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Moderate deviations for estimators of quadratic variational process of diffusion with compound Poisson jumps

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ABSTRACT

We consider the stochastic differential equation driven by Lévy processes. Using discrete observations, moderate deviations for the threshold estimator of the quadratic variational process are studied. Moreover, we also obtain the functional moderate deviations.

1. Introduction

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, P)$, we consider the following stochastic differential equation driven by Lévy process

$$\mathrm{d}X_t = \sigma_t \mathrm{d}W_t + bt + \mathrm{d}L_t, \quad X_0 = x_0, t \in [0, 1] \tag{1.1}$$

where b is a constant, W is a standard Brownian motion and $0 \le \sigma_t \in L^2(\mathbb{R}^+, \mathrm{d}t)$. Moreover, L is a compound Poisson process independent of W:

$$L_t = \sum_{i=1}^{N_t} Y_i, \quad t \in [0, 1]$$

where Y_i are i.i.d. real random variables having law $\frac{1}{\lambda}\nu$ with the Lévy measure ν of L and N is a Poisson process, independent of each Y_i and with constant intensity λ.

Assume that σ_t is unknown and we want to estimate it from discrete observations $\{x_0, X_{t_1}, \ldots, X_{t_n}\}$ of the process X with $t_i = i/n$. Exactly, we want to estimate the unknown quadratic variational process of X^c which is the continuous part of X,

$$[X^c]_t = \int_0^t \sigma_s^2 \mathrm{d}s, \quad t \in [0, 1],$$

which is also called integral volatility in the financial econometric literature.

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In the case that X have no jumps, this question has been well investigated. Florens-Zmirou (1993) studied it both from the parametric and nonparametric point of view, and she obtained the consistency and the central limit theorem of her estimators. Djellout et al. (1999) obtained large and moderate deviations. For further references, one can see Avesani and Bertrand (1997), Bertrand (1997), Bercu and Dembo (1997) and Bercu et al. (1997).

If X has jumps defined by (1.1), in order to estimate $[X^c]_t$, the first problem is how to disentangle the jump part from the diffusion part, since the latter provides information on $[X^c]_t$, while the jump part introduces noise. By the method of threshold criterion, Mancini (2006) provided the following threshold estimator

$$V_t^n(X) := \sum_{i=1}^{[nt]}(X_{t_i} - X_{t_{i-1}})^2 I_{\{(X_{t_i}-X_{t_{i-1}})^2 \le r(1/n)\}},$$

where $r(1/n)$ satisfies that

$$\lim_{n\to\infty} r(1/n) = 0, \qquad \lim_{n\to\infty} \frac{\log n}{nr(1/n)} = 0.$$

He has shown that $V_t^n(X)$ is consistent estimators of $[X^c]_t$, and has some asymptotic normality respectively (c.f. Mancini, 2006, 2008). Furthermore, when $\sigma_t \equiv \sigma$, Mancini (2008) studied the large deviation of the threshold estimator. For more references, one can see Mancini (2004). In our article, by the method as in Mancini (2008) and Djellout et al. (1999), we consider moderate deviations and functional moderate deviations for estimators $V_t^n(X)$.

More precisely, we are interested in the estimations of

$$P\left(\frac{\sqrt{n}}{b(n)}\left(V_t^n(X) - \int_0^t \sigma_s^2 \mathrm{d}s\right) \in A\right),$$

where A is a given domain of deviation, $(b(n), n > 0)$ is some sequence denoting the scale of deviation. When $b(n) = 1$, this is exactly the estimation of central limit theorem. When $b(n) = \sqrt{n}$, it becomes the large deviations. Furthermore, when $b(n) \to \infty$ and $b(n) = o(\sqrt{n})$, this is the so called moderate deviations. In other words, the moderate deviations investigate the convergence speed between the large deviations and central limit theorem.

Let us first recall some basic definitions in large deviations theory (c.f. Dembo and Zeitouni, 1997). Let $\{\mu_T, T > 0\}$ be a family of probability on a topological space $(S, \mathcal{S})$ where $\mathcal{S}$ is a σ-algebra on S and let $\lambda(T)$ be a nonnegative function on $[1, +\infty)$ such that $\lim_{n\to\infty} \lambda(T) = +\infty$. A function $I : S \to [0, +\infty]$ is said to be a rate function if it is lower semicontinuous and it is said to be a good rate function if its level set $\{x \in S : I(x) \le a\}$ is compact for all $a \ge 0$. $\{\mu_T, T \ge 1\}$ is said to satisfy a large deviation principle (LDP) with speed $\lambda(T)$ and rate function $I(x)$ if for any closed set $F \in \mathcal{S}$

$$\limsup_{T\to\infty} \frac{1}{\lambda(T)} \log \mu_T(F) \le -\inf_{x\in F} I(x)$$

and for any open set $G \in \mathcal{S}$,

$$\liminf_{T\to\infty} \frac{1}{\lambda(T)} \log \mu_T(G) \ge -\inf_{x\in G} I(x).$$

