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Multiplication rule of similarity transformations in the coherent state representation and its mapping onto quantum optical generalized *ABCD* law

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Abstract

We find that classical similarity transformations in the coherent state representation projects onto the similarity transformation operators (STO), these operators constitute a loyal representation of symplectic group. Remarkably, the multiplication rule of the STOs naturally leads to the quantum optical generalized *ABCD* law, which is the quantum mechanical correspondence of the classical optical *ABCD* law. Throughout the whole derivation, the technique of integration within an ordered product (*IWOP*) of operators is employed.

Keywords: Generalized *ABCD* law; Matrix optics; Similarity transformation; *IWOP* technique

1. Introduction

In classical optics, ray-transfer matrices, $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $AD - BC = 1$, have been used to describe the geometrical formation of images by a centred lens system. For an optical ray (a centred spherical wavefront) passing through optical instruments there is a famous law, i.e. the *ABCD* law [1,2], governing the relation between input ray (r_1, α_1) and output ray (r_2, α_2), i.e.

$$\begin{pmatrix} r_2 \\ \alpha_2 \end{pmatrix} = M \begin{pmatrix} r_1 \\ \alpha_1 \end{pmatrix}, \tag{1}$$

where r_1 is the ray height from the optical axis, and α_1 is named the optical direction-cosine, $r_1/\alpha_1 \equiv R_1$ specifies the ray's wavefront shape. Eq. (1) implies

$$R_2 = \frac{r_2}{\alpha_2} = \frac{AR_1 + B}{CR_1 + D}. \tag{2}$$

This law is the core of matrix optics, since it tells us how the curvature of a centred spherical wavefront changes from one reference plane to the next. There are several literatures dealing with *ABCD* optical systems and the *ABCD* law is used in wild areas [3–9]. In addition, the multiplication rule of matrix optics implies that if the ray-transfer matrices of the n optical components are $M_1, M_2, M_3, \ldots, M_n$, respectively, then the whole system is determined by a matrix $M = M_1 M_2 M_3 \ldots M_n$. One of the remarkable things of modern optics is the case with which geometrical ray-transfer methods,

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constituting the matrix optics, can be adapted to describe the generation and propagation of laser beams. In 1965 Kogelnik [10] pointed out that propagation of Gaussian beam also obeys the *ABCD* law via optical diffraction integration, i.e. the input light field $f(x_1)$ and the output light field $g(x_2)$ are related to each other by the so-called Fresnel integration [11,12]

$$g(x_2) = \int_{-\infty}^{\infty} \kappa(A,B,C;x_2,x_1) f(x_1)\,dx_1, \tag{3}$$

where

$$\kappa(A,B,C;x_2,x_1) = \frac{1}{\sqrt{2\pi iB}} \exp\left[\frac{i}{2B}(Ax_1^2 - 2x_2x_1 + Dx_2^2)\right]. \tag{4}$$

The *ABCD* law for Gaussian beam passing through an optical system is [13,14]

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}, \tag{5}$$

where q_1 (q_2) represents the complex curvature of the input (output) Gaussian beam, Eq. (5) has the form similar to Eq. (2). Thus an interesting and important question naturally arises: Dose *ABCD* law also exhibit in quantum optics? Since classical similarity transform should have its quantum optical counterpart (we name it the similarity transformation operator (STO) [15]), this question also challenges us if there exist corresponding multiplication rule of the STO which corresponds to $M = M_1M_2M_3\ldots M_n$?

2. The multiplication rule for similarity transformation operator in the coherent state representation

In the following we derive the generalized *ABCD* law in quantum optics by introducing the appropriate STO and exhibiting its multiplication rule. We begin with mapping the symplectic transform in complex (z, z^*) space $(z, z^*) \to (\tau z - vz^*, \mu z^* - \sigma z)$ onto operator W by virtue of the coherent state representation [16–18]

$$W = \tau^{1/2} \int \frac{d^2 z}{\pi} \left| \begin{pmatrix} \tau & -v \\ -\sigma & \mu \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} z \\ z^* \end{pmatrix} \right|, \tag{6}$$

where

$$\begin{aligned} &\left| \begin{pmatrix} \tau & -v \\ -\sigma & \mu \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \right\rangle \\ &\quad = \left| \begin{pmatrix} \tau z - vz^* \\ \mu z^* - \sigma z \end{pmatrix} \right\rangle \\ &\quad = \exp\left[(\tau z - vz^*)a^\dagger - (\mu z^* - \sigma z)a\right]|0\rangle, \end{aligned} \tag{7}$$

and μ, v, σ and τ are complex numbers satisfying the unimodularity condition

$$\mu\tau - \sigma v = 1, \tag{8}$$

$a^\dagger$ is the Bose creation operator, $[a, a^\dagger] = 1$. Using the vacuum projector $|0\rangle\langle 0|$ in normal ordering of boson operators

