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Uniform attractor for non-autonomous hyperbolic equation with critical exponent ☆

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ABSTRACT

In this paper, we study the uniform attractor for semilinear wave equation with mixed differential quotient terms and critical nonlinearity. We prove the existence of the uniform attractor in $H_0^1(\Omega) \times L^2(\Omega)$.

1. Introduction

In this paper, we consider the following weakly damped non-autonomous wave equation involving mixed differential quotient terms [12–14]

$$\begin{cases} u_{tt} + \beta u_t - \Delta u + \alpha \sum_{i=1}^{3} D_i u_t + f(u) = g(x,t), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u(x,\tau) = u_\tau^0(x), \quad u_t(x,\tau) = u_\tau^1(x), \end{cases} \tag{1.1}$$

here Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary, α, β are positive constants and the functions f and g satisfy the following conditions:

$$f \in C^1(\mathbb{R}), \quad |f'(s)| \leqslant C(1 + |s|^2), \tag{1.2}$$

$$f(s)s - m_1 F(s) \geqslant -m_2, \tag{1.3}$$

$$g(x,t) \in L^\infty(\mathbb{R}; L^2(\Omega)) \tag{1.4}$$

and

$$\partial_t g \in L_b^r(\mathbb{R}; L^r(\Omega)) \quad \text{with} \ r > \frac{6}{5}, \tag{1.5}$$

where m_1 and m_2 are positive constants. The number 2 is called the critical exponent, since the nonlinearity f is not compact in this case. It is clear that the external forces g satisfying (1.4) and (1.5) is translation bounded but not translation compact (the definition can be found in the beginning of the next section).

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In this paper, we consider the non-autonomous system (1.1) via the uniform attractors of the corresponding family of processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, the feature of the model (1.1) is that: (i) the wave equation involves mixed differential quotient terms, (ii) the nonlinearity $f(u)$ has critical exponent, and (iii) the external forcing $g(x,t)$ is not translation compact in $L^2_{\rm loc}(\mathbb{R};L^2(\Omega))$.

Let us recall some relevant research in this area. For the autonomous case, the global attractors for wave equations were studied in [1–7,9] for the linear damping case and [17–19,21,23,24] for the nonlinear damping case. In the case of non-autonomous system, in Chepyzhov and Vishik [7], the authors obtained the existence of a uniformly attractor when g is translation compact and nonlinearity f is subcritical. In Zelik [8], by use of a bootstrap argument together with a sharp use of Gronwall-type lemmas, when $g,\ \partial_t g\in L^\infty(\mathbb{R};L^2(\Omega))$, $f\in C^2(\mathbb{R})$, $f(0)=0$ and f is critical exponent, the author obtained some regularity estimates for the solutions, which implies naturally the existence of a uniform attractor. Under the assumptions that g and $\partial_t g$ are both in the space of bounded continuous functions $C_b(\mathbb{R};L^2(\Omega))$, Zhou and Wang [11] have proved the existence of kernel sections and obtained the estimation of the Hausdorff dimension of the kernel sections. Caraballo et al. [10] have discussed the pullback attractors for the case of subcritical nonlinearity.

The study of wave equations has attracted much attention and has made fast progress in recent years, the qualitative theories for quasilinear hyperbolic equations with damping were considered in [25–29]. But few of the wave equations involving mixed differential quotient terms, especially for the non-autonomous case. For the autonomous case, recently, Kurt [12] proved the existence of an absorbing set of (1.1) in $H^1_0(\Omega)\times L^2(\Omega)$ in the one-dimensional case. Subsequently, in [13], Zhang and Zhong obtained the global attractor of (1.1) in $H^1_0(\Omega)\times L^2(\Omega)$ for the one-dimensional case. In [14], Zhang showed the global attractor for the problem (1.1) in $H^1_0(\Omega)\times L^2(\Omega)$ with subcritical nonlinearity. In the case of non-autonomous systems, to our best knowledge, the problem is less clear. Therefore, it is necessary to extensively search. Our main goal in this paper is to prove the existence of a uniform attractor for the problem (1.1)–(1.5) in $H^1_0(\Omega)\times L^2(\Omega)$ and improve the result of [12–14]. We state our main result in Section 4, namely Theorem 4.6.

