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# A fourth-order method of the convection–diffusion equations with Neumann boundary conditions

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ABSTRACT

In this paper, we have developed a fourth-order compact finite difference scheme for solving the convection–diffusion equation with Neumann boundary conditions. Firstly, we apply the compact finite difference scheme of fourth-order to discrete spatial derivatives at the interior points. Then, we present a new compact finite difference scheme for the boundary points, which is also fourth-order accurate. Finally, we use a Padé approximation method for the resulting linear system of ordinary differential equations. The presented scheme has fifth-order accuracy in the time direction and fourth-order accuracy in the space direction. It is shown through analysis that the scheme is unconditionally stable. Numerical results show that the compact finite difference scheme gives an efficient method for solving the convection–diffusion equations with Neumann boundary conditions.



## 1. Introduction

In practical engineering applications, convection–diffusion equations are generally used to describe the transport processes involving fluid motion, heat transfer, astrophysics, oceanography, meteorology, semiconductors, hydraulics, pollutant and sediment transport, and chemical engineering. With the rapid progress in computer power, the differential convection–diffusion equations can be analytically studied by pursuing the numerical solution of their discretized counterparts. Many numerical methods have been developed to solve the convection–diffusion equations with Dirichlet boundary conditions, see [1–13].

However, fewer difference schemes have been developed to solve the convection–diffusion equation with Neumann boundary conditions, which are much more difficult to handle than Dirichlet conditions. Even for those less compact difference schemes involving Neumann boundary conditions, very often, the schemes are fourth or sixth order accurate at the interior points, but only first-order or second-order at the boundary points. We have found that, when the first-order accurate scheme for the Neumann boundary conditions is employed, it affects the accuracy of the overall numerical solution even if a second-order numerical method is constructed at the interior grid points.

Recently, some authors pay attention to the numerical method of the following diffusion equations with Neumann boundary conditions:

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant L, \; 0 \leqslant t \leqslant T,$$
$$u(x,0) = \phi(x), \quad 0 \leqslant x \leqslant L,$$
$$\frac{\partial u(0,t)}{\partial x} = g_0(t), \quad \frac{\partial u(L,t)}{\partial x} = g_1(t), \quad t \in [0,T].$$

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To the above equations, Zhao et al. [14] presented a fourth-order compact difference scheme at the interior points and a second-order difference scheme at the boundary points. Sun [15] gave a fourth-order difference scheme for the boundary points.

In this paper, we consider the numerical method of the following one-dimensional convection–diffusion equation with Neumann boundary conditions:

$$\frac{\partial u}{\partial t}+\varepsilon\frac{\partial u}{\partial x}=\gamma\frac{\partial^2 u}{\partial x^2},\quad 0\leqslant x\leqslant L,\ 0\leqslant t\leqslant T, \tag{1}$$

$$u(x,0)=\phi(x),\quad 0\leqslant x\leqslant L, \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x}=g_0(t),\quad \frac{\partial u(L,t)}{\partial x}=g_1(t),\quad t\in[0,T]. \tag{3}$$

We propose a scheme for solving Eqs.(1)–(3) which is unconditionally stable and is fifth-order accurate in the time direction and fourth-order accurate in the space direction. Specifically speaking, we first use the compact finite difference approximation of fourth-order for discretizing spatial derivatives of convection–diffusion. It is very important that the new scheme is fourth-order accurate in space at all grid points, both interior and boundary points. Secondly, we use the Taylor expansion and the Padé approximation method for the resulting linear system of ordinary differential equations. Finally, we give three numerical examples to verify the effectivity of the present scheme.

## 2. Spatial discretizations

### 2.1. Preliminaries

The solution domain $[0,L]\times[0,T]$ of the problem is covered by a mesh of grid lines

$$x_i=ih,\quad i=0,1,2,\ldots,n,$$
$$t_j=jk,\quad j=0,1,2,\ldots,m$$

parallel to the space and time coordinate axes, respectively. Approximations $u_i^j$ to $u(x_i,t_j)$ are calculated at the intersection of these lines, and $(ih,jk)$ is referred to as the $(i,j)$ grid-point. The constant spatial and temporal grid spacings are $h=L/n$ and $k=T/m$, respectively. In the next two subsections, we will construct fourth-order approximations to Eq. (1) in space.