Throughout this paper, let $b(n), n \ge 1$ be positive numbers satisfying

$$b(n) \to \infty \quad \text{and} \quad \frac{b(n)}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty.$$

Theorem 1.1. *Given* (X_t) *by* (1.1). *Assume* $\sigma_t^2 \in L^2([0, 1], \mathrm{d}t)$, $r(1/n) = o\left(\frac{1}{\sqrt{n}b(n)}\right)$ *and* σ_t *satisfy:*

$$\frac{r(1/n)}{\left(\log \frac{n}{b^2(n)}\right) \max_{1\le k\le n} \int_{(k-1)/n}^{k/n} \sigma_s^2 \mathrm{d}s} \to +\infty. \tag{1.2}$$

Then

$$\left\{P\left(\frac{\sqrt{n}}{b(n)}\left(V_t^n(X) - \int_0^t \sigma_s^2 \mathrm{d}s\right) \in \cdot\right), n \ge 1\right\}$$

satisfies the large deviations with speed $b^2(n)$ *and rate function given by*

$$I_t(x) = \frac{x^2}{4\int_0^t \sigma_s^4 \mathrm{d}s}. \quad \square$$

Remark 1.1. (1) Under the conditions of Theorem 1.1, we have

$$\sqrt{n}b(n) \max_{1\le k\le n} \int_{(k-1)/n}^{k/n} \sigma_s^2 \mathrm{d}s \to 0, \quad n \to \infty.$$

(2) Letting $r(1/n) = n^{-0.9}$, $b(n) = n^{-0.3}$, $\sigma_t \equiv 1$, the conditions of Theorem 1.1 are satisfied.

(3) If for some $p > 2$, $\sigma_t \in L^p([0,1], \mathrm{d}t)$, then $\max_{1\le k\le n}\int_{(k-1)/n}^{k/n}\sigma_s^2\mathrm{d}s = o\left(n^{\frac{1}{p}-1}\right)$. Consequently, taking $r(1/n) = O\left(\frac{\log n}{n^{1-\frac{1}{p}}}\right)$, we can easily obtain the conditions of Theorem 1.1. $\square$

Let $D_0[0,1]$ be the space of real right continuous functions x with left limit on $[0,1]$ and $x(0)=0$, equipped with the uniform topology. $\mathcal{B}^s$ is the σ-field generated by the coordinates $\{x(t), 0\le t\le 1\}$.

Theorem 1.2. *Given (X_t) by (1.1). Assume $\sigma_t^2 \in L^2([0,1], \mathrm{d}t)$ and (1.2) holds. Then in the space $(D_0[0,1], \mathcal{B}^s)$,*

$$\left\{P\left(\frac{\sqrt{n}}{b(n)}\left(V_\cdot^n(X) - \int_0^\cdot \sigma_s^2\mathrm{d}s\right)\in\cdot\right), 0\le t\le 1, n\ge 1\right\}$$

satisfies the large deviations with speed $b^2(n)$ and rate function given by

$$J(x) = \begin{cases}\displaystyle\int_0^1 \frac{\dot{x}^2(t)}{4\sigma_t^4} I_{\{t:\sigma_t>0\}}\mathrm{d}t, & \text{if } \mathrm{d}x \ll \sigma_t^4\mathrm{d}t;\\ +\infty, & \text{otherwise}, \end{cases} \quad \square$$

where $\mathrm{d}x \ll \sigma_t^4\mathrm{d}t$ means that $\mathrm{d}x$ is absolutely continuous with respect to the measure $\sigma_t^4\mathrm{d}t$.

2. Moderate deviations of $V_t^n(X)$

We first calculate the logarithm moment generating function of $V_t^n(X)$:

Lemma 2.1. *For any $\theta\in\mathbb{R}$, set $\Lambda_t(n,\theta) = \log E\exp\left(\sqrt{n}b(n)\theta\left(V_t^n(X) - \int_0^t\sigma_s^2\mathrm{d}s\right)\right)$. Under the conditions of Theorem 1.1, we have*

$$\lim_{n\to\infty}\frac{1}{b^2(n)}\Lambda_t(n,\theta) = \theta^2\int_0^t\sigma_s^4\mathrm{d}s.$$

Proof. Since the process X has independent increments, we have

$$\frac{1}{b^2(n)}\log E\exp\left(\sqrt{n}b(n)\theta\left(V_t^n(X) - \int_0^t\sigma_s^2\mathrm{d}s\right)\right)$$
$$= \frac{1}{b^2(n)}\left(-\sqrt{n}b(n)\theta\int_0^t\sigma_s^2\mathrm{d}s + \sum_{i=1}^{[nt]}\log E\exp\left(\sqrt{n}b(n)\theta(X_{t_i}-X_{t_{i-1}})^2 I_{\left\{(X_{t_i}-X_{t_{i-1}})^2\le r(1/n)\right\}}\right)\right).$$