$$|0\rangle\langle 0| =: e^{-a^\dagger a}:, \tag{9}$$

and the technique of integration within an ordered product (*IWOP*) of operators [19–21] we can directly perform the integration in Eq. (6) and obtain

$$\begin{aligned} W &= \tau^{1/2} \int \frac{d^2 z}{\pi} : \exp\left[-\mu\tau|z|^2 + z\tau a^\dagger + z^*(a - va^\dagger) \right. \\ &\qquad \left. + \frac{1}{2}\mu v z^{*2} + \frac{1}{2}\tau\sigma z^2 - a^\dagger a \right] : \\ &= \mu^{-1/2} : \exp\left\{ -\frac{v}{2\mu}a^{\dagger 2} + \left(\frac{1}{\mu} - 1\right)a^\dagger a + \frac{\sigma}{2\mu}a^2 \right\} : \\ &= \mu^{-1/2} \exp\left(-\frac{v}{2\mu}a^{\dagger 2} \right) \exp(-a^\dagger a \ln\mu) \exp\left(\frac{\sigma}{2\mu}a^2 \right), \end{aligned} \tag{10}$$

which is named the STO. We can easily verify that W generates the general linear similarity transformation of a and $a^\dagger$

$$d = WaW^{-1} = \mu a + va^\dagger, \quad g^\dagger = Wa^\dagger W^{-1} = \sigma a + \tau a^\dagger. \tag{11}$$

It is easily seen that the similarity transformation W preserves the commutator $[d, g^\dagger] = 1$ even though d and $g^\dagger$ are not generally Hermitian conjugates. Using the overlap

$$\langle z|z'\rangle = \exp\left[-\frac{|z|^2 + |z'|^2}{2} + z^* z' \right] \tag{12}$$

and the *IWOP* technique as well as the formula

$$\begin{aligned} &\int \frac{d^2 z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^* + fz^2 + gz^{*2}) \\ &\quad = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left(\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg} \right), \end{aligned} \tag{13}$$

where the convergent condition is either

$$\mathrm{Re}(\zeta + f + g) < 0, \quad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta + f + g} \right) < 0, \tag{14}$$

or

$$\mathrm{Re}(\zeta - f - g) < 0, \quad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta - f - g} \right) < 0, \tag{15}$$

we can calculate

$$
\begin{aligned}
WW' &= \tau^{1/2}\int\frac{d^2z}{\pi}\left|\begin{pmatrix}\tau & -v\\ -\sigma & \mu\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z\\ z^*\end{pmatrix}\right|\tau'^{1/2}\\
&\quad\times\int\frac{d^2z'}{\pi}\left|\begin{pmatrix}\tau' & -v'\\ -\sigma' & \mu'\end{pmatrix}\begin{pmatrix}z'\\ z'^*\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z'\\ z'^*\end{pmatrix}\right|\\
&= (\tau\tau')^{1/2}\int\frac{d^2zd^2z'}{\pi^2}\left|\begin{pmatrix}\tau & -v\\ -\sigma & \mu\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}\right\rangle\\
&\quad\times\langle 0|\exp\Big\{-\mu'\tau'|z'|^2+\tau'z^*z'+(a-v'z^*)z'^*\\
&\quad+\frac{1}{2}\tau'\sigma'z'^2+\frac{1}{2}\mu'v'z'^{*2}\Big\}\\
&= \left(\tau/\mu'\right)^{1/2}\int\frac{d^2z}{\pi}\left|\begin{pmatrix}\tau & -v\\ -\sigma & \mu\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}\right\rangle\\
&\quad\times\langle 0|\exp\left\{-\frac{1}{2}|z|^2+\frac{a}{\mu'}z^*-\frac{v'}{2\mu'}z^{*2}+\frac{\sigma'}{2\mu'}a^2\right\}\\
&= \frac{1}{\sqrt{\mu'\mu+\sigma v'}}:\exp\Big\{-\frac{\mu'v+v'\tau}{2(\mu'\mu+\sigma v')}a^{\dagger 2}\\
&\quad+\left(\frac{1}{\mu'\mu+\sigma v'}-1\right)a^\dagger a+\frac{\tau'\sigma+\sigma'\mu}{2(\mu'\mu+\sigma v')}a^2\Big\}:\\
&= \frac{1}{\sqrt{\mu''}}:\exp\Big\{-\frac{v''}{2\mu''}a^{\dagger 2}+\left(\frac{1}{\mu''}-1\right)a^\dagger a\\
&\quad+\frac{\sigma''}{2\mu''}a^2\Big\}:,
\end{aligned}
\tag{16}
$$