### 2.2. The interior points

We consider the following differential equation:

$$-\gamma\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2}+\varepsilon\frac{\mathrm{d}y(x)}{\mathrm{d}x}=g(x). \tag{4}$$

If we denote the central difference schemes of order two for the second and first derivatives of $y$ as $\delta_x^2 y=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$ and $\delta_x y=\frac{y_{i+1}-y_{i-1}}{2h}$, respectively, where $y(x_i)=y_i$. Similar to the methods presented in [13,16], it is easy to derive a three-point fourth-order difference scheme for Eq. (4) as

$$\left[-\left(\gamma+\frac{\varepsilon^2h^2}{12\gamma}\right)\delta_x^2+\varepsilon\delta_x\right]y_i=\left[1+\frac{h^2}{12}\left(\delta_x^2-\frac{\varepsilon}{\gamma}\delta_x\right)\right]g_i+O(h^4), \tag{5}$$

where $\delta_x^2$ and $\delta_x$ are the second-order and first-order center difference operators.

If we discretize Eq. (1) in space at point $x_i$ by the compact finite scheme (5), we can obtain

$$L_x^{-1}A_xu_i(t)=-\frac{\partial u_i(t)}{\partial t}+O(h^4), \tag{6}$$

where $L_x=1+\frac{h^2}{12}\left(\delta_x^2-\frac{\varepsilon}{\gamma}\delta_x\right)$ and $A_x=-\left(\gamma+\frac{\varepsilon^2h^2}{12\gamma}\right)\delta_x^2+\varepsilon\delta_x$ are two difference operators, $u_i(t)=u(x_i,t)$.

Denote

$$v_i(t)=\frac{\partial u_i(t)}{\partial t}. \tag{7}$$

Then we have

$$L_x^{-1}A_xu_i(t)=-v_i(t)+O(h^4). \tag{8}$$

Neglecting the error term $O(h^4)$ of (8) and then rewriting it we get:

$$\left(\frac{1}{12}+\frac{h\varepsilon}{24\gamma}\right)v_{i-1}(t)+\frac{5}{6}v_i(t)+\left(\frac{1}{12}-\frac{h\varepsilon}{24\gamma}\right)v_{i+1}(t)=\left(\frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h}\right)u_{i-1}(t)+\left(-\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma}\right)u_i(t)+\left(\frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h}\right)u_{i+1}(t). \quad (9)$$

*2.3. The boundary points*

**Lemma 1** [17]. *Let $G = [g_{ij}]_{n\times n}$ be strictly diagonally dominant. Then*

(a) *G is invertible.*
(b) *If all main diagonal entries of G are positive, then all the eigenvalues of G have positive real parts.*
(c) *If G is Hermitian and all main diagonal entries of G are positive, then all the eigenvalues of G are real and positive.*

**Theorem 1.** *Assume that $\mu(x)\in C^5[0,L]$, then*

$$\mu'(x_0)=\frac{\mu(x_1)-\mu(x_0)}{h}-\frac{5}{12}h\mu''(x_0)-\frac{1}{12}h\mu''(x_1)-\frac{1}{12}h^2\mu^{(3)}(x_0)+O(h^4), \quad (10)$$

$$\mu'(x_n)=\frac{\mu(x_n)-\mu(x_{n-1})}{h}+\frac{5}{12}h\mu''(x_n)+\frac{1}{12}h\mu''(x_{n-1})-\frac{1}{12}h^2\mu^{(3)}(x_n)+O(h^4). \quad (11)$$

**Proof.** Denote $D^j\mu(x) = \mu^{(j)}(x)$. It is clear that

$$\begin{aligned}\frac{\mu(x_1)-\mu(x_0)}{h}-\frac{5}{12}h\mu''(x_0)-\frac{1}{12}h\mu''(x_1)&=\left(D+\frac{h}{2!}D^2+\frac{h^2}{3!}D^3+\frac{h^3}{4!}D^4+\frac{h^4}{5!}D^5+\cdots\right)\mu(x_0)\\&\quad-\left(\frac{h}{2}D^2+\frac{h^2}{12}D^3+\frac{h^3}{24}D^4+\frac{h^4}{72}D^5+\cdots\right)\mu(x_0)\\&=\mu'(x_0)+\frac{1}{12}h^2\mu^{(3)}(x_0)-\frac{1}{180}h^4\mu^{(5)}(x_0)+\cdots\end{aligned}$$

Thus Eq. (10) holds. Similarly we can prove Eq. (11).