This paper is organized as follows: in Section 2, we give some preparations for our consideration; in Section 3, we prove the existence of an uniformly absorbing set in $H^1_0(\Omega)\times L^2(\Omega)$; in the last Section, we derive uniform asymptotic compactness of the corresponding family of processes $\{U_\sigma(t,\tau)\},\sigma\in\Sigma$ generated by problem (1.1).

2. Preliminaries

In this section, we first recall some basic concepts about non-autonomous systems, we refer to [7] for more details.

Space of translation bounded functions in $L^r_{\rm loc}(\mathbb{R};L^k(\Omega))$, with $r,k\geqslant 1$

$$L^r_b(\mathbb{R};L^k(\Omega))=\left\{g\in L^r_{\rm loc}(\mathbb{R};L^k(\Omega)):\sup_{t\in\mathbb{R}}\int_t^{t+1}\left(\int_\Omega|g(x,s)|^k dx\right)^{\frac{r}{k}}{\rm d}s<\infty\right\}.$$

Space of translation compact functions in $L^2_{\rm loc}(\mathbb{R};L^2(\Omega))$

$$\begin{aligned}L^2_c(\mathbb{R};L^2(\Omega))=\Big\{&g\in L^2_{\rm loc}(\mathbb{R};L^2(\Omega)):\ \text{For any interval } [t_1,t_2]\subset\mathbb{R},\\ &\{g(x,h+s):h\in\mathbb{R}\}|_{[t_1,t_2]}\ \text{is precompact in } L^2(t_1,t_2;L^2(\Omega))\Big\}.\end{aligned}$$

Let X be a Banach space, and Σ be a parameter set.

The operators $\{U_\sigma(t,\tau)\},\sigma\in\Sigma$ are said to be a family of processes in X with symbol space Σ if for any $\sigma\in\Sigma$

$$U_\sigma(t,s)\circ U_\sigma(s,\tau)=U_\sigma(t,\tau)\quad\forall t\geqslant s\geqslant\tau,\ \ \tau\in\mathbb{R},\tag{2.1}$$
$$U_\sigma(\tau,\tau)=Id(\text{identity})\quad\forall\tau\in\mathbb{R}.\tag{2.2}$$

Let $\{T(s)\}_{s\geqslant 0}$ be the translation semigroup on Σ, we say that a family of processes $\{U_\sigma(t,\tau)\},\sigma\in\Sigma$ satisfies the translation identity if

$$U_\sigma(t+s,\tau+s)=U_{T(s)\sigma}(t,\tau)\quad\forall\sigma\in\Sigma,\ \ t\geqslant\tau,\ \ \tau\in\mathbb{R},\ \ s\geqslant 0,\tag{2.3}$$
$$T(s)\Sigma=\Sigma\quad\forall s\geqslant 0.\tag{2.4}$$

By $\mathscr{B}(X)$ we denote the collection of the bounded sets of X, and $\mathbb{R}_\tau=\{t\in\mathbb{R},t\geqslant\tau\}$.

Definition 2.1 [7]. A bounded set $B_0\in\mathscr{B}(X)$ is said to be a bounded uniformly (w.r.t. $\sigma\in\Sigma$) absorbing set for $\{U_\sigma(t,\tau)\},\sigma\in\Sigma$ if for any $\tau\in\mathbb{R}$ and $B\in\mathscr{B}(X)$ there exists $T_0=T_0(B,\tau)$ such that $\bigcup_{\sigma\in\Sigma}U_\sigma(t;\tau)B\subset B_0$ for all $t\geqslant T_0$.