Then

$$E\exp\left(\sqrt{n}b(n)\theta(X_{t_i}-X_{t_{i-1}})^2 I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n)\right\}}\right)$$
$$= E\left(\exp\left(\sqrt{n}b(n)\theta(X_{t_i}-X_{t_{i-1}})^2\right) I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n)\right\}}\right) + P\left(\left(X_{t_i}-X_{t_{i-1}}\right)^2 > r(1/n)\right).$$

Hence

$$E\exp\left(\sqrt{n}b(n)\theta(X_{t_i}-X_{t_{i-1}})^2 I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n)\right\}}\right)$$
$$= E\left(\mathrm{e}^{\sqrt{n}b(n)\theta\left(\int_{t_{i-1}}^{t_i}\sigma_s\mathrm{d}W_s+\frac{b}{n}\right)^2}\,\middle|\, N_{t_i}-N_{t_{i-1}}=0\right)P(N_{t_i}-N_{t_{i-1}}=0)$$
$$-\ E\left(\mathrm{e}^{\sqrt{n}b(n)\theta\left(\int_{t_{i-1}}^{t_i}\sigma_s\mathrm{d}W_s+\frac{b}{n}\right)^2} I_{\left\{\left(\int_{t_{i-1}}^{t_i}\sigma_s\mathrm{d}W_s+\frac{b}{n}\right)^2>r(1/n)\right\}}\,\middle|\, N_{t_i}-N_{t_{i-1}}=0\right)P(N_{t_i}-N_{t_{i-1}}=0)$$
$$+E\left(\mathrm{e}^{\sqrt{n}b(n)\theta\left(X_{t_i}-X_{t_{i-1}}\right)^2} I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n), N_{t_i}-N_{t_{i-1}}\ne 0\right\}}\right)$$
$$+\ P\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s\mathrm{d}W_s+\frac{b}{n}\right)^2 > r(1/n)\,\middle|\, N_{t_i}-N_{t_{i-1}}=0\right)P(N_{t_i}-N_{t_{i-1}}=0)$$

$$+\ P\left(\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s + L_{t_i} - L_{t_{i-1}} + \frac{b}{n}\right)^2 > r(1/n)\,\middle|\, N_{t_i} - N_{t_{i-1}} \neq 0\right) P(N_{t_i} - N_{t_{i-1}} \neq 0)\right.$$

$$:= M_1\left(i, \frac{1}{n}\right) - M_2\left(i, \frac{1}{n}\right) + M_3\left(i, \frac{1}{n}\right) + M_4\left(i, \frac{1}{n}\right) + M_5\left(i, \frac{1}{n}\right).$$

Since N is independent of W, for n large enough, we have

$$M_1\left(i, \frac{1}{n}\right) = \mathrm{e}^{-\lambda/n}\frac{1}{\sqrt{1 - 2\sqrt{n}b(n)\theta\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}}\exp\left(\frac{\sqrt{n}b(n)b^2\theta/n^2}{1 - 2\sqrt{n}b(n)\theta\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}\right). \tag{2.1}$$

By the maximal inequality of martingales,

$$M_4\left(i, \frac{1}{n}\right) = \mathrm{e}^{-\lambda/n}P\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s + \frac{b}{n}\right)^2 > r(1/n)\right) \le \mathrm{e}^{-\lambda/n}\exp\left(-\frac{r(1/n) - \frac{b^2}{n^2}}{2\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}\right) \tag{2.2}$$

and

$$M_5\left(i, \frac{1}{n}\right) \le P(N_{t_i} - N_{t_{i-1}} \neq 0) = 1 - \mathrm{e}^{-\lambda/n}. \tag{2.3}$$

Now, we estimate $M_2(i, \frac{1}{n})$ as follows: In the case $\theta \le 0$,

$$M_2\left(i, \frac{1}{n}\right) = \mathrm{e}^{-\lambda/n}P\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s + \frac{b}{n}\right)^2 > r(1/n)\right) \le \mathrm{e}^{-\lambda/n}\exp\left(-\frac{r(1/n) - \frac{b^2}{n^2}}{2\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}\right).$$