Assume that $u(x,t)$ is the exact solution of problem (1), then, according to Theorem 1, we have

$$\frac{\partial u(x_0,t)}{\partial x}=\frac{u(x_1,t)-u(x_0,t)}{h}-\frac{5}{12}h\frac{\partial^2u(x_0,t)}{\partial x^2}-\frac{1}{12}h\frac{\partial^2u(x_1,t)}{\partial x^2}-\frac{1}{12}h^2\frac{\partial^3u(x_0,t)}{\partial x^3}+O(h^4), \quad (12)$$

$$\frac{\partial u(x_n,t)}{\partial x}=\frac{u(x_n,t)-u(x_{n-1},t)}{h}+\frac{5}{12}h\frac{\partial^2u(x_n,t)}{\partial x^2}+\frac{1}{12}h\frac{\partial^2u(x_{n-1},t)}{\partial x^2}-\frac{1}{12}h^2\frac{\partial^3u(x_n,t)}{\partial x^3}+O(h^4). \quad (13)$$

From Eq. (1), we get

$$\frac{\partial^2u(x_0,t)}{\partial x^2}=\frac{1}{\gamma}\left[\varepsilon\frac{\partial u(x_0,t)}{\partial t}+\frac{\partial u(x_0,t)}{\partial x}\right], \quad (14)$$

$$\frac{\partial^2u(x_1,t)}{\partial x^2}=\frac{1}{\gamma}\left[\varepsilon\frac{\partial u(x_1,t)}{\partial t}+\frac{\partial u(x_1,t)}{\partial x}\right], \quad (15)$$

$$\frac{\partial^3u(x_0,t)}{\partial x^3}=\frac{1}{\gamma}\left[\varepsilon\frac{\partial^2u(x_0,t)}{\partial t\partial x}+\frac{\partial^2u(x_0,t)}{\partial x^2}\right], \quad (16)$$

substituting (14) into (16) yields

$$\frac{\partial^3u(x_0,t)}{\partial x^3}=\frac{\varepsilon}{\gamma}\frac{\partial^2u(x_0,t)}{\partial t\,\partial x}+\frac{1}{\gamma^2}\left[\varepsilon\frac{\partial u(x_0,t)}{\partial t}+\frac{\partial u(x_0,t)}{\partial x}\right]. \quad (17)$$

By using the Taylor series, we obtain

$$\frac{\partial u(x_1,t)}{\partial x}=\frac{\partial u(x_0,t)}{\partial x}+h\frac{\partial^2u(x_0,t)}{\partial x^2}+\frac{h^2}{2}\frac{\partial^3u(x_0,t)}{\partial x^3}+O(h^3). \quad (18)$$

Substituting Eqs. (14), (15), (17) and (18) into (12)

$$\left(\frac{5h}{12\gamma}+\frac{\varepsilon h^2}{6\gamma^2}+\frac{\varepsilon^2 h^3}{24\gamma^3}\right)v_0(t)+\frac{h}{12\gamma}v_1(t)=-\frac{1}{h}u_0(t)+\frac{1}{h}u_1(t)+\left(-1-\frac{\varepsilon h}{2\gamma}-\frac{\varepsilon^2h^2}{6\gamma^2}-\frac{\varepsilon^3h^3}{24\gamma^3}\right)g_0(t) + \left(-\frac{\varepsilon h^3}{24\gamma^2}-\frac{h^2}{12\gamma}\right)g_0'(t). \tag{19}$$

Similar to Eq. (13), we can obtain the difference scheme at the point $x = x_n$

$$\frac{h}{12\gamma}v_{n-1}(t)+\left(\frac{5h}{12\gamma}-\frac{\varepsilon h^2}{6\gamma^2}+\frac{\varepsilon^2 h^3}{24\gamma^3}\right)v_n(t)=\frac{1}{h}u_{n-1}(t)-\frac{1}{h}u_n(t)+\left(1-\frac{\varepsilon h}{2\gamma}+\frac{\varepsilon^2h^2}{6\gamma^2}-\frac{\varepsilon^3h^3}{24\gamma^3}\right)g_1(t) + \left(-\frac{\varepsilon h^3}{24\gamma^2}+\frac{h^2}{12\gamma}\right)g_1'(t). \tag{20}$$

