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Analytical vectorial structure of a Lorentz–Gauss beam in the far field

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Abstract A description of a Lorentz–Gauss beam is made directly starting with the Maxwell equations. Based on the vector angular spectrum representation of the Maxwell equations and the method of stationary phase, the analytical TE and TM terms of a Lorentz–Gauss beam have been presented in the far field. The TE and TM terms are orthogonal to each other in the far field. The energy flux distributions of a Lorentz–Gauss beam and its TE and TM terms are depicted in the far field reference plane. The influences of the different parameters on the energy flux distributions of a Lorentz–Gauss beam and its TE and TM terms are discussed. Moreover, the vectorial structure of a Lorentz–Gauss beam is also compared with that of a Gaussian beam. This research is useful to the descriptions and applications of highly divergent laser beams.

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1 Introduction

Lorentz–Gauss beams have been recently introduced as a new kind of realizable beams [1]. The Lorentz beam can be regarded as a special case of Lorentz–Gauss beams. With the spatial extension being the same, the angular spreading of a Lorentz–Gaussian distribution is higher than that of a Gaussian description [2]. Therefore, the Lorentz–Gauss beam provides an appropriate model to describe some certain laser sources, e.g., double heterojunction $Ga_{1\text{-}x}Al_xAs$ lasers, which produce highly divergent fields [2, 3]. Moreover, the closed-form scalar representation of a Lorentz–Gauss beam on a general transverse plane has been presented [1]. While the purpose of this paper is just to provide a novel approach to reveal the vectorial composition of a Lorentz–Gauss beam. Here, the description of a Lorentz–Gauss beam is made directly starting with the Maxwell equations. Moreover, the vector angular spectrum method is used to resolve the Maxwell equations. As the vector angular spectrum can be separated into two terms in the frequency domain, the Lorentz–Gauss beam is decomposed into the TE and TM terms. The TE term denotes the electric field transverse to the propagation axis, and the TM term means the associated magnetic field transverse to the propagation axis [4–8]. Since the divergence condition of the electric field should be satisfied and the polarization direction of every plane wave component must be perpendicular to its own wave vector, the TE and TM terms of a Lorentz–Gauss beam is unique. The TE and TM terms are orthogonal to each other in the far field. Meanwhile, some researches and applications are conducted in the far field. By means of the method of stationary phase, therefore, the analytical expressions of the TE and TM terms will be presented in the far field. The influences of the different parameters on the energy flux distributions of a Lorentz–Gauss beam and its TE and TM terms are discussed. The vectorial structure of a Lorentz–Gauss beam is also compared with that of a Gaussian beam.

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2 Analytical vectorial structure in the far field

In the Cartesian coordinate system, the z-axis is taken to be the propagation axis. In the present paper, the description of a Lorentz–Gauss beam is directly derived from the Maxwell equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) - ik\boldsymbol{H}(\boldsymbol{r}) = 0, \tag{1}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) + ik\boldsymbol{E}(\boldsymbol{r}) = 0, \tag{2}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = 0, \tag{3}$$

where $\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $k = 2\pi/\lambda$; $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$ are the propagating electromagnetic fields. When the Maxwell equations are transformed from the space domain into the frequency domain, (1)–(3) become

$$\boldsymbol{L} \times \tilde{\boldsymbol{E}}(p,q,z) - ik\tilde{\boldsymbol{H}}(p,q,z) = 0, \tag{4}$$

$$\boldsymbol{L} \times \tilde{\boldsymbol{H}}(p,q,z) + ik\tilde{\boldsymbol{E}}(p,q,z) = 0, \tag{5}$$

$$\boldsymbol{L} \cdot \tilde{\boldsymbol{E}}(p,q,z) = \boldsymbol{L} \cdot \tilde{\boldsymbol{H}}(p,q,z) = 0, \tag{6}$$

where $\boldsymbol{L} = ikp\mathbf{i} + ikq\mathbf{j} + \partial/\partial z\mathbf{k}$, p/λ and q/λ are the transverse frequencies, and $\tilde{\boldsymbol{E}}(p,q,z)$ and $\boldsymbol{E}(\boldsymbol{r})$ are the spatial Fourier transform pair [9–11]:

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\boldsymbol{E}}(p,q,z)\exp\big[ik(px+qy)\big]\,dp\,dq. \tag{7}$$

$\tilde{\boldsymbol{H}}(p,q,z)$ and $\boldsymbol{H}(\boldsymbol{r})$ are also the spatial Fourier transform pair:

$$\boldsymbol{H}(\boldsymbol{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\boldsymbol{H}}(p,q,z)\exp\big[ik(px+qy)\big]\,dp\,dq. \tag{8}$$