In the case $\theta > 0$, by Hölder inequality with $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have for n large enough

$$\begin{aligned}
M_2\left(i, \frac{1}{n}\right) &= \mathrm{e}^{-\lambda/n}E\left(\mathrm{e}^{\sqrt{n}b(n)\theta\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s + \frac{b}{n}\right)^2} I_{\left\{\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s + \frac{b}{n}\right)^2 > r(1/n)\right\}}\right)\\
&\le \mathrm{e}^{-\frac{\lambda}{n}}\left(E\exp\left(\sqrt{n}b(n)p\theta\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s + \frac{b}{n}\right)^2\right)\right)^{\frac{1}{p}} P^{\frac{1}{q}}\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s + \frac{b}{n}\right)^2 > r(1/n)\right)\\
&\le \frac{\mathrm{e}^{-\lambda/n}}{\left(1 - 2\sqrt{n}b(n)p\theta\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s\right)^{1/2p}}\exp\left(-\frac{r(1/n) - \frac{b^2}{n^2}}{2q\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s} + \frac{\sqrt{n}b(n)b^2\theta/n^2}{1 - 2\sqrt{n}b(n)p\theta\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}\right).
\end{aligned}$$

Therefore,

$$M_2\left(i, \frac{1}{n}\right) \le \begin{cases} \dfrac{\mathrm{e}^{-\lambda/n}}{\left(1 - 2\sqrt{n}b(n)p\theta\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s\right)^{1/2p}} & \\ \quad\times\ \exp\left(-\dfrac{r(1/n) - \frac{b^2}{n^2}}{4q\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s} + \dfrac{\sqrt{n}b(n)b^2\theta/n^2}{1 - 2\sqrt{n}b(n)p\theta\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}\right), & \text{if } \theta > 0,\\ \mathrm{e}^{-\lambda/n}\exp\left(-\dfrac{r(1/n) - \frac{b^2}{n^2}}{4\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}\right), & \text{if } \theta \le 0. \end{cases} \tag{2.4}$$

Then, we only have to estimate $M_3(i, \frac{1}{n})$. In fact,

$$\begin{aligned}
M_3\left(i, \frac{1}{n}\right) &= E\left(\mathrm{e}^{\sqrt{n}b(n)\theta(X_{t_i} - X_{t_{i-1}})^2} I_{\left\{\left(X_{t_i} - X_{t_{i-1}}\right)^2 \le r(1/n), N_{t_i} - N_{t_{i-1}} \neq 0\right\}}\right)\\
&= E\left(\mathrm{e}^{\sqrt{n}b(n)\theta(X_{t_i} - X_{t_{i-1}})^2} I_{\left\{\left(X_{t_i} - X_{t_{i-1}}\right)^2 \le r(1/n), N_{t_i} - N_{t_{i-1}} \neq 0, L_{t_i} - L_{t_{i-1}} = 0\right\}}\right)
\end{aligned}$$

$$+E\left(\mathrm{e}^{\sqrt{n}b(n)\theta(X_{t_i}-X_{t_{i-1}})^2} I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n), N_{t_i}-N_{t_{i-1}}\neq 0, L_{t_i}-L_{t_{i-1}}\neq 0\right\}}\right).$$

By the estimation of $M_1(i, \frac{1}{n})$ and $M_2(i, \frac{1}{n})$, we have

$$\begin{aligned}
&E\left(\mathrm{e}^{\sqrt{n}b(n)\theta\left(X_{t_i}-X_{t_{i-1}}\right)^2} I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n), N_{t_i}-N_{t_{i-1}}\neq 0, L_{t_i}-L_{t_{i-1}}=0\right\}}\right)\\
&\quad\le E\left(\mathrm{e}^{\sqrt{n}b(n)\theta\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2} I_{\left\{\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2\le r(1/n)\right\}}\right) P(N_{t_i}-N_{t_{i-1}}\neq 0)\\
&\quad=\left(M_1\left(i,\frac{1}{n}\right)-M_2\left(i,\frac{1}{n}\right)\right)(\mathrm{e}^{\lambda/n}-1).
\end{aligned}$$

Furthermore, for $\theta > 0$ and n large enough,

$$\begin{aligned}
&E\left(\mathrm{e}^{\sqrt{n}b(n)\theta\left(X_{t_i}-X_{t_{i-1}}\right)^2} I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n), N_{t_i}-N_{t_{i-1}}\neq 0, L_{t_i}-L_{t_{i-1}}\neq 0\right\}}\right)\\
&\quad\le \mathrm{e}^{\sqrt{n}b(n)\theta r(1/n)} P\left(\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n), N_{t_i}-N_{t_{i-1}}\neq 0, L_{t_i}-L_{t_{i-1}}\neq 0\right)\\
&\quad= \mathrm{e}^{\sqrt{n}b(n)\theta r(1/n)} \lim_{\varepsilon\to 0} P\left(\left(X_{t_i}-X_{t_{i-1}}\right)^2\le r(1/n), |L_{t_i}-L_{t_{i-1}}|>\varepsilon\right)\\
&\quad\le \mathrm{e}^{\sqrt{n}b(n)\theta r(1/n)} \lim_{\varepsilon\to 0} P\left(\left|\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right|\ge \varepsilon-\sqrt{r(1/n)}\right)\\
&\quad\le \mathrm{e}^{\sqrt{n}b(n)\theta r(1/n)} \exp\left(-\frac{r(1/n)+\frac{b^2}{n^2}}{2\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s}\right).
\end{aligned}$$