From Eqs. (9), (19) and (20), we obtain a system of ordinary differential equations which is as follows:

$$\begin{cases} A\mathbf{V}(t)=B\mathbf{U}(t)+\mathbf{G}(t),\\ \mathbf{U}(0)=\mathbf{U}_0, \end{cases} \tag{21}$$

where

$$\begin{aligned}
\mathbf{V}(t)&=[v_0(t),v_1(t),\ldots,v_{n-1}(t),v_n(t)]^T,\\
\mathbf{U}(t)&=[u_0(t),u_1(t),\ldots,u_{n-1}(t),u_n(t)]^T,\\
\mathbf{G}(t)&=\Bigg[\left(-1-\frac{\varepsilon h}{2\gamma}-\frac{\varepsilon^2h^2}{6\gamma^2}-\frac{\varepsilon^3h^3}{24\gamma^3}\right)g_0(t)+\left(-\frac{\varepsilon h^3}{24\gamma^2}-\frac{h^2}{12\gamma}\right)g_0'(t),\underbrace{0,\ldots,0}_{n-1},\left(1-\frac{\varepsilon h}{2\gamma}+\frac{\varepsilon^2h^2}{6\gamma^2}-\frac{\varepsilon^3h^3}{24\gamma^3}\right)g_1(t)\\
&\qquad+\left(-\frac{\varepsilon h^3}{24\gamma^2}+\frac{h^2}{12\gamma}\right)g_1'(t)\Bigg]^T,\\
\mathbf{U}(0)&=[\phi(x_0),\phi(x_1),\ldots,\phi(x_{n-1}),\phi(x_n)]^T,
\end{aligned}$$

and $A$ and $B$ are the tri-diagonal matrix of order $n + 1$ as below

$$A=\begin{pmatrix}
\frac{5h}{12\gamma}+\frac{\varepsilon h^2}{6\gamma^2}+\frac{\varepsilon^2h^3}{24\gamma^3} & \frac{h}{12\gamma} & 0 & \cdots & 0 & 0\\
\frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} & 0 & \cdots & 0\\
0 & \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} & 0 & \cdots\\
\vdots & \ddots & \ddots & \ddots & & \vdots\\
\cdots & 0 & \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} & 0\\
0 & \cdots & 0 & \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma}\\
0 & 0 & \cdots & 0 & \frac{h}{12\gamma} & \frac{5h}{12\gamma}-\frac{\varepsilon h^2}{6\gamma^2}+\frac{\varepsilon^2h^3}{24\gamma^3}
\end{pmatrix},$$

$$B=\begin{pmatrix}
-\frac{1}{h} & \frac{1}{h} & 0 & \cdots & 0 & 0\\
\frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} & 0 & \cdots & 0\\
0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} & 0 & \cdots\\
\vdots & \ddots & \ddots & \ddots & & \vdots\\
\cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h} & 0\\
0 & \cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{2h} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{2h}\\
0 & 0 & \cdots & 0 & \frac{1}{h} & -\frac{1}{h}
\end{pmatrix}.$$

We always let $h<\frac{10\gamma}{\varepsilon}$, then one can obtain that matrix $A$ is strictly diagonally dominant. So, from Lemma 1, we can get that $A$ is invertible, and (21) can be written as follows:

$$\begin{cases} \frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t}=A^{-1}B\mathbf{U}(t)+A^{-1}\mathbf{G}(t),\\ \mathbf{U}(0)=\mathbf{U}_0. \qquad \square \end{cases} \tag{22}$$

### 2.4. Time discretization

The exact solution of Eq. (22) for any interior computational node at $t + k$, given the solution at time $t$, can be expressed by the following Taylor series:

$$U(t+k) = \left(1 + k\frac{d}{dt} + \frac{k^2}{2!}\frac{d^2}{dt^2} + \cdots\right)U(t) = \exp\left(k\frac{d}{dt}\right)U(t), \tag{23}$$

where $k$, $k = 0, 1, 2, \ldots$, is the time step length.

To do so, we employ the Padé approximation to $e^z$. The [2,2] Padé approximation to $e^z$ is

$$e^z \approx \frac{12 + 6z + z^2}{12 - 6z + z^2}. \tag{24}$$

Approximation (24) satisfies

$$e^z - \frac{12 + 6z + z^2}{12 - 6z + z^2} = O(z^5). \tag{25}$$

Now, we use $\frac{12+6k\frac{d}{dt}+k^2\frac{d^2}{dt^2}}{12-6k\frac{d}{dt}+k^2\frac{d^2}{dt^2}}$ to approximate the $e^{k\frac{d}{dt}}$ in Eq. (23), then we can get

$$\left(12 - 6k\frac{d}{dt} + k^2\frac{d^2}{dt^2}\right)U(t+k) = \left(12 + 6k\frac{d}{dt} + k^2\frac{d^2}{dt^2}\right)U(t). \tag{26}$$