The solutions of (4)–(6) can be expressed in the form

$$\tilde{\boldsymbol{E}}(p,q,z) = \boldsymbol{A}(p,q)\exp(ikmz), \tag{9}$$

$$\tilde{\boldsymbol{H}}(p,q,z) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\big[\boldsymbol{s} \times \boldsymbol{A}(p,q)\big]\exp(ikmz), \tag{10}$$

where $\boldsymbol{A}(p,q)$ is the vector angular spectrum, $m = (1 - p^2 - q^2)^{1/2}$ for homogeneous waves and $m = i(p^2 + q^2 - 1)^{1/2}$ for evanescent waves, and $\boldsymbol{s} = p\mathbf{i} + q\mathbf{j} + m\mathbf{k}$. ε_0 and μ_0 are the electric permittivity and the magnetic permeability in vacuum, respectively. The vector angular spectrum $\boldsymbol{A}(p,q)$ reads as

$$\boldsymbol{A}(p,q) = A_x(p,q)\mathbf{i} + A_y(p,q)\mathbf{j} + A_z(p,q)\mathbf{k}. \tag{11}$$

The transverse components of the vector angular spectrum $A_x(p,q)$ and $A_y(p,q)$ are given by the Fourier transform of the input condition [12]

$$A_x(p,q) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(x_0,y_0,0) \times \exp\big[-ik(px_0+qy_0)\big]\,dx_0\,dy_0, \tag{12}$$

$$A_y(p,q) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(x_0,y_0,0) \times \exp\big[-ik(px_0+qy_0)\big]\,dx_0\,dy_0, \tag{13}$$

where $E_x(x_0,y_0,0)$ and $E_y(x_0,y_0,0)$ are the input transverse electric fields of a Lorentz–Gauss beam. The longitudinal component of the vector angular spectrum $A_z(p,q)$ is given by the orthogonal relation $\boldsymbol{s} \cdot \boldsymbol{A}(p,q) = 0$ and turns out to be

$$A_z(p,q) = -\big[pA_x(p,q) + qA_y(p,q)\big]/m. \tag{14}$$

In the frequency domain, two unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ can be defined by [4–8]

$$\boldsymbol{e}_1 = \frac{q}{b}\mathbf{i} - \frac{p}{b}\mathbf{j}, \qquad \boldsymbol{e}_2 = \frac{pm}{b}\mathbf{i} + \frac{qm}{b}\mathbf{j} - b\mathbf{k}, \tag{15}$$

where $b = (p^2 + q^2)^{1/2}$. Thus, the three unit vectors $\boldsymbol{s}$, $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$ form the mutually perpendicular right-handed system

$$\boldsymbol{s} \times \boldsymbol{e}_1 = \boldsymbol{e}_2, \qquad \boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{s}, \qquad \boldsymbol{e}_2 \times \boldsymbol{s} = \boldsymbol{e}_1. \tag{16}$$

In the frequency domain, the vector angular spectrum $\boldsymbol{A}(p,q)$ can be decomposed into two terms:

$$\boldsymbol{A}(p,q) = \big[\boldsymbol{A}(p,q).\boldsymbol{e}_1\big]\boldsymbol{e}_1 + \big[\boldsymbol{A}(p,q).\boldsymbol{e}_2\big]\boldsymbol{e}_2. \tag{17}$$

Therefore, (9) and (10) can be rewritten as follows:

$$\tilde{\boldsymbol{E}}(p,q,z) = \big[\boldsymbol{A}(p,q).\boldsymbol{e}_1\big]\boldsymbol{e}_1\exp(ikmz) + \big[\boldsymbol{A}(p,q).\boldsymbol{e}_2\big]\boldsymbol{e}_2\exp(ikmz), \tag{18}$$

$$\tilde{\boldsymbol{H}}(p,q,z) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\big\{\big[\boldsymbol{A}(p,q).\boldsymbol{e}_1\big]\boldsymbol{e}_2 - \big[\boldsymbol{A}(p,q).\boldsymbol{e}_2\big]\boldsymbol{e}_1\big\}\exp(ikmz). \tag{19}$$

Accordingly, the propagating electric field of a Lorentz–Gauss beam can be decomposed into the TE and TM terms

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r}) + \boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r}) \tag{20}$$

with $\boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r})$ and $\boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r})$ given by

$$\boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[\boldsymbol{A}(p,q).\boldsymbol{e}_1\big]\boldsymbol{e}_1\exp(ik\boldsymbol{r}.\boldsymbol{s})\,dp\,dq, \tag{21}$$

$$\boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[\boldsymbol{A}(p,q).\boldsymbol{e}_2\big]\boldsymbol{e}_2\exp(ik\boldsymbol{r}.\boldsymbol{s})\,dp\,dq. \tag{22}$$

Similarly, the propagating magnetic field of a Lorentz–Gauss beam can also be expressed as the sum of the corresponding TE and TM terms

$$\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}_{\mathrm{TE}}(\boldsymbol{r}) + \boldsymbol{H}_{\mathrm{TM}}(\boldsymbol{r}) \tag{23}$$

with $\boldsymbol{H}_{\mathrm{TE}}(\boldsymbol{r})$ and $\boldsymbol{H}_{\mathrm{TM}}(\boldsymbol{r})$ given by

$$\boldsymbol{H}_{\mathrm{TE}}(\boldsymbol{r}) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[\boldsymbol{A}(p,q).\boldsymbol{e}_1\big]\boldsymbol{e}_2 \times \exp(ik\boldsymbol{r}.\boldsymbol{s})\,dp\,dq, \tag{24}$$

$$\boldsymbol{H}_{\mathrm{TM}}(\boldsymbol{r}) = -\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\boldsymbol{A}(p,q).\boldsymbol{e}_2]\boldsymbol{e}_1 \times \exp(ik\boldsymbol{r}.\boldsymbol{s})\, dp\, dq. \quad (25)$$