For $\theta \le 0$, similar to the case of $\theta > 0$, we have

$$E\left(\mathrm{e}^{\sqrt{n}b(n)\theta(X_{t_i}-X_{t_{i-1}})^2} I_{\{(X_{t_i}-X_{t_{i-1}})^2\le r(1/n), N_{t_i}-N_{t_{i-1}}\neq 0, L_{t_i}-L_{t_{i-1}}\neq 0\}}\right)\le \exp\left(-\frac{r(1/n)+\frac{b^2}{n^2}}{2\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s}\right).$$

By the above estimations, we can easily follow from (1.2) and the fact that $r(1/n) = o\left(\frac{1}{\sqrt{n}b(n)}\right)$,

$$\max_i \frac{n}{b^2(n)}\cdot\frac{M_j\left(i,\frac{1}{n}\right)}{M_1\left(i,\frac{1}{n}\right)}\to 0,\quad n\to\infty, j=2,3,4,5.$$

Therefore, by Taylor formula, we can obtain that

$$\begin{aligned}
\Lambda_t(\theta) &= \lim_{n\to\infty}\frac{1}{b^2(n)}\left(-\sqrt{n}b(n)\theta\int_0^t\sigma_s^2 \mathrm{d}s-\frac{1}{2}\sum_{i=1}^{[nt]}\log\left(1-2\sqrt{n}b(n)\theta\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s\right)\right)\\
&= \lim_{n\to\infty}\theta^2\sum_{i=1}^{[nt]}(1+\eta(n))n\left(\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s\right)^2,
\end{aligned}$$

where $|\eta(n)| \le C|\theta|\sqrt{n}b(n)\max_{1\le i\le n}\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s$.

Next, we show that:

$$\sum_{i=1}^{[nt]} n\left(\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s\right)^2\to\int_0^t\sigma_s^4 \mathrm{d}s,\quad n\to\infty. \tag{2.5}$$

Indeed, by Jensen's inequality

$$\sum_{i=1}^{[nt]} n\left(\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s\right)^2=\sum_{i=1}^{[nt]}\frac{1}{n}\left(\frac{\int_{t_{i-1}}^{t_i}\sigma_s^2 \mathrm{d}s}{1/n}\right)^2\le\int_0^t\sigma_s^4 \mathrm{d}s.$$

On the other hand, denote by $\alpha_i^n = \frac{i}{2^n}t$, $0 \le i \le 2^n$ and $g(s) = \int_0^s \sigma_u^2 \mathrm{d}u$. By the convergence of martingales,

$$\psi^n(s) = \sum_{i=0}^{2^n} \frac{g(\alpha_{i+1}^n) - g(\alpha_i^n)}{\frac{T}{2^n}} I_{[\alpha_i^n, \alpha_i^n)}(s) \to g'(s) = \sigma_s^2, \quad a.s.$$

with Fatou lemma,

$$\int_0^t \sigma_s^4 \mathrm{d}s = \int_0^t (g'(s))^2 \mathrm{d}s \le \limsup_{n\to\infty} \int_0^t |\psi^n(s)|^2 \mathrm{d}s \le \limsup_{n\to\infty} \sum_{i=1}^{[nt]} n \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \mathrm{d}s \right)^2 .$$

Therefore, (2.5) holds. □

Proof of Theorem 1.1. By Lemma 2.1, this theorem can be obtained by Gätner–Ellis theorem. □

3. Functional moderate deviations

In this section, we prove the functional moderate deviation theorem (Theorem 1.2). It is well known that the LDP of finite dimensional distributions

$$P\left(\frac{\sqrt{n}}{b_n} \left(V_{s_1}^n(X) - \int_0^{s_1} \sigma_s^2 \mathrm{d}s, \ldots, V_{s_k}^n(X) - \int_0^{s_k} \sigma_s^2 \mathrm{d}s \right) \right), \quad 0 < s_1 < \cdots < s_k \le 1, k \ge 1$$

and the following exponential tightness: for any $s \in [0, 1]$ and $\eta > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \frac{1}{b^2(n)} \log P\left(\frac{\sqrt{n}}{b(n)} \sup_{s \le t \le s+\delta} \left| V_t^n(X) - V_s^n(X) - \int_s^t \sigma_u^2 \mathrm{d}u \right| > \eta \right) = -\infty$$

are sufficient for the large deviation of $\{V_t^n, t \in [0, 1]\}$ (cf. Dembo and Zeitouni, 1997; Wu, 1997).