From Eqs. (22) and (26), we can obtain the following difference scheme for the numerical solution of (22):

$$U_{j+1} = M^{-1}NU_j + M^{-1}[(6kI - k^2A^{-1}B)\mathbf{p}_{j+1} + (6kI + k^2A^{-1}B)\mathbf{p}_j + k^2(\mathbf{p}'_j - \mathbf{p}'_{j+1})], \tag{27}$$

where $M = 12I - 6kA^{-1}B + k^2(A^{-1}B)^2$, $N = 12I + 6kA^{-1}B + k^2(A^{-1}B)^2$, $p(t) = A^{-1}G(t)$, $U_j$ is the numerical solution of $U(t_j)$, $I$ is the $(n+1)\times(n+1)$ identity matrix. The accuracy order of the difference scheme (27) is $O(k^5)$ in the time direction because of (25), and $O(h^4)$ in the space direction, so that it is $O(k^5 + h^4)$ accurate.

## 3. Stability analysis

For the homogeneous boundary conditions, the proposed method (27) can be written as

$$U_{j+1} = \varphi U_j, \quad j = 0, 1, 2, \ldots, \tag{28}$$

where the amplification matrix is given by

$$\varphi = [12I - 6kA^{-1}B + k^2(A^{-1}B)^2]^{-1}[12I + 6kA^{-1}B + k^2(A^{-1}B)^2]. \tag{29}$$

For unconditional stability of method (27) it is necessary that the absolute value of the eigenvalues of the application matrix $(\varphi)$ be less than one.

Note that if $\lambda$ is an eigenvalue of $A^{-1}B$, then $(12 - 6k\lambda + k^2\lambda^2)^{-1}(12 + 6k\lambda + k^2\lambda^2)$ is an eigenvalue of matrix (29) having the same corresponding eigenvectors. Thus, in order to prove that $|(12 - 6k\lambda + k^2\lambda^2)^{-1}(12 + 6k\lambda + k^2\lambda^2)| < 1$, we only need to show that each $\lambda$ is real and $\lambda \leqslant 0$.

In order to prove it, we need the following results.

**Lemma 2** [12]. *If the real part of z is non-positive, then*

$$\left|\frac{12 + 6z + z^2}{12 - 6z + z^2}\right| \leqslant 1. \tag{30}$$

**Theorem 2.** *Difference scheme (27) is unconditionally stable.*

**Proof.** Let $\lambda_A$ and $\lambda_B$ be the eigenvalues of $A$ and $B$, respectively. From the condition $h < \frac{10\gamma}{\varepsilon}$, we can obtain that $A$ is strictly diagonally dominant, so, from Lemma 1, we can conclude that $\lambda_A$ is positive, and from the *Geršchgorin* theorem, we can get that $\lambda_B$ is non-positive. So, the eigenvalue of matrix $A^{-1}B$ is also non-positive

Let $\lambda_\varphi$ be the eigenvalues of $\varphi$; one can see that the eigenvalues of $\varphi$ are

$$(\lambda_\varphi)_i = \frac{12 + 6k\lambda_i + k^2\lambda_i^2}{12 - 6k\lambda_i + k^2\lambda_i^2}, \tag{31}$$

where $i = 1, 2, \ldots, n + 1$.

From Lemmas 1 and 2, we have

$$\left|\frac{12+6k\lambda_i+k^2\lambda_i^2}{12-6k\lambda_i+k^2\lambda_i^2}\right| \leqslant 1, \quad i=1,2,\ldots,n+1. \tag{32}$$

Thus, the difference scheme (27) is unconditionally stable. □

## 4. Numerical experiments and discussion

In this section, we present the numerical results of the new method (27) on several problems. We tested the accuracy and stability of method (27) presented in this paper by performing the mentioned method for different values of $h$, $k$, $\varepsilon$ and $\gamma$. The Péclet number is defined as $P_e = \frac{\varepsilon}{\gamma}$. When the Péclet number is high, the convection term dominates and when the Péclet number is low the diffusion term dominates.