Now, we consider the special case that a Lorentz–Gauss beam is polarized along the $x - z$ plane and propagates toward the half free space $z \geq 0$. At the $z = 0$ source plane, the input transverse electric field of a Lorentz–Gauss beam takes the form [1]

$$\begin{pmatrix} E_x(x_0,y_0,0) \\ E_y(x_0,y_0,0) \end{pmatrix} = \begin{pmatrix} \frac{1}{w_{0x}w_{0y}} \frac{1}{1+(x_0/w_{0x})^2} \frac{1}{1+(y_0/w_{0y})^2} \exp(-\frac{x_0^2+y_0^2}{w_0^2}) \\ 0 \end{pmatrix}, \quad (26)$$

where w_{0x} and w_{0y} are the parameters related to the beam widths in the x and y directions, respectively, and w_0 is the waist of the Gaussian beam. The time dependent factor $\exp(-\mathrm{i}\omega t)$ is omitted in (26), and ω is the circular frequency. According to (12), the x component of the vector angular spectrum $A_x(p,q)$ turns out to be

$$A_x(p,q) = \frac{\pi^2}{4\lambda^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right) \times \left\{\exp\left(-\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}-\frac{p}{2f}\right)\right] + \exp\left(\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}+\frac{p}{2f}\right)\right]\right\} \times \left\{\exp\left(-\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}-\frac{q}{2f}\right)\right] + \exp\left(\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}+\frac{q}{2f}\right)\right]\right\}, \quad (27)$$

where

$$f = 1/kw_0, \qquad f_x = 1/kw_{0x}, \qquad f_y = 1/kw_{0y}. \quad (28)$$

The y component of the vector angular spectrum $A_y(p,q)$ is verified to be zero. After lengthy calculation, the TE and TM terms of the propagating electric field for a Lorentz–Gauss beam are found to be

$$\boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r}) = \frac{\pi^2}{4\lambda^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{q}{b^2} \times \left\{\exp\left(\frac{-p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}-\frac{p}{2f}\right)\right] + \exp\left(\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}+\frac{p}{2f}\right)\right]\right\} \times \left\{\exp\left(\frac{-q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}-\frac{q}{2f}\right)\right] + \exp\left(\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}+\frac{q}{2f}\right)\right]\right\}(q\mathbf{i}-p\mathbf{j}) \times \exp(ik\boldsymbol{r}.\boldsymbol{s})\, dp\, dq, \quad (29)$$

$$\boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r}) = \frac{\pi^2}{4\lambda^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p}{b^2 m} \times \left\{\exp\left(\frac{-p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}-\frac{p}{2f}\right)\right] + \exp\left(\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}+\frac{p}{2f}\right)\right]\right\} \times \left\{\exp\left(\frac{-q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}-\frac{q}{2f}\right)\right] + \exp\left(\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}+\frac{q}{2f}\right)\right]\right\} \times \left(pm\mathbf{i}+qm\mathbf{j}-b^2\mathbf{k}\right)\exp(ik\boldsymbol{r}.\boldsymbol{s})\, dp\, dq, \quad (30)$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-s^2)\, ds \quad (31)$$

is the usual error function [13]. Similarly, the corresponding TE and TM terms of the propagating magnetic field for a Lorentz–Gauss beam read as

$$\boldsymbol{H}_{\mathrm{TE}}(\boldsymbol{r}) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi^2}{4\lambda^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{q}{b^2} \times \left\{\exp\left(-\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}-\frac{p}{2f}\right)\right] + \exp\left(\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}+\frac{p}{2f}\right)\right]\right\} \times \left\{\exp\left(-\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}-\frac{q}{2f}\right)\right] + \exp\left(\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}+\frac{q}{2f}\right)\right]\right\} \times \left(pm\mathbf{i}+qm\mathbf{j}-b^2\mathbf{k}\right)\exp(ik\boldsymbol{r}.\boldsymbol{s})\, dp\, dq, \quad (32)$$

$$\boldsymbol{H}_{\mathrm{TM}}(\boldsymbol{r}) = -\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi^2}{4\lambda^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p}{b^2 m} \times \left\{\exp\left(-\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}-\frac{p}{2f}\right)\right] + \exp\left(\frac{p}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}+\frac{p}{2f}\right)\right]\right\} \times \left\{\exp\left(-\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}-\frac{q}{2f}\right)\right]\right.$$

$$+\exp\left(\frac{q}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}+\frac{q}{2f}\right)\right]\Bigg\}$$
$$\times (q\mathbf{i}-p\mathbf{j})\exp(ik\boldsymbol{r}.\boldsymbol{s})\,dp\,dq. \tag{33}$$

Here, the TE and TM terms denote that the longitudinal component of the electric and the magnetic fields is equal to zero, respectively. The Lorentz–Gauss beam can be decomposed into the transverse and longitudinal components in the spatial domain, while the decomposition of Lorentz–Gauss beam into the TE and TM terms is carried out in terms of the frequency domain.

