}(2n)!}{2^n n! x^{n+1}}. \tag{24}$$

According to Eqs. (10) and (24), the following can be obtained

$$T_0 \approx \frac{\mathrm{i}(k_0 a)^3}{3}\left(\frac{E_0}{E} - 1\right), \quad T_1 \approx \frac{\mathrm{i}(k_0 a)^3}{3}\frac{\rho_0 - \rho}{\rho_0 + 2\rho}. \tag{25}$$

In this case, one can also have

$$L_p(k_0, K, b) \approx -\frac{\mathrm{i}}{k_0(K^2 - k_0^2)}\left(\frac{K}{k_0}\right)^p - \delta_{p0}\frac{b^3}{3}. \tag{26}$$

Only the $p = 0$ term in L_p has a real part, because if $p \geq 1$, the real part is of an order smaller than $O(k_0^3 a^3)$ of the imaginary part.

Similarly,

$$M_p(k_0, K, b) = \delta_{p0}\int_b^\infty r^2\big[g(r) - 1\big]\mathrm{d}r. \tag{27}$$

Only the real part of $M_p(k_0, K, b)$ for $p = 0$ contributes. Other real and imaginary terms are smaller than the corresponding real and imaginary terms of $L_p(k_0, K, b)$. Thus, in the low frequency limit, only the knowledge of an integral of $g(r) - 1$ is required, rather than a detailed behavior of the correlation function as a function of r. According to Eq. (6), one can obtain

$$M_p(k_0, K, b) = \frac{\delta_{p0}}{4\pi n_0}\left[\frac{(1-f)^4}{(1+2f)^2} + 8f - 1\right]. \tag{28}$$

Substituting Eqs. (26) and (28) into Eq. (21), the following can be obtained

$$A_0^0 = \frac{4\pi n_0}{ik_0(K^2 - k_0^2)}\left\{T_0 A_0^0 \frac{1}{4\pi n_0}\left[\frac{(1-f)^4}{(1+2f)^2} + 8f - 1\right] + 3T_1 A_1^0 (K/k_0)\right\}. \tag{29}$$

$$A_1^0 = \frac{4\pi n_0}{ik_0(K^2 - k_0^2)}\left\{T_0 A_0^0 (K/k_0) + T_1 A_1^0\left\{\frac{1}{4\pi n_0}\left[\frac{(1-f)^4}{(1+2f)^2} + 8f - 1\right] + 2K^2/k_0^2\right\}\right\}. \tag{30}$$

$$\mathrm{i}k_0^2(K - k_0)/2\pi n_0 = T_0 A_0^0 + 3T_1 A_1^0. \tag{31}$$

Let the determinant of Eqs. (29) and (30) be zero, the relation between K and k_0 can be obtained as follows:

$$\left(\frac{K}{k_0}\right)^2 = \frac{\big[1 + 3T_0 S/\mathrm{i}(k_0 a)^3\big]\big[1 - 3T_1 S/\mathrm{i}(k_0 a)^3\big]}{1 + 6T_1 f/\mathrm{i}(k_0 a)^3}. \tag{32}$$

where $S = \frac{(1-f)^4}{(1+2f)^2} + 8f - 1$.

Equation (32) is the closed form solution for the effective propagation wave number K. The phase velocity V^* of the elastic waves in composite materials is connected with the effective wave number K by the equation

$$V^* = \frac{k_0}{\mathrm{Re}(K)}. \tag{33}$$

Here $\mathrm{Re}(K)$ denotes the real part of K.