Lemma 3.1. *Under the assumption of Theorem 1.1, we have*

$$\lim_{n\to\infty} \frac{\sqrt{n}}{b(n)} \sup_{t\in[0,1]} \left| EV_t^n(X) - \int_0^t \sigma_s^2 \mathrm{d}s \right| = 0.$$

Proof. Since N is independent of W, we have

$$\begin{aligned} EV_t^n(X) &= \sum_{i=1}^{[nt]} E\left((X_{t_i} - X_{t_{i-1}})^2 I_{\left\{ \left(X_{t_i} - X_{t_{i-1}} \right)^2 \le r(1/n), N_{t_i} - N_{t_i} \ne 0 \right\}} \right) \\ &\quad + \sum_{i=1}^{[nt]} E\left(\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 I_{\left\{ \left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 \le r(1/n), N_{t_i} - N_{t_{i-1}} = 0 \right\}} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} EV_t^n(X) &\le \sum_{i=1}^{[nt]} E\left(\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 I_{\left\{ \left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 \le r(1/n), N_{t_i} - N_{t_{i-1}} \ne 0, L_{t_i} - L_{t_{i-1}} = 0 \right\}} \right) \\ &\quad + \sum_{i=1}^{[nt]} E\left((X_{t_i} - X_{t_{i-1}})^2 I_{\left\{ \left(X_{t_i} - X_{t_{i-1}} \right)^2 \le r(1/n), N_{t_i} - N_{t_{i-1}} \ne 0, L_{t_i} - L_{t_{i-1}} \ne 0 \right\}} \right) \\ &\quad + \sum_{i=1}^{[nt]} E\left(\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 I_{\left\{ \left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 \ge r(1/n) \right\}} \right) \mathrm{e}^{-\lambda/n} \\ &\quad + \sum_{i=1}^{[nt]} E\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 \mathrm{e}^{-\lambda/n}. \end{aligned}$$

By Chebyshev inequality,

$$E\left(\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 I_{\left\{ \left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s + \frac{b}{n} \right)^2 \ge r(1/n) \right\}} \right)$$

$$\leq \left(E\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^4\right)^{1/2} P\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2\geq r(1/n)\right)^{1/2}$$
$$\leq \sqrt{6}\left(\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s\right)\exp\left(-\frac{r(1/n)-\frac{b^2}{n^2}}{8\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s}\right).$$

Then

$$\lim_{n\to\infty}\frac{\sqrt{n}}{b(n)}\sum_{i=1}^{n}E\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2 I_{\left\{\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2\geq r(1/n)\right\}}\right)\leq \sqrt{6}\lim_{n\to\infty}\sqrt{n}b(n)\left(\max_{1\leq i\leq n}\int_{t_{i-1}}^{t_i}\sigma_s^2\mathrm{d}s\right)=0.$$

And

$$\lim_{n\to\infty}\frac{\sqrt{n}}{b(n)}\sup_{t\in[0,1]}\left|\sum_{i=1}^{[nt]}E\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2-\int_0^t\sigma_s^2\mathrm{d}s\right|\leq \lim_{n\to\infty}\frac{1}{b(n)}\max_{1\leq i\leq n}\sqrt{\int_{t_{i-1}}^{t_i}\sigma_s^4\mathrm{d}s}\to 0.$$

Furthermore,

$$\sum_{i=1}^{[nt]}E\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2 I_{\left\{\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2\leq r(1/n),N_{t_i}-N_{t_{i-1}}\neq 0,L_{t_i}-L_{t_{i-1}}=0\right\}}\right)$$
$$\leq \sum_{i=1}^{[nt]}E\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2 P(N_{t_i}-N_{t_{i-1}}\neq 0)$$
$$+\sum_{i=1}^{[nt]}E\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2 I_{\left\{\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2\geq r(1/n)\right\}}\right)P(N_{t_i}-N_{t_{i-1}}\neq 0).$$

Therefore,

$$\lim_{n\to\infty}\frac{\sqrt{n}}{b(n)}\sup_{t\in[0,1]}\sum_{i=1}^{[nt]}E\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s+\frac{b}{n}\right)^2 I_{\left\{\left(\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s\right)^2\leq r(1/n),N_{t_i}-N_{t_{i-1}}\neq 0,L_{t_i}-L_{t_{i-1}}=0\right\}}\right)=0.$$

By the same method as in Lemma 2.1, we have

$$\sup_{t\in[0,1]}\sum_{i=1}^{[nt]}E\left(\left(X_{t_i}-X_{t_{i-1}}\right)^2 I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\leq r(1/n),N_{t_i}-N_{t_{i-1}}\neq 0,L_{t_i}-L_{t_{i-1}}\neq 0\right\}}\right)$$
$$\leq r(1/n)\sup_{t\in[0,1]}\sum_{i=1}^{[nt]}P\left(\left(X_{t_i}-X_{t_{i-1}}\right)^2\leq r(1/n),N_{t_i}-N_{t_{i-1}}\neq 0,L_{t_i}-L_{t_{i-1}}\neq 0\right)$$
$$\leq r(1/n)b^2(n).$$