When z is larger than several light wavelengths, the evanescent waves have almost fallen down. As z is large enough in the far field, the evanescent waves have completely disappeared. Thus, the integral in (29), (30), (32), and (33) should be restrained within the range of $0 \le b \le 1$. In the far field regime, the condition $kr = k(x^2 + y^2+z^2)^{1/2} \to \infty$ is satisfied. By means of the method of stationary phase, therefore, the analytical propagating electromagnetic fields of the TE and TM terms in the far field can be presented. According to the method of stationary phase [14, 15], the surface integral of (29) is shown to have the asymptotic value [16]

$$\boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r}) = \frac{i\lambda}{r}\sum_{j=1}^{n}\frac{\varepsilon_j}{(|\alpha_j\beta_j-\gamma_j^2|)^{1/2}}\boldsymbol{M}(p_j,q_j)$$
$$\times\exp\big[ikrF(p_j,q_j,x,y)\big]\quad \text{as } kr\to\infty, \tag{34}$$

where

$$\boldsymbol{M}(p_j,q_j) = \frac{\pi^2}{4\lambda^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)\frac{q_j}{p_j^2+q_j^2}$$
$$\times\left\{\exp\left(\frac{-p_j}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}-\frac{p_j}{2f}\right)\right]\right.$$
$$\left.+\exp\left(\frac{p_j}{f_x}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}+\frac{p_j}{2f}\right)\right]\right\}$$
$$\times\left\{\exp\left(\frac{-q_j}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}-\frac{q_j}{2f}\right)\right]\right.$$
$$\left.+\exp\left(\frac{q_j}{f_y}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}+\frac{q_j}{2f}\right)\right]\right\}$$
$$\times (q_j\mathbf{i}-p_j\mathbf{j}) \tag{35}$$

and

$$F(p_j,q_j,x,y) = \big[p_jx+q_jy+\big(1-p_j^2-q_j^2\big)^{1/2}z\big]/r. \tag{36}$$

n is the number of stationary points. The stationary points (p_j,q_j) are solutions of the equations

$$\left.\frac{\partial F(p,q,x,y)}{\partial p}\right|_{\substack{p=p_j\\q=q_j}} = \frac{1}{r}\left(x-\frac{zp_j}{(1-p_j^2-q_j^2)^{1/2}}\right)=0, \tag{37}$$

$$\left.\frac{\partial F(p,q,x,y)}{\partial q}\right|_{\substack{p=p_j\\q=q_j}} = \frac{1}{r}\left(y-\frac{zq_j}{(1-p_j^2-q_j^2)^{1/2}}\right)=0. \tag{38}$$

By solving (37) and (38), one can obtain

$$p_1 = x/r,\qquad q_1=y/r. \tag{39}$$

There is only one stationary point, which means $n=1$. As a result, the parameters $\alpha_1,\beta_1,\gamma_1$, and ε_1 turn out to be

$$\alpha_1 = \left.\frac{\partial F^2(p,q,x,y)}{\partial p^2}\right|_{\substack{p=p_1\\q=q_1}} = \frac{-z}{r(1-p_1^2-q_1^2)^{1/2}}$$
$$\times\left(1+\frac{p_1^2}{1-p_1^2-q_1^2}\right) = -\left(1+\frac{x^2}{z^2}\right), \tag{40}$$

$$\beta_1 = \left.\frac{\partial F^2(p,q,x,y)}{\partial q^2}\right|_{\substack{p=p_1\\q=q_1}} = \frac{-z}{r(1-p_1^2-q_1^2)^{1/2}}$$
$$\times\left(1+\frac{q_1^2}{1-p_1^2-q_1^2}\right) = -\left(1+\frac{y^2}{z^2}\right), \tag{41}$$

$$\gamma_1 = \left.\frac{\partial F^2(p,q,x,y)}{\partial p\partial q}\right|_{\substack{p=p_1\\q=q_1}} = -\frac{zp_1q_1}{r(1-p_1^2-q_1^2)^{3/2}}$$
$$= -\frac{xy}{z^2}, \tag{42}$$

$$\varepsilon_1 = -1. \tag{43}$$

Substituting (35)–(43) into (34), the analytical electric field of TE term in the far field is found to be

$$\boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r}) = -\frac{i\pi^2yz}{4\lambda\rho^2r^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)$$
$$\times\left\{\exp\left(-\frac{x}{f_xr}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]\right.$$
$$\left.+\exp\left(\frac{x}{f_xr}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}$$
$$\times\left\{\exp\left(-\frac{y}{f_yr}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]\right.$$
$$\left.+\exp\left(\frac{y}{f_yr}\right)\left[1-\mathrm{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}$$
$$\times (y\mathbf{i}-x\mathbf{j})\exp(ikr), \tag{44}$$

where $\rho=(x^2+y^2)^{1/2}$. The corresponding magnetic field of the TE term in the far field turns out to be

$$\boldsymbol{H}_{\mathrm{TE}}(\boldsymbol{r}) = -\frac{i\pi^2yz}{4\lambda\rho^2r^3}\sqrt{\frac{\varepsilon_0}{\mu_0}}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)$$

$$\times\left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]+\exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}$$
$$\times\left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]+\exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}$$
$$\times\left(xz\mathbf{i}+yz\mathbf{j}-\rho^2\mathbf{k}\right)\exp(ikr). \tag{45}$$