The dynamic effective elastic modulus can be easily obtained from the phase velocity V^* as follows:

$$E^e = E_0(\rho^e/\rho_0)(V^*)^2, \tag{34}$$

where the average mass density ρ^e is written as

$$\rho^e = \rho_0(1-f) + \rho f. \tag{35}$$

6 The phase velocity at higher frequencies

In the region of higher frequencies, closed form solutions are not accessible. Equation (21) has to be solved numerically. A moderately wide range of frequencies $k_0 a$ from 0.05 to 2.5 is considered. The range of f is 0 to 0.6. For these values of $k_0 a$ and f, the determinant of the coefficient of A_n^0 is computed numerically by retaining a maximum of 8 simultaneous homogeneous complex equations for A_0, A_1, A_2, A_3 and A_4. The elements of $F_p(k_0, K, b)$ for $p = 0, 1, 2, \ldots, 7, 8$ are computed by numerically evaluating the integral in Eq. (23) for $r \in [b, 6b]$. As can be seen from Fig. 2 that for f between 0 and 0.6, the value of $g(r) - 1$ is practically zero for r larger than $6b$.

For given values of $k_0 a$ and f, the roots of the determinant are searched in the complex K plane $(K_r + \mathrm{i}K_i)$ using Mueller's method. There are two good initial guesses. One is provided by Eq. (32) at low values of $k_0 a$. The other is the result under Foldy's approximation, i.e.,

$$K^F = k_0 - \frac{\mathrm{i}\pi n_0}{k_0^2}\sum_{n=0}^{\infty}(2n+1)T_0 A_0^0, \tag{36}$$

where K^F denotes effective propagation constant under Foldy's approximation. These two guesses can be applied systematically to obtain the quick convergence of roots at increasing values of $k_0 a$.

7 Numerical examples and discussion

To validate the present method, the numerical examples are given. In the following analysis, it is convenient to make the variables dimensionless. To accomplish this step, we may introduce a representative length scale a, where a is the radius of the reinforcing particles. The following dimensionless variables and quantities have been chosen for computation: $k_0 a = 0.05 - 2.5$, $E/E_0 = 0.1 - 5.0$, $\rho/\rho_0 = 0.1 - 2.0$, $v = v_0 = 0.3$.

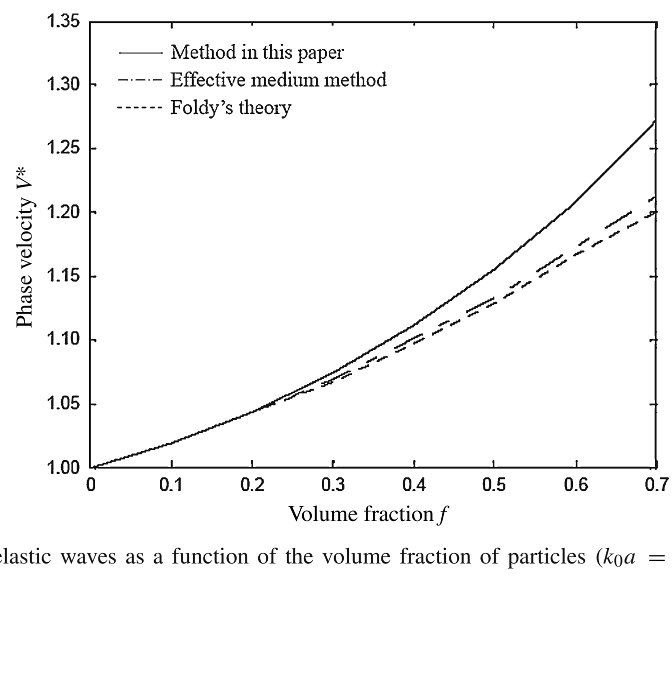

Fig. 3 Phase velocity of elastic waves as a function of the volume fraction of particles ($k_0 a = 0.1$, $E/E_0 = 5.0$, $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$)