Definition 2.2 [7]. A set $\mathscr{A}\subset X$ is said to be uniformly (w.r.t $\sigma\in\Sigma$) attracting for the family of processes $\{U_\sigma(t,\tau)\},\sigma\in\Sigma$ if for any fixed $\tau\in\mathbb{R}$ and any $B\in\mathscr{B}(X)$

$$\lim_{t\to+\infty}\left(\sup_{\sigma\in\Sigma}{\rm dist}(U_\sigma(t;\tau)B;\mathscr{A})\right)=0,$$

here ${\rm dist}(\cdot,\cdot)$ is the usual Hausdorff semidistance in X between two sets.

In particular, a closed uniformly attracting set $\mathscr{A}_\Sigma$ is said to be the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ if it is contained in any closed uniformly attracting set (minimality property).

Similar to the autonomous cases (e.g., see [23]), we have the following existence and uniqueness results, the proof is based on the Galerkin approximation method (e.g., see [22]), the time-dependent terms make no essential complications.

Lemma 2.3. *Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary, f satisfies (1.2) and (1.3), $g \in L^\infty(\mathbb{R}; L^2(\Omega))$. Then for any initial data $(u_\tau^0, u_\tau^1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem (1.1) has a unique solution $u(t)$ satisfies $(u(t), u_t(t)) \in C(\mathbb{R}_\tau; H_0^1(\Omega) \times L^2(\Omega))$ and $\partial_{tt} u(t) \in L^2_{\mathrm{loc}}(\mathbb{R}_\tau; H^{-1}(\Omega))$.*

For convenience, hereafter, the norm and scalar product in $L^2(\Omega)$ are denoted by $\|\cdot\|$ and $(,)$, respectively. and C denotes a general positive constant, which may be different in different estimates.

We use the notations as in Chepyzhov and Vishik [7]: let $y(t) = (u(t), u_t(t))$, $y_\tau = (u_\tau^0, u_\tau^1), E_0 = H_0^1(\Omega) \times L^2(\Omega)$ with finite energy norm

$$\|y\|_{E_0} = \{\|\nabla u\|^2 + \|u_t\|^2\}^{\frac{1}{2}}.$$

Then system (1.1) is equivalent to the following system:

$$\begin{aligned} &\partial_t u_t = \Delta u - \alpha \sum\nolimits_{i=1}^{3} D_i u_t - \beta u_t - f(u) + g(x,t) \quad \text{for any } t \geqslant \tau, \\ &u|_{\partial\Omega} = 0, u(x,\tau) = u_\tau^0(x), u_t(x,\tau) = u_\tau^1(x), \end{aligned} \tag{2.5}$$

which can be rewritten in the operator form

$$\partial_t y = A_{\sigma(t)}(y), \quad y|_{t=\tau} = y_\tau, \tag{2.6}$$

where $\sigma(s) = g(x,s)$ is symbol of Eq. (2.6).

We now define the symbol space for (2.6). Taking a fixed symbol $\sigma_0(s) = g_0(x,s)$, $g_0 \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega))$ for some $r > \frac{6}{5}$. Set

$$\Sigma_0 = \{(x,t) \mapsto g_0(x, t+h) : h \in \mathbb{R}\} \tag{2.7}$$

and

$$\Sigma \text{ be the } * - \text{weakly closure of } \Sigma_0 \text{ in } L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega)). \tag{2.8}$$

Then we have the following simple properties:

Proposition 2.4. *Σ is bounded in $L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega))$, and for any $\sigma \in \Sigma$, the following estimate holds:*

$$\|\sigma\|_{L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega))} \leqslant \|g_0\|_{L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega))}.$$

Thus, from Lemma 2.3, we know that (1.1) is well posed for all $\sigma(s) \in \Sigma$ and generates a family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ given by the formula $U_\sigma(t,\tau) y_\tau = y(t)$, where $y(t)$ is the solution of (1.1)–(1.6), and $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ satisfies (2.1) and (2.2). At the same time, by the unique solvability, we know $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ satisfies the translation identity (2.3).

In what follows, we denote by $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ the family of processes generated by (2.6)–(2.8).

3. Uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set in E_0

Theorem 3.1. *Under assumptions (1.2)–(1.5), the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ corresponding to (1.1) has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0 in E_0.*

Proof. From the definition of Σ we know that for all $\sigma \in \Sigma$

$$\|\sigma\|^2_{L_b^2} \leqslant \|g_0\|^2_{L_b^2}. \tag{3.1}$$

Without loss of generality, we assume that $\alpha = \beta \equiv 1$. Taking the scalar product with $v = u_t + \delta u$ in L^2, where $0 < \delta \leqslant \delta_0$ which will be determined later, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\mathrm{d}x\right) + (1-\delta)\|v\|^2 + \delta\|\nabla u\|^2 - \delta(1-\delta)(u,v) - \delta\left(\sum_{i=1}^{3} D_i u, v\right) + \delta \int_\Omega f(u) u \,\mathrm{d}x = (g, v). \tag{3.2}$$

From (1.3) we know that there is $C_0 > 0$ such that

$$(f(u), u) \geqslant m_1 \int_\Omega F(u)\mathrm{d}x - C_0$$

and we obtain the estimate by using Hölder inequality and Young inequality

$$\|v\|^2 + \|\nabla u\|^2 + \int_\Omega 2F(u)\mathrm{d}x \geqslant \eta(\|v\|^2 + \|\nabla u\|^2) - C_1, \tag{3.3}$$

where η is a positive constant and

$$(1-\delta)\|v\|^2 + \delta\|\nabla u\|^2 - \delta(1-\delta)(u,v) - \delta\left(\sum_{i=1}^{3} D_i u, v\right) \geqslant \frac{1}{2}\|v\|^2 + \frac{\delta}{2}\|\nabla u\|^2. \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\mathrm{d}x\right) + \frac{1}{2}\|v\|^2 + \frac{\delta}{2}\|\nabla u\|^2 + \delta m_1\int_\Omega F(u)\mathrm{d}x \leqslant \delta C_0 + \frac{1}{4}\|v\|^2 + \|g\|^2, \tag{3.5}$$

choose $\gamma = \min\{\frac{1}{2}, \delta, \delta m_1\}$, set $z(t) = \|v\|^2 + \|\nabla u\|^2 + \int_\Omega 2F(u)\mathrm{d}x$, by Gronwall's inequality, we get

$$y(t) \leqslant z(t) \leqslant z(\tau)\exp(-\gamma(t-\tau)) + (1+\gamma^{-1})\left(2\delta C_0 + 2\|g\|^2_{L^2_b}\right) \tag{3.6}$$

from (3.1), we know

$$\|g\|^2_{L^2_b} \leqslant \|g_0\|^2_{L^2_b} \quad \text{for all } g \in \Sigma.$$

Therefore, we obtain the uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0 in E_0.

$$B_0 = \left\{y = (u, u_t) \,|\, \|y\|^2 \leqslant \rho_0^2\right\},$$

where $\rho_0 = 2(1+\gamma^{-1})(2\delta C_0 + 2\|g_0\|^2_{L^2_b})$, i.e., for any bounded subset B in E_0, there exists a $t_0 = t_0(\tau, B) \geqslant \tau$ such that

$$\bigcup_{g\in\Sigma} U_g(t,\tau)B \subset B_0 \quad \forall t \geqslant t_0. \qquad \square$$

4. Uniform (w.r.t. $\sigma\in$) asymptotic compactness in E_0

In this section, we will first give some useful preliminaries, then obtain some a priori estimates about the energy inequalities on account of the idea presented in [15–21]. Finally, we use Theorem 4.5 to establish the uniform (w.r.t. $\sigma \in \Sigma$) asymptotic compactness in E_0.

Hereafter, we always denote by B_0 the bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set obtained in Theorem 3.1.

4.1. Preliminaries

Firstly, we recall the simply criterion developed in [16], the following results are similar to that in [20,21] for autonomous cases.