Similarly, the analytical TM term in the far field reads as

$$\boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r})=-\frac{i\pi^2 x}{4\lambda\rho^2 r^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)$$
$$\times\left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]+\exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}$$
$$\times\left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]+\exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}$$
$$\times(xz\mathbf{i}+yz\mathbf{j}-\rho^2\mathbf{k})\exp(ikr), \tag{46}$$

$$\boldsymbol{H}_{\mathrm{TM}}(\boldsymbol{r})=\frac{i\pi^2 x}{4\lambda\rho^2 r}\sqrt{\frac{\varepsilon_0}{\mu_0}}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)$$
$$\times\left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]+\exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}$$
$$\times\left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]+\exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}$$
$$\times(y\mathbf{i}-x\mathbf{j})\exp(ikr). \tag{47}$$

Apparently, the TE and TM terms are orthogonal to each other in the far field. The exclusive condition under which the method of stationary phase can be applied is $kr\to\infty$ and independent of the parameters w_0, w_{0x}, and w_{0y}. In other words, the three parameters w_0, w_{0x}, and w_{0y} in (44)–(47) may take any values of physical meaning. Therefore, the analytical TE and TM terms of a Lorentz–Gauss beam obtained here are applicable not only to the paraxial case but also to the non-paraxial case.

The propagating electric field of a Lorentz–Gauss beam is given by

$$\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r})+\boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r})=-\frac{i\pi^2 z}{4\lambda r^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)$$
$$\times\left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]+\exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}$$
$$\times\left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]+\exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}$$
$$\times\left(\mathbf{i}-\frac{x}{z}\mathbf{k}\right)\exp(ikr). \tag{48}$$

In the paraxial case, the electric field is confined to the paraxial realm. Therefore, the above equation can be further simplified to be

$$\boldsymbol{E}(\boldsymbol{r})=-\frac{i\pi^2 z}{4\lambda r^2}\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right)$$
$$\times\left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]+\exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}$$
$$\times\left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]+\exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}$$
$$\times\mathbf{i}\exp(ikr). \tag{49}$$

The closed-form scalar representation of a Lorentz–Gauss beam on a general transverse plane has been presented as follows [1]:

$$\boldsymbol{E}(\boldsymbol{r})=-\frac{i\pi^2}{4\lambda z}\exp\left[ikz-\frac{x^2+y^2}{w_0^2(1+iz/l)}\right]$$
$$\times\left[V_x^+\left(\frac{x}{1+iz/l},\frac{z}{1+iz/l}\right)+V_x^-\left(\frac{x}{1+iz/l},\frac{z}{1+iz/l}\right)\right]$$
$$\times\left[V_y^+\left(\frac{y}{1+iz/l},\frac{z}{1+iz/l}\right)+V_y^-\left(\frac{y}{1+iz/l},\frac{z}{1+iz/l}\right)\right], \tag{50}$$

where $l = kw_0^2/2$, and $V_c^{\pm}(c,z)$ is given by

$$V_c^{\pm}(c,z) = \exp\left(\frac{k}{2iz}(w_{0c} \pm ic)^2\right) \times \left[1 - \operatorname{erf}\left(\sqrt{\frac{k}{2iz}}(w_{0c} \pm ic)\right)\right], \quad (51)$$

where $c = x$ or y. In the far field, (50) reduces to be

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) = & -\frac{i\pi^2}{4\lambda z}\exp(ikz)\exp\left(\frac{f^2}{f_x^2}+\frac{f^2}{f_y^2}\right) \\ & \times \left\{\exp\left(-\frac{x}{f_x z}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fz}\right)\right]\right. \\ & \left. + \exp\left(\frac{x}{f_x z}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fz}\right)\right]\right\} \\ & \times \left\{\exp\left(-\frac{y}{f_y z}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fz}\right)\right]\right. \\ & \left. + \exp\left(\frac{y}{f_y z}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fz}\right)\right]\right\}. \end{aligned} \quad (52)$$

As to the paraxial case, $r = (z^2+\rho^2)^{1/2} = z + \frac{\rho^2}{2z} \approx z$. In the paraxial case, therefore, the vector Lorentz–Gauss beam obtained here is approximately equal to the scalar Lorentz–Gauss beam in the far field.