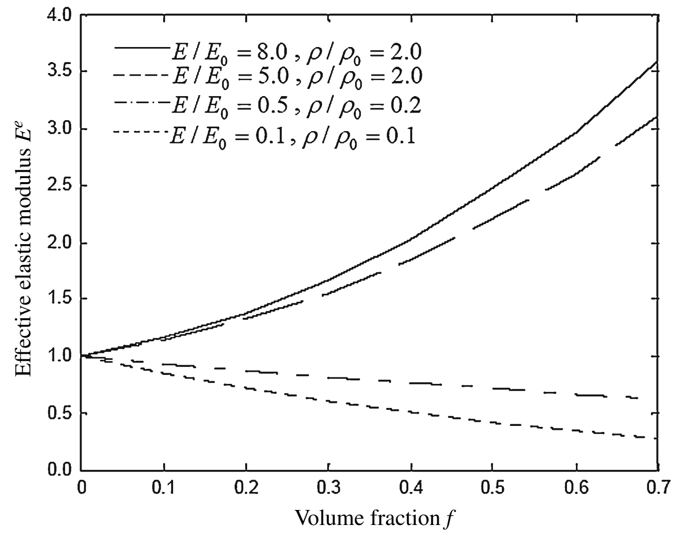

Fig. 4 Dynamic effective elastic modulus as a function of the volume fraction of particles ($k_0a = 0.1$, $v/v_0 = 1.0$)

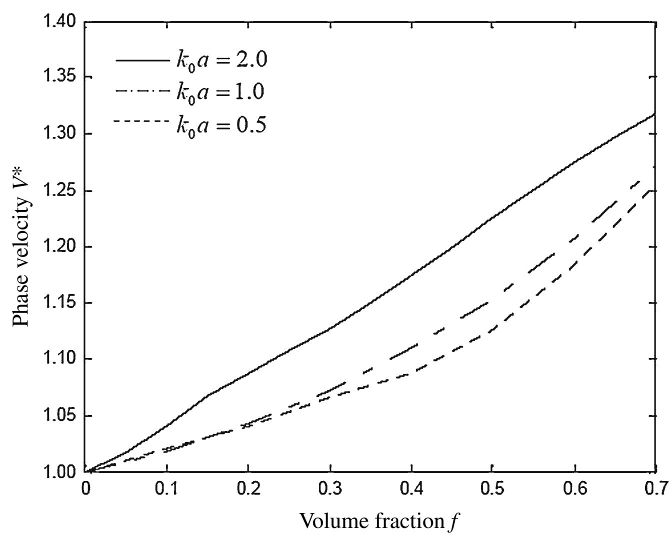

Fig. 5 Phase velocity of elastic waves as a function of the volume fraction of particles ($E/E_0 = 5.0$, $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$)

Figure 3 illustrates the phase velocity V^* of the compressional wave as a function of the volume fraction f with parameters: $k_0a = 0.1$, $E/E_0 = 5.0$, $\rho/\rho_0 = 2.0$ and $v/v_0 = 1.0$. It can be seen that when the volume fraction f is less than 0.3, the results obtained by the method in this paper are in good agreement with those obtained from effective medium method [9]. However, when the volume fraction is greater than 0.3, the phase velocities are greater than those obtained by using effective medium method. It is known from Ref. [10] that the numerical results are more accurate than those obtained from Foldy's theory and effective medium method.

To examine the effect of the material properties on the dynamic effective elastic modulus, we show the dynamic effective elastic modulus as a function of the volume fraction with parameters: $k_0a = 0.1$, $v/v_0 = 1.0$ in Fig. 4. It can be seen that when the material properties contrast ratios E^* and ρ^* are greater than 1, the dynamic effective elastic modulus increases with the increase of the volume fraction. However, if the material properties contrast ratios E^* and ρ^* are smaller than 1, the dynamic effective elastic modulus decreases with the increase of the volume fraction.

Figure 5 presents the phase velocity as a function of the volume fraction with parameters: $E/E_0 = 5.0$, $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$. It is observed that in the low frequency region, when the volume fraction is small, the phase velocity nearly expresses no variation with the volume fraction; when the volume fraction is great, the variation of the phase velocity with the volume fraction is great. However, in the high frequency