Definition 4.1 [16]. Let X be a Banach space and B be a bounded subset of X, Σ be a symbol (or parameter) space. We call a function $\phi(\cdot,\cdot;\cdot,\cdot)$, defined on $(X\times X)\times(\Sigma\times\Sigma)$, to be a contractive function on $B\times B$ if for any sequence $\{x_n\}_{n=1}^\infty \subset B$ and any $\{\sigma_n\}\subset\Sigma$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and $\{\sigma_{n_k}\}_{k=1}^\infty \subset \{\sigma_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi(x_{n_k}, x_{n_l}; \sigma_{n_k}, \sigma_{n_l}) = 0.$$

We denote the set of all contractive functions on $B\times B$ by $\mathrm{Contr}(B,\Sigma)$.

Theorem 4.2 [16]. *Let $\{U_\sigma(t,\tau)\}, \sigma\in\Sigma$ be a family of processes satisfies the translation identity 2.3, 2.4 on Banach space X and has a bounded uniformly (w.r.t. $\sigma\in\Sigma$) absorbing set $B_0\subset X$. Moreover, assume that for any $\varepsilon > 0$ there exist $T = T(B_0,\varepsilon)$ and $\phi_T \in \mathrm{Contr}(B_0,\Sigma)$ such that*

$$\|U_{\sigma_1}(T,0)x - U_{\sigma_2}(T,0)y\| \leqslant \varepsilon + \phi_T(x,y;\sigma_1,\sigma_2) \quad \forall x,y\in B_0 \quad \forall \sigma_1,\sigma_2\in\Sigma.$$

Then $\{U_\sigma(t,\tau)\}, \sigma\in\Sigma$ is uniformly (w.r.t. $\sigma\in\Sigma$) asymptotically compact in X.

We will use the following Proposition on external forcing g from [16].

Proposition 4.3 [16]. *Let $g\in L^\infty(\mathbb{R};L^2(\Omega))\cap W_b^{1,r}(\mathbb{R};L^r(\Omega))(r>\frac{6}{5})$. Then there is an $M>0$ such that*

$$\sup_{t\in\mathbb{R}}\|g(x,t+s)\|_{L^2(\Omega)} \leqslant M \quad \textit{for all } s\in\mathbb{R}.$$

Proposition 4.4 [16]. *Let $s_i \in \mathbb{R}(i = 1, 2, \ldots)$, $g \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}; L^r(\Omega))(r > \frac{6}{5})$, $\{u_n(t) | t \geqslant 0, n = 1, 2, \ldots\}$ is bounded in $H_0^1(\Omega)$, and for any $T_1 > 0$, $\{u_{n_t}(t) \mid n = 1, 2, \ldots\}$ is bounded in $L^\infty(0, T_1; L^2(\Omega))$. Then for any $T > 0$, there exist subsequences $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and $\{s_{n_k}\}_{k=1}^\infty$ of $\{s_n\}_{n=1}^\infty$ such that*

$$\lim_{k\to\infty} \lim_{l\to\infty} \int_0^T \int_s^T \int_\Omega (g(x, \tau + s_{n_k}) - g(x, \tau + s_{n_l}))(u_{n_k} - u_{n_l})_t(\tau) \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}s = 0.$$

4.2. A priori estimates

The main purpose of this part is to establish (4.10)–(4.12), which will be used to obtain the uniform (w.r.t. $\sigma \in \Sigma$) asymptotic compactness. Without loss of generality, we deal only with the strong solutions in the following, the generalized solution case then follows easily by a density argument.