3 Energy flux distributions in the far field

According to the above analytical electromagnetic representations of the TE and TM terms, their energy flux distributions can be investigated in the far field. The energy flux distribution at the plane z =const is described by the z component of the time-average Poynting vector

$$\langle S_z \rangle = \frac{1}{2}\mathrm{Re}\left[\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})\right]_z, \quad (53)$$

where Re means taking the real part, and the asterisk denotes complex conjugation. Therefore, the energy flux distributions of the TE and TM terms at the far field plane z = const are given by

$$\begin{aligned} \langle S_z \rangle_{\mathrm{TE}} = & \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi^4 y^2 z^3}{32\lambda^2\rho^2 r^5}\exp\left(\frac{2f^2}{f_x^2}+\frac{2f^2}{f_y^2}\right) \\ & \times \left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}^2 \\ & \times \left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}^2, \end{aligned} \quad (54)$$

$$\begin{aligned} \langle S_z \rangle_{\mathrm{TM}} = & \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi^4 x^2 z}{32\lambda^2\rho^2 r^3}\exp\left(\frac{2f^2}{f_x^2}+\frac{2f^2}{f_y^2}\right) \\ & \left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}^2 \\ & \times \left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}^2. \end{aligned} \quad (55)$$

Since the TE and TM terms are orthogonal to each other in the far field, the energy flux distribution of a Lorentz–Gauss beam yields

$$\begin{aligned} \langle S_z \rangle = \langle S_z \rangle_{\mathrm{TE}} + \langle S_z \rangle_{\mathrm{TM}} = & \sqrt{\frac{\varepsilon_0}{\mu_0}}\exp\left(\frac{2f^2}{f_x^2}+\frac{2f^2}{f_y^2}\right) \\ & \times \left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}^2 \\ & \times \frac{\pi^4 z}{32\lambda^2\rho^2 r^3}\left(x^2+\frac{z^2}{r^2}y^2\right) \\ & \times \left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}^2. \end{aligned} \quad (56)$$

In the paraxial case, (56) reduces to be

$$\begin{aligned} \langle S_z \rangle = & \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi^4 z}{32\lambda^2 r^3}\exp\left(\frac{2f^2}{f_x^2}+\frac{2f^2}{f_y^2}\right) \\ & \times \left\{\exp\left(-\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}-\frac{x}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{x}{f_x r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_x}+\frac{x}{2fr}\right)\right]\right\}^2 \\ & \times \left\{\exp\left(-\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}-\frac{y}{2fr}\right)\right]\right. \\ & \left. + \exp\left(\frac{y}{f_y r}\right)\left[1-\operatorname{erf}\left(\frac{f}{f_y}+\frac{y}{2fr}\right)\right]\right\}^2. \end{aligned} \quad (57)$$

For the sake of intuition, Figs. 1 and 2 show the energy flux distributions of the TE term, the TM term, and the whole beam of a Lorentz–Gauss beam at the far field plane