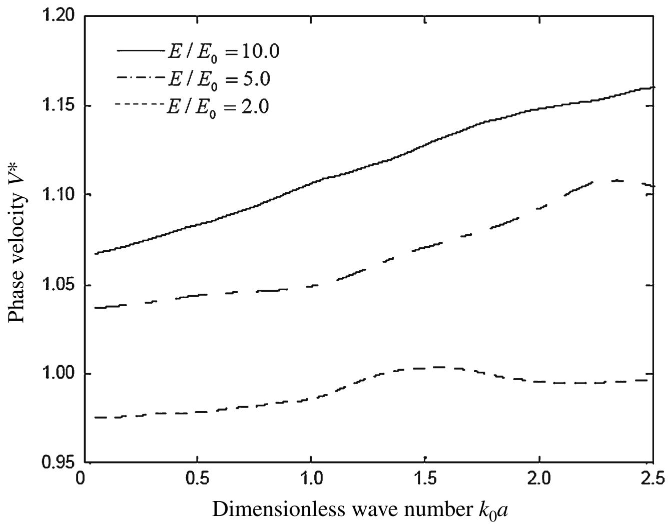

Fig. 6 Phase velocity of elastic waves as a function of dimensionless wave number ($\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$, $f = 0.2$)

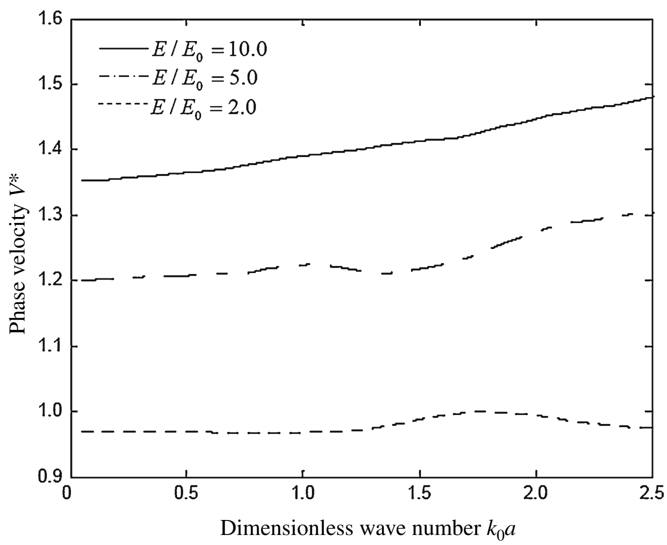

Fig. 7 Phase velocity of elastic waves as a function of dimensionless wave number ($\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$, $f = 0.6$)

region, the variation of the phase velocity with the volume fraction is great over the full range of volume fraction.

Figure 6 illustrates the phase velocity as a function of dimensionless wave number with parameters: $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$, $f = 0.2$. It can be seen that the phase velocity first increases with wave number, then saturates and tends to be invariable as wave number further increases. When the elastic modulus contrast ratio E^* is small, the variation of the phase velocity with dimensionless wave number is little. The dimensionless wave number corresponding to the maximum phase velocity increases with the increase of the elastic modulus contrast ratio E^*.

Figure 7 illustrates the phase velocity as a function of dimensionless wave number with parameters: $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$, $f = 0.6$. In contrast to Fig. 6, it is clear that the dimensionless wave number corresponding to the maximum phase velocity increases with the increase of the volume fraction of the particles. The effect of the elastic modulus contrast ratio E^* on the phase velocity increases with an increase of the volume fraction.

The variation of the phase velocity with the dimensionless wave number for $E/E_0 = 5.0$, $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$ is shown in Fig. 8. It can be seen that the variation of the phase velocity is little if the

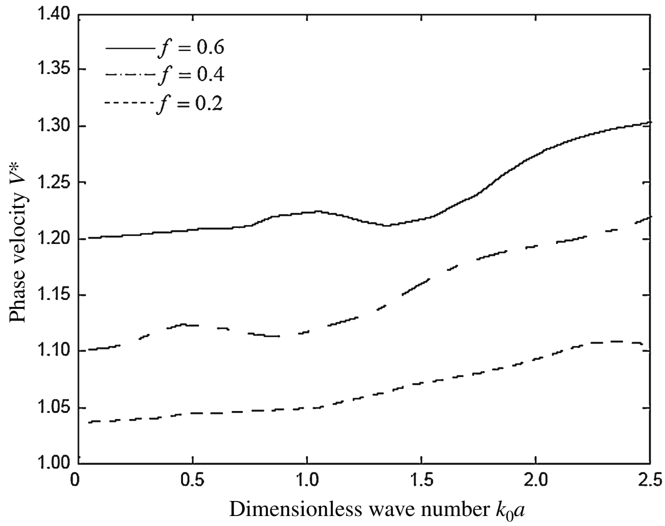

Fig. 8 Phase velocity of elastic waves as a function of dimensionless wave number ($E/E_0 = 5.0$, $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$)