Consequently,

$$\lim_{n\to\infty}\frac{\sqrt{n}}{b(n)}E\left(\left(X_{t_i}-X_{t_{i-1}}\right)^2 I_{\left\{\left(X_{t_i}-X_{t_{i-1}}\right)^2\leq r(1/n),N_{t_i}-N_{t_{i-1}}\neq 0,L_{t_i}-L_{t_{i-1}}\neq 0\right\}}\right)\leq \lim_{n\to\infty}\sqrt{n}b(n)r(1/n)=0.$$

By the above estimations, we complete the proof of this lemma. □

Lemma 3.2. *For any* $0=s_0<s_1<\cdots<s_k\leq 1$, $k\geq 1$,

$$P\left(\frac{\sqrt{n}}{b_n}\left(V_{s_1}^n(X)-\int_0^{s_1}\sigma_s^2\mathrm{d}s,\ldots,V_{s_k}^n(X)-\int_0^{s_k}\sigma_s^2\mathrm{d}s\right)\right),$$

satisfies large deviation with rate $b^2(n)$ *and rate function given by*

$$I_{s_1,\ldots,s_k}(x_1,\ldots,x_k)=\sum_{i=1}^{k}\frac{(x_k-x_{k-1})^2}{4\int_{s_{k-1}}^{s_k}\sigma_s^4\mathrm{d}s}$$

where $x_0=0$.

Proof. For n large enough we have $1 < [nt_1] < \cdots < [nt_k] < n$, so by applying the homomorphism

$$\Gamma : (x_1, x_2, \ldots, x_k) \to (x_1, x_2 - x_1, \ldots, x_k - x_{k-1})$$

$Z_n = \left(V^n_{s_1}(X) - \int_0^{s_1} \sigma_s^2 \mathrm{d}s, \ldots, V^n_{s_k}(X) - \int_0^{s_k} \sigma_s^2 \mathrm{d}s\right)$ can be mapped to the random vector $U_n = \Gamma Z_n$ with independent components. Then we consider the large deviations of $\frac{\sqrt{n}}{b(n)}(U_n - EU_n)$. Similar to the proof of Theorem 1.1, one can get that for any $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$,

$$\Lambda_{s_1,s_2,\ldots,s_k}(\theta) = \lim_{n\to\infty} \frac{1}{b^2(n)} \log E\mathrm{e}^{b(n)\sqrt{n}\theta\langle\lambda, U_n - EU_n\rangle} = \sum_{i=1}^{k} \theta_i^2 \int_{s_{i-1}}^{s_i} \sigma_s^4 \mathrm{d}s$$

By Gärtner–Ellis theorem,

$$\left\{P\left(\frac{n}{b_n}(U_n - EU_n(s_1, s_2, \ldots, s_k)) \in \cdot\right), n \geq 1\right\}$$

satisfies the large deviations in $\mathbb{R}^k$ with the speed $\frac{1}{b^2(n)}$ and the rate function

$$\Lambda^*_{s_1,s_2,\ldots,s_k}(x) = \frac{1}{4}\sum_{i=1}^{k} \frac{x_i^2}{\int_{s_{i-1}}^{s_i} \sigma_s^4 \mathrm{d}s}.$$

Then by the inverse contraction principle,

$$\left\{P\left(\frac{n}{b_n}(Z_n - EZ_n) \in \cdot\right), n \geq 1\right\}$$

satisfies large deviations with speed $b^2(n)$ and the rate function $I_{s_1,s_2,\ldots,s_k}(x)$. □

Lemma 3.3. *For any $s \in [0, 1]$ and $\eta > 0$,*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{b^2(n)} \log P\left(\frac{\sqrt{n}}{b(n)} \sup_{s\leq t\leq s+\delta} \left|V_t^n(X) - V_s^n(X) - \int_s^t \sigma_u^2 \mathrm{d}u\right| > \eta\right) = -\infty.$$

Proof. By Lemma 3.1, we only have to show that

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{b^2(n)} \log P\left(\frac{\sqrt{n}}{b(n)} \sup_{s\leq t\leq s+\delta} \left|\Delta_s^t V^n(X) - E\Delta_s^t V^n(X)\right| > \eta\right) = -\infty \tag{3.1}$$

where $\Delta_s^t V^n(X) = V_t^n(X) - V_s^n(X)$.