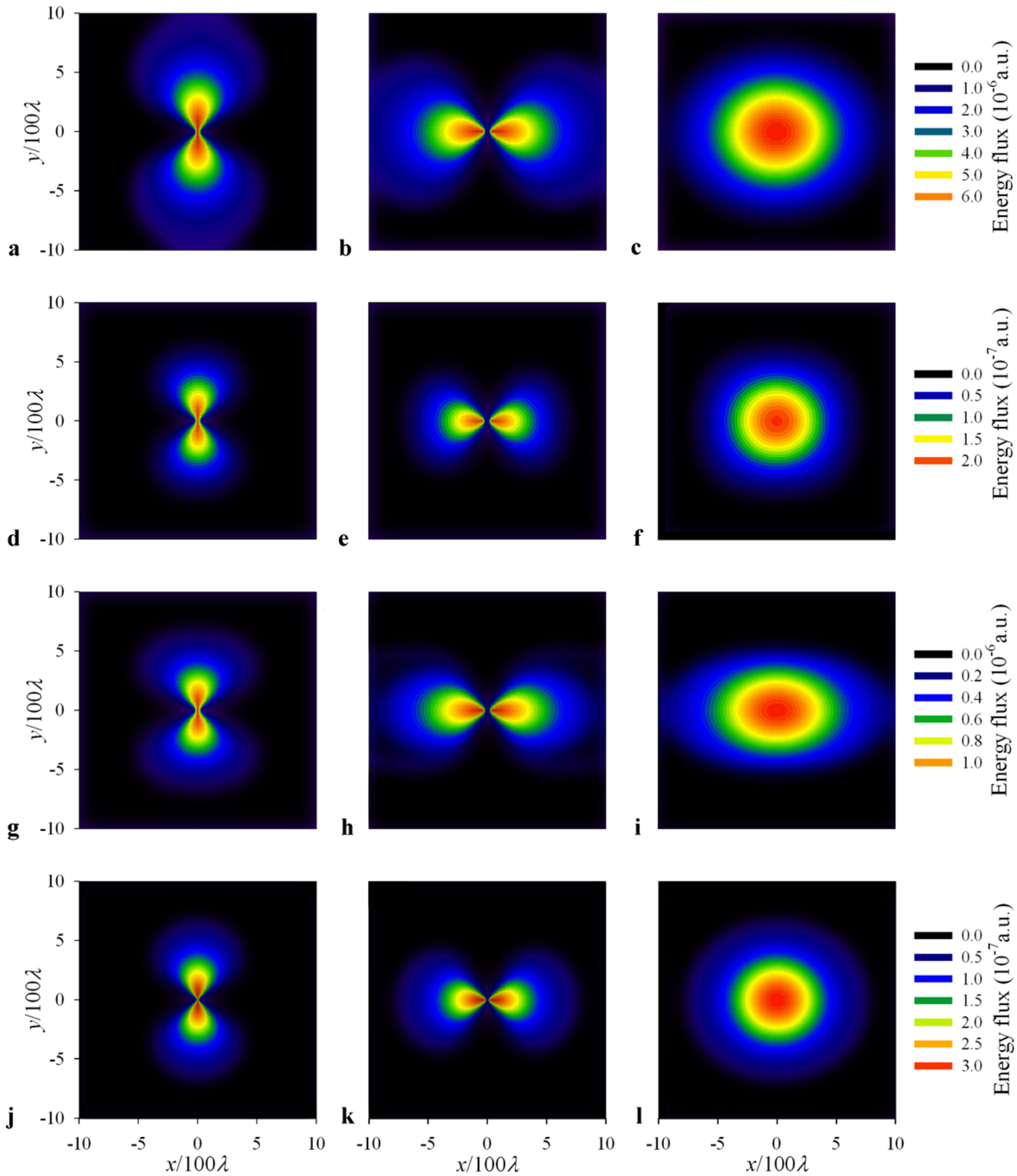

Fig. 1 The energy flux distribution of a Lorentz–Gauss beam (the *first*, the *second*, and the *third rows*) and a TE Gaussian beam (the *bottom row*) at the reference plane $z = 1000\lambda$. $w_0 = 0.5\lambda$. (**a**, **d**, **g**, **j**) The TE term. (**b**, **e**, **h**, **k**) The TM term. (**c**, **f**, **i**, **l**) The whole beam. $w_{0x} = w_{0y} = 0.25\lambda$ in (**a**)–(**c**), and $w_{0x} = w_{0y} = 1\lambda$ in (**d**)–(**f**). In (**g**)–(**i**), $w_{0x} = 0.25\lambda$, and $w_{0y} = 1\lambda$

$z = 1000\lambda$. For simplicity, ε_0/μ_0 is set to be unity. In Figs. 1 and 2, subfigures (a), (d), (g), and (j) represent the energy flux distributions of the TE term, and subfigures (b), (e), (h), and (k) the energy flux distributions of the TM term. Subfigures (c), (f), (i), and (l) refer to the energy flux distribution of the whole beam. In Fig. 1, $w_0 = 0.5\lambda$, which corresponds to the non-paraxial case. In Fig. 2, $w_0 = 5\lambda$, which stands for the paraxial case. $w_{0x} = w_{0y} = 0.25\lambda$ in Figs. 1(a)–1(c), and $w_{0x} = w_{0y} = 1\lambda$ in Figs. 1(d)–1(f). In Figs. 1(g)–1(i), $w_{0x} = 0.25\lambda$, and $w_{0y} = 1\lambda$. $w_{0x} = w_{0y} = 2.5\lambda$ in Figs. 2(a)–2(c), and $w_{0x} = w_{0y} = 10\lambda$ in Figs. 2(d)–2(f). In Figs. 2(g)–2(i), $w_{0x} = 2.5\lambda$, and $w_{0y} = 10\lambda$. For comparison, the energy flux distributions of a TE Gaussian beam, its TE and TM terms are also depicted in Figs. 1(j)–1(l) and 2(j)–2(l). Apparently, the TE and TM terms are orthogonal to each other in the far field. As to the TE term, the divergence in the x-direction is smaller than that in y-direction. While for the TM term, the divergence in the x-direction