Let $U_i^j$ and $u(x_i, t_j)$ be the numerical solution and the exact solution of problem (1)–(3). The maximum of $l_2$-norm errors of the numerical solutions as compared with the exact solution was computed for $0 \leqslant t \leqslant T$ based on the formula

$$E(h,k) = \max_{0\leqslant jk\leqslant T}\sqrt{h\sum_{i=1}^{n}[U_i^j - u(x_i,t_j)]^2}. \tag{33}$$

To obtain the convergence rate with respect to the spatial variable, we may assume that $E(h,k) = O(k^p + h^q)$. If $k$ is small enough, then $E(h,k) \approx O(h^q)$. Consequently, $\frac{E(h,k)}{E(2h,k)} \approx 2^q$, where $q \approx \log_2 \frac{E(h,k)}{E(2h,k)}$ is the convergence rate with respect to the spatial variable. Similarly, $p \approx \log_2 \frac{E(h,2k)}{E(h,k)}$ is the convergence rate with respect to the time variable.

**Problem 1.** We consider the following equation [8]:

$$\frac{\partial u}{\partial t} + \varepsilon\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant 2,\ 0 \leqslant t \leqslant 1, \tag{34}$$

with initial conditions

$$u(x,0) = \sin x, \quad 0 \leqslant x \leqslant 2,$$

and boundary conditions

$$\frac{u(0,t)}{\partial x} = \exp(-\gamma t)\cos(\varepsilon t), \quad \frac{u(2,t)}{\partial x} = \exp(-\gamma t)\cos(2-\varepsilon t).$$

The exact solution of the above problem is $u(x,t) = \exp(-\gamma t)\sin(x - \varepsilon t)$. Obviously, the exact solution $u(x,t)$ becomes very small when $t$ is large.

In our computation, we first choose $\varepsilon = 1$, $P_e = 10$, $k = 10^{-4}$. In Table 1, the maximum of $l_2$-norm errors of numerical solutions at the final time $t = 1$ with different $h$ are obtained by using the present scheme (27). Table 1 shows the numerical result when $k = 10^{-4}$, $h$ = 1/10, 1/20, 1/40, 1/80, 1/160. It can be seen from the table that the convergence rate of the present scheme (27) is 4 with respect to the spatial variable.

In Fig. 1 we show the numerical solutions and exact solutions obtained for Problem 1 at the final time $t = 1$ with $h = 1/16$ and several values of $P_e$. We see, the numerical solutions of Problem 1 approximate the exact solutions very well.

**Problem 2.** We consider the convection–diffusion equation [8]:

$$\frac{\partial u}{\partial t} + \varepsilon\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant 2,\ 0 \leqslant t \leqslant 2, \tag{35}$$

with initial conditions

**Table 1**
Maximum $l_2$-norm errors $E(h,k)$ and convergence rates when $k = 10^{-4}$ for Problem 1.

| $h$ | The present scheme (27) | Convergence rate |
|---|---|---|
| 1/10 | 2.8194 (−06) | 3.94 |
| 1/20 | 1.8372 (−07) | 3.96 |
| 1/40 | 1.1808 (−08) | 3.98 |
| 1/80 | 7.5042 (−10) | 3.99 |
| 1/160 | 4.7340 (−11) | – |