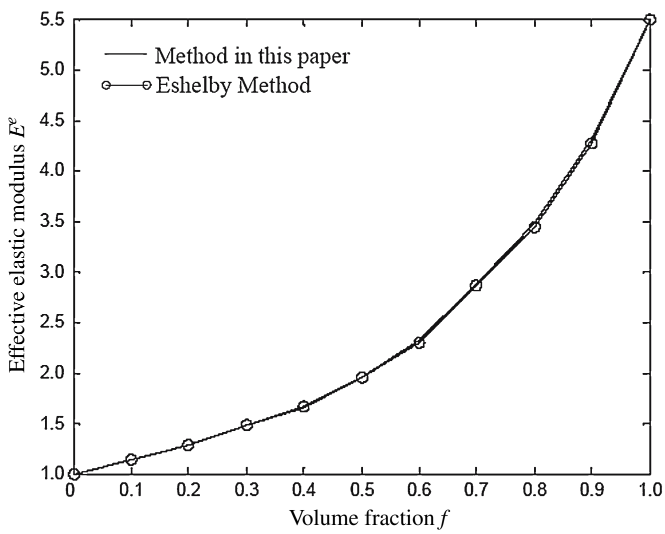

Fig. 9 Effective elastic modulus as a function of the volume fraction of particles ($k_0a = 0.0$, $E/E_0 = 5.5$, $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$)

volume fraction and the dimension wave number are small. The variation of the phase velocity with wave number increases with the increase of the volume fraction.

Finally, we consider the static effective elastic modulus of materials. As $k_0a \to 0$, the dynamic effective elastic modulus tends to be the static solutions. The expression for the static effective elastic modulus of composites is given in Appendix 2. Figure 9 shows the variation of the static effective elastic modulus with the volume fraction with $k_0a = 0.0$, $E/E_0 = 5.5$, $\rho/\rho_0 = 2.0$, $v/v_0 = 1.0$. A comparison with the static effective elastic modulus obtained from Eshelby method [20] is made. It should be emphasized that the results agree well with those from Eshelby method over the full range of the volume fraction.

8 Conclusion

In this study, a new and simple method (QA–PY) is presented to analyze the propagation of a plane compressional wave in matrix composite materials with high volume concentrations of particles. An analytical solution for effective wave number in the low frequency limit is derived. At higher frequencies, numerical solutions are constructed. Numerical examples show that the phase velocities in composite materials with low volume fractions obtained from this method are in good agreement with those in previous literatures. The accuracy in the case of high volume concentration increases greatly. So the efficacy of the quasicrystalline approximation increases greatly when combing with the Percus–Yevick correlation function.

It has been found that the phase velocity in materials is dependent on the incident wave number, the material properties contrast ratios and the volume fraction of particles, which is consistent with the result in Refs. [3,4]. The phase velocity increases with the increase of the incident wave number, the volume fraction of particles and the material properties contrast ratios of the particles and matrix. The greater the volume fraction of particles and the incident wave number, the greater the effect of the material properties contrast ratios on the phase velocity and the dynamic effective elastic modulus.

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Appendix 1

The derivation of $\int_{\tau-\tau_e} e^{\mathrm{i}Kz} h_p(k_0|\mathbf{r}_2-\mathbf{r}_1|)P_p(\cos\theta_{r_2r_1})\mathrm{d}\tau_2$ is given by

Making the change of variable $\mathbf{r}_2-\mathbf{r}_1=\mathbf{r}$ inside the integral yields the following

$$\int\limits_{\tau} e^{\mathrm{i}Kz} h_p(k_0|\mathbf{r}_2-\mathbf{r}_1|)P_p(\cos\theta_{r_2r_1})\mathrm{d}\tau_2 = \int\limits_{\tau-\tau_e} e^{\mathrm{i}Kz} w_p(k_0 r)\mathrm{d}\tau. \tag{1}$$

where $w_p(k_0r)=h_p(k_0r)P_p(\cos\theta)=(-\mathrm{i})^p P_{pz}h_0(k_0r)$.