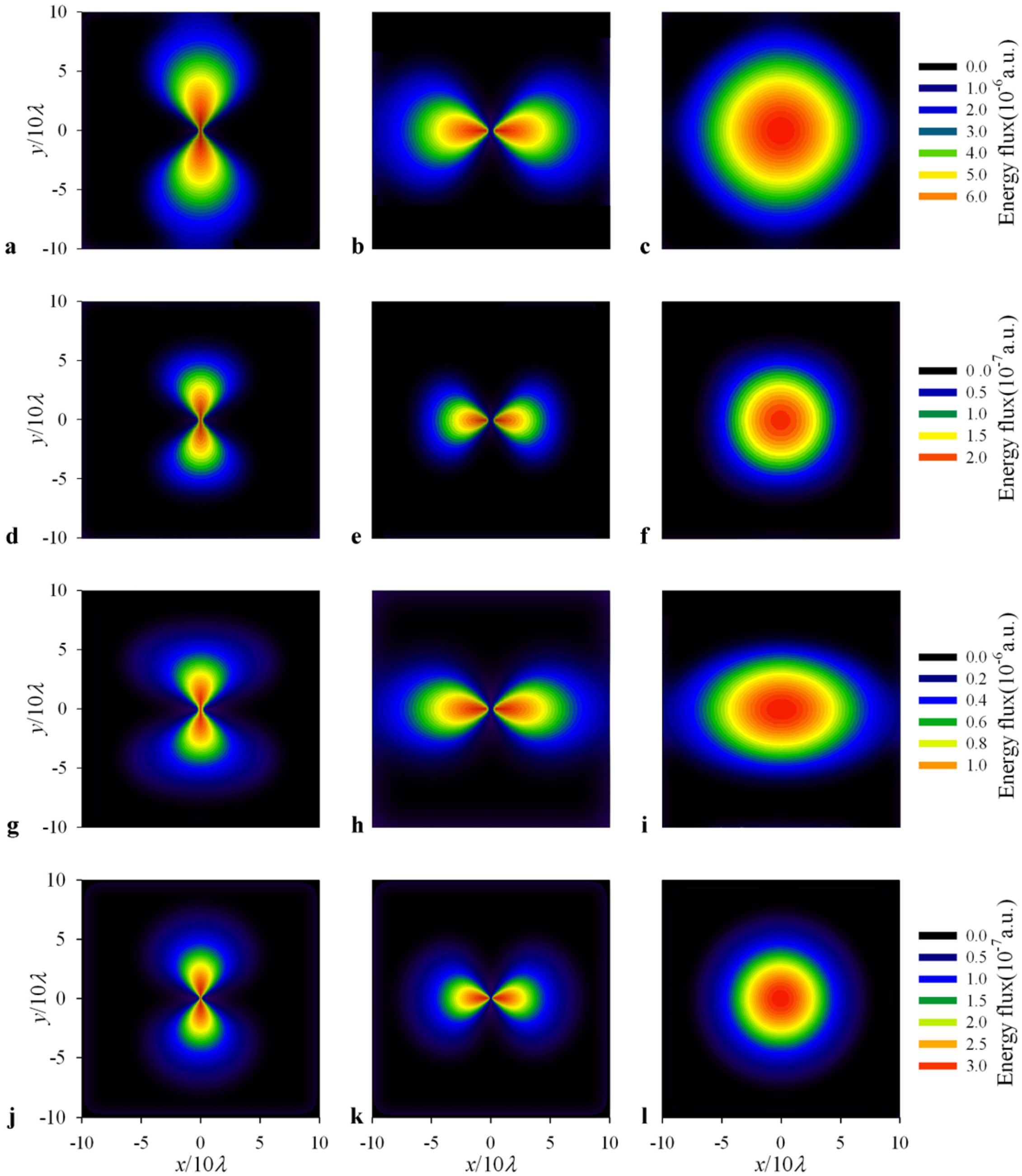

Fig. 2 The energy flux distribution of a Lorentz–Gauss beam (the *first*, the *second*, and the *third rows*) and a TE Gaussian beam (the *bottom row*) at the reference plane $z = 1000\lambda$. $w_0 = 5\lambda$. (**a**, **d**, **g**, **j**) The TE term. (**b**, **e**, **h**, **k**) The TM term. (**c**, **f**, **i**, **l**) The whole beam. $w_{0x} = w_{0y} = 2.5\lambda$ in (**a**)–(**c**), and $w_{0x} = w_{0y} = 10\lambda$ in (**d**)–(**f**). In (**g**)–(**i**), $w_{0x} = 2.5\lambda$, and $w_{0y} = 10\lambda$

is larger than that in y-direction. When w_{0x} and w_{0y} are both larger than w_0, the divergence a Lorentz–Gauss beam is slightly larger than that of a Gaussian beam. In this case, the beam spot of a Lorentz–Gauss beam is close to that of a Gaussian beam. When w_{0x} and w_{0y} are both smaller than w_0, the divergence of a Lorentz–Gauss beam is apparently larger than that of a Gaussian beam. In the non-paraxial case, the whole beam spot of a Lorentz–Gauss beam is elliptical though $w_{0x} = w_{0y}$. If $w_{0x} \neq w_{0y}$, the whole beam spot is elliptical. Moreover, the long-axis is located along the direction where the corresponding value of w_{0x} or w_{0y} is smaller. The beam spot shape of a Lorentz–Gauss beam depends on the choices of three parameters w_{0x}, w_{0y} and w_0. According to the practical conditions, therefore, the three parameters should be optimized. The degree of paraxiality has been introduced to evaluate the paraxiality of a monochromatic light beam [17]. Since the length of the paper is limited, the relation between the degree of paraxiality of a Lorentz–Gauss beam and the three parameters w_{0x}, w_{0y}, and w_0 will be reported elsewhere.

4 Conclusions

A description of a Lorentz–Gauss beam is made directly starting with the Maxwell equations. According to the vectorial structure of electromagnetic beam and the method of stationary phase, the analytical TE and TM terms of a Lorentz–Gauss beam have been presented in the far field. The TE and TM terms are orthogonal to each other in the far field. As to the TE term, the divergence in the x-direction is smaller than that in y-direction, while for the TM term, the divergence in the x-direction is larger than that in y-direction. The energy flux distributions of a Lorentz–Gauss beam, the TE term and the TM term are depicted in the far field reference plane, which distinctly reveals the vectorial composition of the Lorentz–Gauss beam. The influences of the different parameters on the energy flux distributions of a Lorentz–Gauss beam and its TE and TM terms are discussed. Moreover, the vectorial structure of a Lorentz–Gauss beam is also compared with that of a Gaussian beam. Compared with the Gaussian beam, the Lorentz–Gauss beam provides a better appropriate model to describe some certain laser sources with high divergence. The method of decomposition of a Lorentz–Gauss beam used here is different from that of [18], where the transverse electric fields are decomposed into two orthogonal components, and the longitudinal component keeps invariable. This research is useful for descriptions and applications of highly divergent laser beams.

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