For any $(u_0^i, v_0^i) \in B_0$, let $(u_i(t), u_{i_t}(t))$ be the corresponding solution to σ_i with respect to initial data (u_0^i, v_0^i), $i = 1, 2$, that is, $(u_i(t), u_{i_t}(t))$ is the solution of the following equation:

$$\begin{cases} u_{tt} + \beta u_t + \alpha \sum_{i=1}^3 D_i u_t - \Delta u + f(u(t)) = \sigma_i(x, t), \\ u|_{\partial\Omega} = 0, \\ (u(0), u_t(0)) = (u_0^i, v_0^i). \end{cases} \tag{4.1}$$

For convenience, we denote

$$g_i(t) = \sigma_i(x, t), \quad t \geqslant 0, \quad i = 1, 2$$

and

$$w(t) = u_1(t) - u_2(t).$$

Then $w(t)$ satisfies

$$\begin{cases} w_{tt} + \beta w_t + \alpha \sum_{i=1}^3 D_i w_t - \Delta w + f(u_1(t)) - f(u_2(t)) = g_1(t) - g_2(t), \\ w|_{\partial\Omega} = 0, \\ (w(0), w_t(0)) = (u_0^1, v_0^1) - (u_0^2, v_0^2). \end{cases} \tag{4.2}$$

Set

$$E_w(t) = \frac{1}{2} \int_\Omega |w_t(t)|^2 + \frac{1}{2} \int_\Omega |\nabla w(t)|^2.$$

At first, without loss of generality, we assume that $\alpha = \beta \equiv 1$, multiplying (4.2) by w_t and integrating over $[s, T] \times \Omega$, we get

$$E_w(T) - E_w(s) + \int_s^T \int_\Omega |w_t(\tau)|^2 \, \mathrm{d}x \, \mathrm{d}\tau + \int_s^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau))) w_t(\tau) \mathrm{d}x \, \mathrm{d}\tau = \int_s^T \int_\Omega (g_1(\tau) - g_2(\tau)) w_t(\tau) \mathrm{d}x \, \mathrm{d}\tau, \tag{4.3}$$

where $0 \leqslant s \leqslant T$. Then we have

$$\int_0^T \int_\Omega |w_t(\tau)|^2 \mathrm{d}x \, \mathrm{d}\tau \leqslant \int_0^T \int_\Omega (g_1(\tau) - g_2(\tau)) w_t(\tau) \mathrm{d}x \, \mathrm{d}\tau + E_w(0) - \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau))) w_t(\tau) \mathrm{d}x \, \mathrm{d}\tau. \tag{4.4}$$

Secondly, multiplying (4.2) by w and integrating over $[0, T] \times \Omega$, we obtain

$$\begin{aligned} \int_0^T \int_\Omega |\nabla w(s)|^2 \mathrm{d}x \, \mathrm{d}s = {} & \sum_{i=1}^3 \int_0^T \int_\Omega D_i w w_t \mathrm{d}x \, \mathrm{d}s + \int_0^T \int_\Omega |w_t(s)|^2 \mathrm{d}x \, \mathrm{d}s - \frac{1}{2} \|w(T)\|^2 + \frac{1}{2} \|w(0)\|^2 + (w_t(0), w(0)) \\ & - (w_t(T), w(T)) - \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s))) w(s) \mathrm{d}x \, \mathrm{d}s + \int_0^T \int_\Omega (g_1 - g_2) w \, \mathrm{d}x \, \mathrm{d}s. \end{aligned} \tag{4.5}$$

Using Young inequality, we tackle with $\sum_{i=1}^3 \int_0^T \int_\Omega D_i w w_t \, \mathrm{d}x \, \mathrm{d}s$.

$$\sum_{i=1}^3 \int_0^T \int_\Omega D_i w w_t \, \mathrm{d}x \, \mathrm{d}s \leqslant \int_0^T \left(\frac{1}{6} \left\| \sum_{i=1}^3 D_i w \right\|^2 + \frac{3}{2} \|w_t\|^2 \right) \mathrm{d}s \leqslant \frac{1}{2} \int_0^T \int_\Omega |\nabla w(s)|^2 \mathrm{d}x \, \mathrm{d}s + \frac{3}{2} \int_0^T \int_\Omega |w_t(s)|^2 \mathrm{d}x \, \mathrm{d}s. \tag{4.6}$$