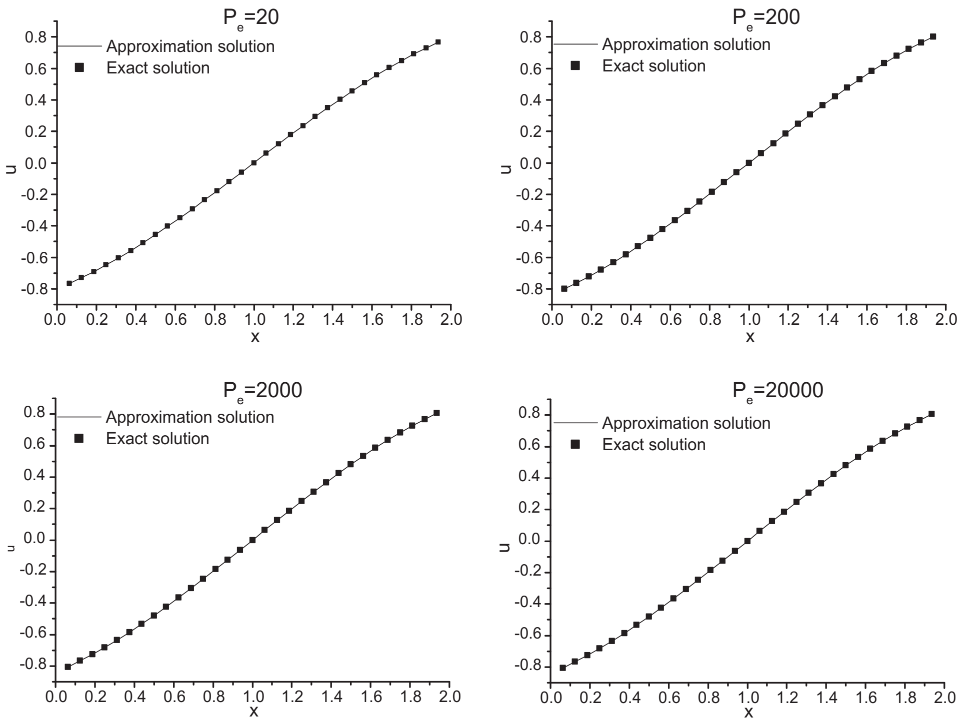


**Fig. 1.** Exact solution and approximate solutions of Problem 1 for several values of $P_e$ with $h = 1/16$ at $t = 1$.

$$u(x,0) = \exp\left\{-\frac{(x-\varepsilon)^2}{4\gamma}\right\}, \quad 0 \leqslant x \leqslant 2,$$

and boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\varepsilon}{2\gamma\sqrt{1+t}}\exp\left\{-\frac{(1+t)^2\varepsilon^2}{4\gamma(1+t)}\right\}, \quad \frac{\partial u(2,t)}{\partial x} = -\frac{2-(1+t)\varepsilon}{2\gamma(1+t)\sqrt{1+t}}\exp\left\{-\frac{(2-(1+t)\varepsilon)^2}{4\gamma(1+t)}\right\}.$$

The exact solution of the above problem is $u(x,t) = \frac{1}{\sqrt{1+t}}\exp\left\{-\frac{(x-(1+t)\varepsilon)^2}{4\gamma(1+t)}\right\}$.

For this test problem we first put $\varepsilon = 0.25$, $k = 10^{-4}$ and $P_e = 25$. In Table 2, we show the numerical solutions at the final time $t = 2$ obtained for solving Problem 2 with the method presented in this paper for different values of $h$. It can be seen from Table 2 that the convergence rate of the present scheme (27) is about 4. In Fig. 2, we show the numerical solutions obtained for Problem 2 at the final time $t = 2$ with $h = 1/128$ and several values of $P_e$.

**Problem 3.** We consider the following convection–diffusion equation:

$$\frac{\partial u}{\partial t} + \varepsilon\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant 1, \tag{36}$$

with initial conditions

$$u(x,0) = a\exp(-cx), \quad 0 \leqslant x \leqslant 1,$$

and boundary conditions

**Table 2**
Maximum $l_2$-norm errors $E(h,k)$ and convergence rates when $k = 10^{-4}$ for Problem 2.

| $h$ | The present scheme (27) | Convergence rate |
|---|---|---|
| 1/10 | 3.3101 (−03) | 3.94 |
| 1/20 | 2.1563 (−04) | 4.00 |
| 1/40 | 1.3452 (−05) | 3.99 |
| 1/80 | 8.4229 (−07) | 3.99 |
| 1/160 | 5.2770 (−08) | – |