In fact, $\left(V_t^n(X) - EV_t^n(X)\right)$ is a $\mathcal{F}_{[nt]/n}$-martingale. Then

$$\exp\left(\theta\left(V_t^n(X) - EV_t^n(X)\right)\right)$$

is a submartingale. By the maximal inequality, we have

$$P\left(\frac{\sqrt{n}}{b(n)} \sup_{s\leq t\leq s+\delta} \left(\Delta_s^t V^n(X) - E\Delta_s^t V^n(X)\right) > \eta\right) \leq \mathrm{e}^{-b^2(n)\theta\eta} E\exp\left(b(n)\sqrt{n}\theta\Delta_s^{s+\delta}(V^n(X) - EV^n(X))\right).$$

By Theorem 1.1, we have

$$\begin{aligned}\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{b^2(n)} \log P\left(\frac{\sqrt{n}}{b(n)} \sup_{s\leq t\leq s+\delta} \left(\Delta_s^t V^n(X) - E\Delta_s^t V^n(X)\right) > \eta\right) &\leq \lim_{\delta\downarrow 0} \inf_{\theta>0}\left\{-\theta\eta + \theta^2 \int_s^{s+\delta} \sigma_u^4 \mathrm{d}u\right\} \\ &= -\lim_{\delta\downarrow 0} \frac{\eta^2}{\int_s^{s+\delta} \sigma_u^4 \mathrm{d}u} = -\infty.\end{aligned}$$

Similarly,

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{b^2(n)} \log P\left(\frac{\sqrt{n}}{b(n)} \inf_{s\leq t\leq s+\delta} \left(\Delta_s^t V^n(X) - E\Delta_s^t V^n(X)\right) < -\eta\right) = -\infty.$$

Hence (3.1) follows from the above estimations. □

Proof of Theorem 1.2. By Lemmas 3.2 and 3.3,

$$\left\{P\left(\frac{\sqrt{n}}{b(n)}\left(V_t^n(X)-\int_0^t \sigma_s^2 \mathrm{d}s\right)_{t\in[0,1]}\in\cdot\right), n\geq 1\right\}$$

satisfies the large deviations with the speed b_n^2 and the rate function

$$\hat{J}(x)=\sup\left\{I_{s_1,\ldots,s_k}\big(x(s_1),\ldots,x(s_k)\big); 0=s_0<s_1<\cdots<s_k\leq 1, k\geq 0\right\},\quad x\in D_0[0,1].$$

It is enough to prove that $\hat{J}(x)=J(x)$. For $x\in D_0[0,1]$, set $F(t)=\int_0^t \sigma_s^4 I_{\{s:\sigma_s>0\}}\mathrm{d}u$ for $s\in[0,1]$. If $J(x)<+\infty$, then for $0=s_0<s_1<\cdots<s_k\leq 1$,

$$\begin{aligned}
\sum_{j=1}^k \frac{(x(s_j)-x(s_{j-1}))^2}{4\int_{s_{j-1}}^{s_j}\sigma_u^4\mathrm{d}u} &= \frac{1}{4}\sum_{j=1}^k\big(F(s_j)-F(s_{j-1})\big)\left(\frac{x(s_j)-x(s_{j-1})}{F(s_j)-F(s_{j-1})}\right)^2\\
&= \frac{1}{4}\sum_{j=1}^k (F(s_j)-F(s_{j-1}))\left(\frac{1}{F(s_j)-F(s_{j-1})}\int_{s_{j-1}}^{s_j}\frac{x'(u)}{\sigma_u^4}\mathrm{d}F(u)\right)^2\\
&\leq \frac{1}{4}\int_0^1 \frac{\big(x'(t)\big)^2}{\sigma_t^4} I_{\{t:\sigma_t>0\}}\mathrm{d}t
\end{aligned}$$

which implies that $\hat{J}(x)\leq J(x)$.

On the other hand, suppose $\hat{J}(x)<+\infty$. Denote by $\alpha_i^n=\frac{i}{2^n}, 0\leq i\leq 2^n$. If $\mathrm{d}x \ll \sigma_t^4\mathrm{d}t$, by the convergence of martingales,

$$\psi^n(s)=\sum_{i=0}^{2^n}\frac{x(\alpha_{i+1}^n)-x(\alpha_i^n)}{\alpha_{i+1}^n-\alpha_i^n} I_{[\alpha_i^n,\alpha_i^n)}(t)\to\frac{x'(t)}{\sigma_t^4},\quad \mathrm{d}F\text{-}a.s.$$

By Fatou lemma,

$$\frac{1}{4}\int_0^1\left(\frac{x'(s)}{\sigma_t^4}\right)^2\mathrm{d}F(t)\leq\frac{1}{4}\limsup_{n\to\infty}\int_0^1|\psi^n(t)|^2\mathrm{d}F(t)\leq\hat{J}(x)<+\infty.$$

Therefore, $J(x)=\hat{J}(x)$. □

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