So from (4.4)–(4.6), we have

$$\int_0^T E_w(s)\mathrm{d}s \leqslant 3\int_0^T\int_\Omega (g_1-g_2)w_t\,\mathrm{d}x\,\mathrm{d}s - 3\int_0^T\int_\Omega (f(u_1(s))-f(u_2(s)))w_t(s)\mathrm{d}x\,\mathrm{d}s + 3E_w(0) - \frac{1}{2}\|w(T)\|^2 + \frac{1}{2}\|w(0)\|^2 + (w_t(0),w(0)) - (w_t(T),w(T)) - \int_0^T\int_\Omega (f(u_1(s))-f(u_2)(s))w(s)\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_\Omega (g_1-g_2)\,w\,\mathrm{d}x\,\mathrm{d}s. \tag{4.7}$$

Integrating (4.3) over $[0,T]$ with respect to s, we obtain that

$$TE_w(T) + \int_0^T\int_s^T\int_\Omega |w_t(\tau)|^2\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s = -\int_0^T\int_s^T\int_\Omega (f(u_1(\tau))-f(u_2(\tau)))w_t(\tau)\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^T\int_s^T\int_\Omega (g_1-g_2)w_t\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^T E_w(s)\mathrm{d}s. \tag{4.8}$$

Therefore, from (4.7) and (4.8), we have

$$TE_w(T) \leqslant 3E_w(0) + \frac{1}{2}\|w(0)\|^2 + (w_t(0),w(0)) - (w_t(T),w(T)) + 3\int_0^T\int_\Omega (g_1-g_2)w_t\,\mathrm{d}x\,\mathrm{d}s - 3\int_0^T\int_\Omega (f(u_1(s))-f(u_2(s)))w_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_\Omega (g_1-g_2)w\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_s^T\int_\Omega (f(u_1(\tau))-f(u_2(\tau)))w_t(\tau)\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^T\int_s^T\int_\Omega (g_1-g_2)w_t\,\mathrm{d}x\,\mathrm{d}\tau\mathrm{d}s - \int_0^T\int_\Omega (f(u_1(s))-f(u_2(s)))w\,\mathrm{d}x\,\mathrm{d}s. \tag{4.9}$$

Set

$$C_M = 3E_w(0) + \frac{1}{2}\|w(0)\|^2 + (w_t(0),w(0)) - (w_t(T),w(T)), \tag{4.10}$$

$$\phi_T((u_0^1,\upsilon_0^1),(u_0^2,\upsilon_0^2);\sigma_1,\sigma_2) = \int_0^T\int_\Omega (g_1-g_2)w\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_s^T\int_\Omega (f(u_1(\tau))-f(u_2(\tau)))w_t(\tau)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s + 3\int_0^T\int_\Omega (g_1-g_2)w_t\,\mathrm{d}x\,\mathrm{d}s - 3\int_0^T\int_\Omega (f(u_1(s))-f(u_2(s)))w_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_s^T\int_\Omega (g_1-g_2)w_t\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s - \int_0^T\int_\Omega (f(u_1(s))-f(u_2(s)))w\,\mathrm{d}x\,\mathrm{d}s. \tag{4.11}$$

Then we have

$$E_w(T) \leqslant \frac{C_M}{T} + \frac{1}{T}\phi_T((u_0^1,\upsilon_0^1),(u_0^2,\upsilon_0^2);\sigma_1,\sigma_2). \tag{4.12}$$

4.3. Uniform asymptotic compactness

In this subsection, we shall prove the uniform (w.r.t. $\sigma\in\Sigma$) asymptotic compactness in $H_0^1(\Omega)\times L^2(\Omega)$, which is given in the following theorem:

Theorem 4.5. *Assume that f satisfies 1.2, 1.3. If $g_0\in L^\infty(\mathbb{R};L^2(\Omega))\cap W_b^{1,r}(\mathbb{R};L^r(\Omega))$ for some $r>\frac{6}{5}$ and Σ is defined by (2.8), then the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ corresponding to (1.1), is uniformly (w.r.t. $\sigma\in\Sigma$) asymptotically compact in $H_0^1(\Omega)\times L^2(\Omega)$.*