In the second equality, Kasterin's representation is used to express the spherical waves with P_{pz} standing for $P_p(1/\mathrm{i}k_0, \partial/\partial z)$. Since $(\nabla^2+k_0^2)w_p(k_0r)=0$ and $(\nabla^2+K^2)e^{\mathrm{i}Kz}=0$, it is possible to obtain

$$e^{\mathrm{i}Kz} w_p(k_0r) = \frac{1}{K^2-k_0^2}\left(e^{\mathrm{i}Kz}\nabla^2 w_p - w_p\nabla^2 e^{\mathrm{i}Kz}\right). \tag{2}$$

By using Green's theorem, the volume integral in Eq. (1) can be transformed into the surface integrals, i.e.,

$$\int\limits_{\tau} e^{\mathrm{i}Kz} w_p(k_0r)\mathrm{d}\tau = \frac{1}{K^2-k_0^2}\int\limits_{S+S_e}\left[e^{iKz}\frac{\partial w_p}{\partial n'} - w_p\frac{\partial e^{iKz}}{\partial n'}\right]\mathrm{d}s, \tag{3}$$

where $S=\lim_{R\to\infty}[S_1(z_1)+S_2]$, S_e is a complete spherical surface of radius b, and n' is the normal outward unit vector. Note that $R\to\infty$ denotes the semi-infinite space.

The surface integral can be split into three integrals over S_1, S_2 and S_e. The first can be calculated in cylindrical coordinate $r=(\rho^2+z^2)^{1/2}$, and then the following is obtained

$$\begin{aligned} C_p(k_0,K) &= \frac{1}{K^2-k_0^2}\int\limits_{s_1}\left[e^{iKz}\frac{\partial w_p}{\partial n'} - w_p\frac{\partial e^{iKz}}{\partial n'}\right]\mathrm{d}s \\ &= \frac{2\pi}{K^2-k_0^2}\left[e^{iKz}\int\limits_0^R\left(iKw_p-\frac{\partial w_p}{\partial z}\right)\rho\mathrm{d}\rho\right]_{z=-z_1} \\ &= -\frac{2\pi e^{-\mathrm{i}Kz_1}}{k_0^2(K^2-k_0^2)}(-\mathrm{i})^p\left(\mathrm{i}K+\frac{\partial}{\partial z_1}\right)P_{p,-z_1}\left[e^{\mathrm{i}k_0(R^2+z_1^2)^{1/2}}-e^{\mathrm{i}k_0z_1}\right]. \end{aligned} \tag{4}$$

When $R\to\infty$, the oscillating term $e^{\mathrm{i}k_0(R^2+z_1^2)^{1/2}}$ disappears. In addition, $P_{p,-z_1}e^{-\mathrm{i}k_0z_1}=P_p(-1)e^{-\mathrm{i}k_0z_1}=(-1)^pe^{-\mathrm{i}k_0z_1}$, therefore,

$$C_p(k_0,K)=\frac{1}{K^2-k_0^2}\int\limits_{s_1}\left[e^{\mathrm{i}Kz}\frac{\partial w_p}{\partial n'} - w_p\frac{\partial e^{\mathrm{i}Kz}}{\partial n'}\right]\mathrm{d}s = \frac{2\pi\mathrm{i}}{k_0^2(K-k_0)}\mathrm{i}^p e^{\mathrm{i}(k_0-K)z_1}. \tag{5}$$