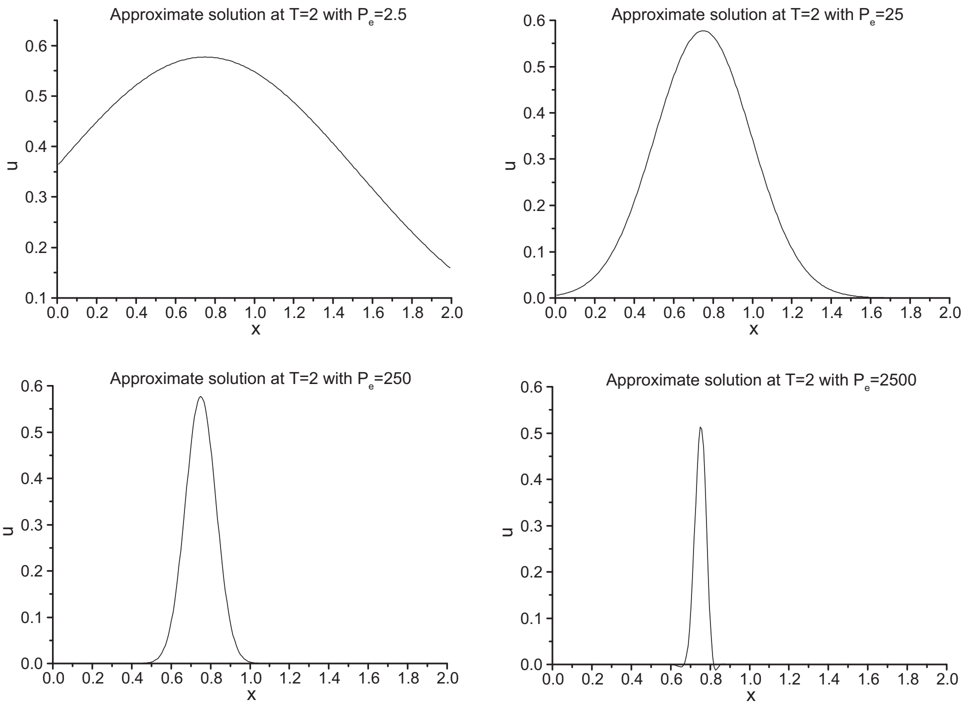


**Fig. 2.** Approximate solutions of Problem 2 for several values of $P_e$ with $h = 1/128$ at $t = 2$.

**Table 3**
Maximum $l_2$-norm errors $E(h,k)$ and convergence rates when $k = 10^{-4}$ for Problem 3.

| $h$ | The present scheme (27) | Convergence rate |
|---|---|---|
| 1/10 | 3.3411(−09) | 3.85 |
| 1/20 | 2.3106(−10) | 4.08 |
| 1/40 | 1.3608(−11) | 3.21 |
| 1/80 | 1.4694(−12) | – |

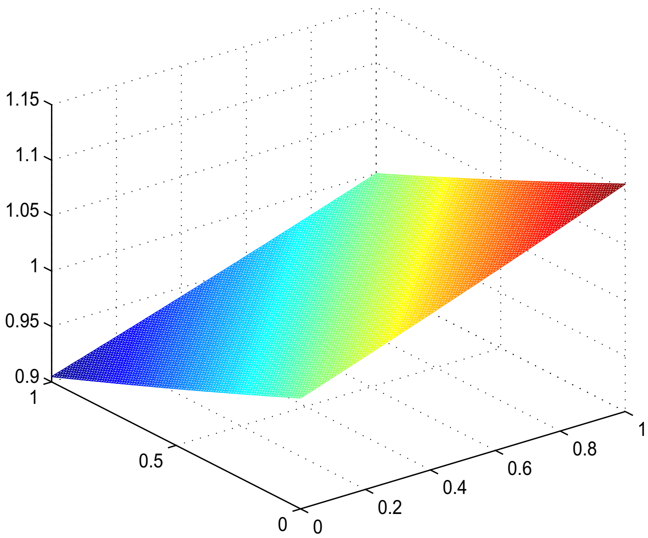


**Fig. 3.** Approximate solutions of Problem 3 for $P_e = 1000$ with $h = k = 1/100$.

$$\frac{\partial u(0,t)}{\partial x} = -ac\exp(bt), \quad \frac{\partial u(1,t)}{\partial x} = -ac\exp(bt-c),$$

where $c = \frac{-\varepsilon+\sqrt{\varepsilon^2+4\gamma b}}{2\gamma}$, $a$ and $b$ are constants.

The exact solution of the above problem is $u(x,t) = a\exp(bt - cx)$. For this problem, let $a = 1$, $b = 0.1$, $\varepsilon = 1$, $P_e = 1000$. In Table 3, we show the maximum $l_2$-norm errors $E(h,k)$ and convergence rates when $k = 10^{-4}$ with different $h$. From Table 3, we can see that the convergence rate of the present scheme (27) is almost 4, it should be mentioned that when the number of grid points increases from 21 to 41, the convergence rate of the present scheme (27) decreases from 4.08 to 3.21. In Fig. 3, we show the numerical solutions of problem 3 for $P_e = 1000$ with $h = k = 1/1000$.

## 5. Concluding remarks

In this paper, a high accuracy difference scheme for solving one-dimensional convection–diffusion equations with Neumann boundary conditions is presented, which is unconditionally stable for all choice of $k$ and $h$. The accuracy of the presented method is $O(k^5 + h^4)$. It is shown from the above numerical results that the presented scheme (27) is fourth-order convergent in space at all grid points, both interior and boundary points.

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