Proof. Since the family of processes $\{U_\sigma(t,\tau)\}\sigma\in\Sigma$ has a bounded uniformly absorbing set, by the definition of C_M, we know that for any fixed $\varepsilon>0$, we can choose T large enough, such that

$$\frac{C_M}{T} \leqslant \varepsilon.$$

Hence, thanks to Theorem 4.2, it is sufficient to prove that $\phi_T(\cdot,\cdot;\cdot,\cdot)\in \mathrm{Contr}(B_0,\Sigma)$ for each fixed T.

From the proof procedure of Theorem 3.1, we can deduce that for any fixed T, we have

$$\bigcup_{\sigma\in\Sigma}\bigcup_{t\in[0,T]} U_\sigma(t,0)B_0 \text{ is bounded in } E_0 \tag{4.13}$$

and the bound depends on T.

Let (u_n,u_{n_t}) be the solutions corresponding to initial data $(u_0^n,v_0^n)\in B_0$ with respect to symbol $\sigma_n\in\Sigma, n=1,2,\ldots$. Then, from (4.13), without loss of generality (at most by passing subsequence), we assume that

$$u_n \to u \quad \star - \text{weakly in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.14}$$
$$u_{n_t} \to u_t \quad \star - \text{weakly in } L^\infty(0,T;L^2(\Omega)), \tag{4.15}$$
$$u_n \to u \quad \text{in } L^2(0,T;H_0^{1-\varepsilon}(\Omega)) \quad \forall \varepsilon \in (0,1], \tag{4.16}$$
$$u_n(0) \to u(0) \quad \text{and} \quad u_n(T) \to u(T) \ \text{ in } \ L^k(\Omega), \tag{4.17}$$

for $k < 6$, where we use the compact embedding $H_0^1 \hookrightarrow L^k$.

Now, we will deal with each term corresponding to that in (4.11) one by one.

At first, from Proposition 4.3 and (4.16), we can obtain that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (g_n(x,s)-g_m(x,s))(u_n(s)-u_m(s))\mathrm{d}x\,\mathrm{d}s = 0 \tag{4.18}$$

and from Proposition 4.4 we can get that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (g_n(x,s)-g_m(x,s))(u_{n_t}(s)-u_{m_t}(s))\mathrm{d}x\,\mathrm{d}s = 0, \tag{4.19}$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_s^T\int_\Omega (g_n(x,\tau)-g_m(x,\tau))(u_{n_t}(\tau)-u_{m_t}(\tau))\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s = 0. \tag{4.20}$$

Secondly, from (4.16), we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (f(u_n(s))-f(u_m(s)))(u_n(s)-u_m(s))\mathrm{d}x\,\mathrm{d}s = 0. \tag{4.21}$$

Finally, by the similar method used in the Proof of Lemma 2.2 in [21], we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (f(u_n(s))-f(u_m(s)))(u_{n_t}(s)-u_{m_t}(s))\mathrm{d}x\,\mathrm{d}s = 0, \tag{4.22}$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_s^T\int_\Omega (f(u_n(\tau))-f(u_m(\tau)))(u_{n_t}(\tau)-u_{m_t}(\tau))\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s = 0. \tag{4.23}$$

Hence, from (4.18)–(4.23) we get $\phi_T(\cdot,\cdot;\cdot,\cdot) \in \mathrm{Contr}(B_0,\Sigma)$ immediately. □

4.4. Existence of uniform attractor

Theorem 4.6. *Assume that f satisfy (1.2) and (1.3). If $g_0 \in L^\infty(\mathbb{R};L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R};L^r(\Omega))$ for some $r > \frac{6}{5}$ and Σ is defined by (2.8), then the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ corresponding to (1.1) has a compact uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathscr{A}_\Sigma$.*

Proof. Theorems 3.1 and 4.5 imply the existence of a uniform attractor immediately. □

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