The remaining integrals

$$D_p(k_0,K,b) = \frac{-1}{K^2-k_0^2}\int\limits_{s_e}\left[e^{\mathrm{i}Kz}\frac{\partial w_p}{\partial n'} - w_p\frac{\partial e^{\mathrm{i}Kz}}{\partial n'}\right]\mathrm{d}s, \tag{6}$$

and

$$E_p = \frac{1}{K^2 - k_0^2} \lim_{R\to\infty} \int_{s_2} \left[e^{\mathrm{i}Kz} \frac{\partial w_p}{\partial n'} - w_p \frac{\partial e^{\mathrm{i}Kz}}{\partial n'} \right] \mathrm{d}s, \tag{7}$$

are independent of z_1, and just constants depending on k_0 and K. On the spherical surface $r = b$ and $x = \cos\theta$, Eq. (6) can be rewritten as

$$D_p(k_0, K, b) = \frac{-2\pi b^2}{K^2 - k_0^2} \int_{-1}^{1} e^{\mathrm{i}Kbx} P_p(x) \left[k_0 h'_p(k_0 b) - h_p(k_0 b)\mathrm{i}Kx \right] \mathrm{d}x. \tag{8}$$

By using the identities [21]

$$\int_{-1}^{1} e^{\mathrm{i}yx} P_p(x)\mathrm{d}x = 2\mathrm{i}^p j_p(y), \tag{9}$$

$$\int_{-1}^{1} \mathrm{i}x e^{\mathrm{i}yx} P_p(x)\mathrm{d}x = 2\mathrm{i}^p j'_p(y), \tag{10}$$

one can obtain

$$D_p(k_0, K, b) = -\frac{4\pi \mathrm{i}^p b^2}{K^2 - k_0^2} \left[k_0 h'_p(k_0 b) j_p(Kb) - K h_p(k_0 b) j'_p(Kb) \right]. \tag{11}$$

While on S_2, by employing $h_p(k_0 R) \approx \mathrm{i}^p (\mathrm{i}k_0 R)^{-1} e^{\mathrm{i}k_0 R}$ for large R, Eq. (7) can be rewritten as

$$E_p = \frac{2\pi \mathrm{i}^{-p}}{K^2 - k_0^2} \lim_{R\to\infty} \left[R e^{\mathrm{i}k_0 R} \int_{0}^{1} e^{\mathrm{i}k_0 Rx} P_p(x)(1 - (K/k_0)x)\mathrm{d}x \right]. \tag{12}$$

In Eq. (12), $P_p(x)[1 - (K/k_0)x] = Q(x)$ is a polynomial function of $p + 1$ degree. Repeating integrations by parts yields the following

$$\begin{aligned} E_p &= \frac{2\pi \mathrm{i}^{-p}}{K^2 - k_0^2} \lim_{R\to\infty} R e^{\mathrm{i}k_0 R} \left[\frac{e^{\mathrm{i}KR} Q(1) - Q(0)}{\mathrm{i}KR} + \frac{e^{\mathrm{i}KR} Q'(1) - Q'(0)}{(KR)^2} + \cdots \right] \\ &= \frac{2\pi \mathrm{i}^{-p}}{K^2 - k_0^2} \lim_{R\to\infty} \frac{e^{\mathrm{i}k_0 R}}{\mathrm{i}K} \left[e^{\mathrm{i}KR} Q(1) - Q(0) \right] = 0. \end{aligned} \tag{13}$$

From Eqs. (5), (11) and (13), the following can be obtained

$$\int_{\tau} e^{\mathrm{i}Kz} h_p(k_0 |r_2 - r_1|) P_p(\cos\theta_{r_2 r_1}) \mathrm{d}\tau_2 = C_p(k_0, K) + D_p(k_0, K, b). \tag{14}$$

Appendix 2

Using the Eshelby method, the effective elastic modulus is written as [20]

$$E^e = (1 - f)E_0 + fE + f(1 - f)\frac{(E - E_0)(1/E - 1/E_0)}{(1 - f)/E_0 + f/E + 4\mu_0/3EE_0}, \tag{15}$$

where $\mu_0 = \frac{E_0}{2(1+\nu_0)}$ is the bulk modulus of the matrix.

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