in the last step we have set

$$\mu''=\mu'\mu+\sigma v',\quad v''=\mu'v+v'\tau, \tag{17}$$

whose matrix form is

$$
\begin{aligned}
\begin{pmatrix}\tau'' & -v''\\ -\sigma'' & \mu''\end{pmatrix} &= \begin{pmatrix}\tau & -v\\ -\sigma & \mu\end{pmatrix}\begin{pmatrix}\tau' & -v'\\ -\sigma' & \mu'\end{pmatrix}\\
&= \begin{pmatrix}\tau\tau'+v\sigma' & -(\tau v'+v\mu')\\ -(\sigma\tau'+\mu\sigma') & \sigma v'+\mu\mu'\end{pmatrix},
\end{aligned}
\tag{18}
$$

from (8) and (18) we see

$$
\begin{aligned}
\tau''\mu''-v''\sigma'' &= (\tau\tau'+v\sigma')(\sigma v'+\mu\mu')\\
&\quad-(\tau v'+v\mu')(\sigma\tau'+\mu\sigma')=1.
\end{aligned}
\tag{19}
$$

Thus we can conclude that

$$
\begin{aligned}
WW' &= \frac{1}{\sqrt{\mu''}}:\exp\left\{-\frac{v''}{2\mu''}a^{\dagger 2}+\left(\frac{1}{\mu''}-1\right)a^\dagger a+\frac{\sigma''}{2\mu''}a^2\right\}:\\
&\equiv W'',
\end{aligned}
\tag{20}
$$

which coincides with

$$
\begin{aligned}
&\tau''^{1/2}\int\frac{d^2z}{\pi}\left|\begin{pmatrix}\tau'' & -v''\\ -\sigma'' & \mu''\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z\\ z^*\end{pmatrix}\right|\\
&\quad= (\tau\tau')^{1/2}\int\frac{d^2zd^2z'}{\pi^2}\left|\begin{pmatrix}\tau & -v\\ -\sigma & \mu\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}\right\rangle\\
&\quad\times\left\langle\begin{pmatrix}z\\ z^*\end{pmatrix}\right|\left(\begin{pmatrix}\tau' & -v'\\ -\sigma' & \mu'\end{pmatrix}\begin{pmatrix}z'\\ z'^*\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z'\\ z'^*\end{pmatrix}\right|.
\end{aligned}
\tag{21}
$$

This is the new product rule of the STO in the coherent state representation, which manifestly exhibits that W constitutes a loyal group representation of the symplectic group. To see the generalized $ABCD$ law more explicitly, we make the identification $z=(1/\sqrt{2})(q+ip)$,

$$
\begin{aligned}
|z\rangle &= \left|\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle=\exp[i(pQ-qP)]|0\rangle\\
&= \exp\left[-\frac{1}{4}\left(p^2+q^2\right)+\frac{q+ip}{\sqrt{2}}a^\dagger\right]|0\rangle,
\end{aligned}
$$

$$Q=\frac{a+a^\dagger}{\sqrt{2}},\quad P=\frac{a-a^\dagger}{i\sqrt{2}}, \tag{22}$$

and

$$
\begin{aligned}
\tau &= \tfrac{1}{2}[A+D-i(B-C)],\\
\mu &= \tfrac{1}{2}[A+D+i(B-C)],\\
v &= -\tfrac{1}{2}[A-D+i(B+C)],\\
\sigma &= -\tfrac{1}{2}[A-D-i(B+C)],
\end{aligned}
\tag{23}
$$

where the unimodularity condition $\mu\tau-\sigma v=1$ becomes $AD-BC=1$, which guarantees the classical Poisson bracket invariant. The reverse relation of (23) is

$$
\begin{aligned}
A &= \tfrac{1}{2}[\tau+\mu-v-\sigma],\\
B &= \frac{i}{2}[\tau+\sigma-\mu-v],\\
C &= -\frac{i}{2}[v+\tau-\mu-\sigma],\\
D &= \tfrac{1}{2}[\tau+\mu+v+\sigma],
\end{aligned}
$$

note that they are complex in general. Accordingly, Eq. (6) can be put into the form

$$
\begin{aligned}
W &= \sqrt{\frac{A+D-i(B-C)}{2}}\int\frac{dp\,dq}{2\pi}\\
&\quad\times\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\ p\end{pmatrix}\right|\equiv W(A,B,C),
\end{aligned}
\tag{24}
$$

and (10) becomes

$$W(A,B,C)=\sqrt{\frac{2}{A+D+i(B-C)}}\times:\exp\left\{\frac{A-D+i(B+C)}{2[A+D+i(B-C)]}a^{\dagger 2}+\left[\frac{2}{A+D+i(B-C)}-1\right]a^{\dagger}a-\frac{A-D-i(B+C)}{2[A+D+i(B-C)]}a^{2}\right\}:,\tag{25}$$

which implies that the classical symplectic transformation in matrix form in the coherent state basis

$$\begin{pmatrix}q\\p\end{pmatrix}\to\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}\equiv\begin{pmatrix}q'\\p'\end{pmatrix}\tag{26}$$

maps onto $W(A,B,C)$. The multiplication rule for W is $W(A',B',C')W(A,B,C)=W(A'',B'',C'')$, where

$$\begin{pmatrix}A''&B''\\C''&D''\end{pmatrix}=\begin{pmatrix}A'&B'\\C'&D'\end{pmatrix}\begin{pmatrix}A&B\\C&D\end{pmatrix}.\tag{27}$$

Eq. (24) clearly reveals the intrinsic relationship between classical symplectic transformation and quantum similarity transformation.

3. Generalized *ABCD* law in quantum optics

Now we can directly use the STO to derive the generalized *ABCD* law in quantum optics. From (25) we see that the STO generates

$$W(A,B,C)|0\rangle=\sqrt{\frac{2}{A+iB-i(C+iD)}}\times\exp\left\{\frac{A-D+i(B+C)}{2[A+D+i(B-C)]}a^{\dagger 2}\right\}|0\rangle\tag{28}$$

if we identify

$$\frac{A-D+i(B+C)}{A+D+i(B-C)}=\frac{q_1-i}{q_1+i},\tag{29}$$

then

$$W(A,B,C)|0\rangle=\sqrt{-\frac{2/(C+iD)}{q_1+i}}\exp\left[\frac{q_1-i}{2(q_1+i)}a^{\dagger 2}\right]|0\rangle.\tag{30}$$

The solution to (29) is

$$q_1=-\frac{A+iB}{C+iD}.\tag{31}$$

Let $W(A,B,C)|0\rangle$ expressed by (30) be an input state for an optical system which is characterized by parameters A',B',C',D', then the quantum optical generalized *ABCD* law states that the output state is

$$W(A',B',C')W(A,B,C)|0\rangle=\sqrt{-\frac{2/(C''+iD'')}{q_2+i}}\exp\left[\frac{q_2-i}{2(q_2+i)}a^{\dagger 2}\right]|0\rangle\tag{32}$$

which has the similar form to (30), where (C'',D'') is determined by (27), and

$$\bar{q}_2=\frac{A'\bar{q}_1+B'}{C'\bar{q}_1+D'},\quad \bar{q}_i\equiv -q_i\ \ (i=1,2)\tag{33}$$

which resembles (5), so Eq. (33) indicates the generalized *ABCD* law in the context of quantum optics. Especially, when we take $\mu=\tau^*$ and $\sigma=\nu^*$ it reduce to the special case in Ref. [22].

Proof. According to the multiplication rule of two STOs and Eqs. (25) and (27) we have

$$\begin{aligned}W(A',B',C')W(A,B,C)|0\rangle&=\sqrt{\frac{2}{A''+D''+i(B''-C'')}}\times\exp\left\{\frac{A''-D''+i(B''+C'')}{2[A''+D''+i(B''-C'')]}a^{\dagger 2}\right\}|0\rangle\\&=\sqrt{\frac{2}{A'(A+iB)+B'(C+iD)-iC'(A+iB)-iD'(C+iD)}}\\&\quad\times\exp\left\{\frac{A'(A+iB)+B'(C+iD)+iC'(A+iB)+iD'(C+iD)}{2[A'(A+iB)+B'(C+iD)-iC'(A+iB)-iD'(C+iD)]}\times a^{\dagger 2}\right\}|0\rangle\\&=\sqrt{-\frac{2/(C+iD)}{A'q_1-B'-i(C'q_1-D')}}\times\exp\left\{\frac{A'q_1-B'+i(C'q_1-D')}{2[A'q_1-B'-i(C'q_1-D')]}a^{\dagger 2}\right\}|0\rangle.\end{aligned}\tag{34}$$

Using (31) we can find $(2/(C+iD))/(C'q_1-D')=-2/(C''+iD'')$ together with (33), we can reach (32), thus the law is proven. Using (31) we can rewrite (33) as

$$q_2=-\frac{A'(A+iB)+B'(C+iD)}{C'(A+iB)+D'(C+iD)}=-\frac{A''+iB''}{C''+iD''},\tag{35}$$

which is consistent with (31). Eqs. (30)–(35) are therefore self-consistent. □

In summary, based on the *IWOP* technique, which exhibits the mapping from classical symplectic transformation to quantum operators in a transparent fashion, we have shown that classical similarity transformations in the coherent state representation projects onto the similarity transformation operators, these operators constitute a loyal group representation of symplectic group. Remarkably, the multiplication rule of the STOs naturally leads to the quantum optical generalized *ABCD* law, which is the quantum mechanical correspondence of the classical optical *ABCD* law. Therefore, the *ABCD* law exists not only in classical optics but also in quantum optics, this is a new resemblance between the two fields.

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