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Limit Set Maps in Impulsive Semidynamical Systems

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Abstract In this paper, we discuss the semicontinuities of orbital and limit set maps in an impulsive semidynamical system and investigate their relationships with the stabilities of orbits. Actually, we only deal with infinite impulsive trajectories under the hypotheses that each prolongational set is compact in the phase space. We prove that if the limit set is stable (eventually stable or eventually weakly stable), then the corresponding limit set map is upper semicontinuous or lower semicontinuous, respectively. And if the limit set map is upper semicontinuous (lower semicontinuous), then the corresponding limit set is stable (eventually stable or eventually weakly stable, respectively). Furthermore, we give several sufficient conditions to guarantee that limit sets are minimal .

Keywords Impulsive semidynamical system · Limit set map · Semicontinuity · Stability

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1 Introduction

Let $X = (X, \rho)$ be a metric space with metric ρ. A semidynamical system on X is a triple $(X, \pi, \mathbb{R}^+)$, where $\mathbb{R}^+$ is the set of all nonnegative reals and $\pi : X \times \mathbb{R}^+ \to X$ is a continuous function satisfying

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(i) $\pi(x,0)=x$ for all $x\in X$ and
(ii) $\pi(\pi(x,s),t)=\pi(x,s+t)$ for all $x\in X$ and $t,s\in\mathbb{R}^+$.

Sometimes, we denote a semidynamical system $(X,\pi,\mathbb{R}^+)$ by (X,π). If $\mathbb{R}^+$ is replaced by $\mathbb{R}$, the triple $(X,\pi,\mathbb{R})$ is a dynamical system. For each $x\in X$, the function $\pi_x:\mathbb{R}^+\to X$ defined by $\pi_x(t)=\pi(x,t)$ is clearly continuous, and we call π_x the trajectory of x. In this paper, we denote $\pi(x,t)$ by xt for brevity. Similarly, for $A\subset X$ and $B\subset\mathbb{R}^+$, set $AB=\{xt: x\in A, t\in B\}$.

Let $(X,\pi,\mathbb{R}^+)$ be a semidynamical system. Given $x\in X$, the set

$$C^+(x)=\{\pi(x,t): t\in\mathbb{R}^+\}=\{x\}\mathbb{R}^+$$

is the *positive orbit* through x, which we also denote by $\pi^+(x)$. The closure of $C^+(x)$ in X is denoted by $K^+(x)$, i.e., $K^+(x)=\overline{C^+(x)}$. Given $x\in X$ and $r\in\mathbb{R}^+$, we set

$$C^+(x,r)=\{xt: 0\le t<r\}.$$

For $t\ge 0$ and $x\in X$, we define

$$F(x,t)=\{y:\pi(y,t)=x\}$$

and, for $\Delta\subset[0,+\infty)$ and $D\subset X$, we set

$$F(D,\Delta)=\cup\{F(x,t): x\in D, t\in\Delta\}.$$

A point $x\in X$ is called an *initial point*, if $F(x,t)=\emptyset$ for all $t>0$.

Now, we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(X,\pi;M,I)$ consists of a semidynamical system (X,π) together with a nonempty closed subset M of X and a continuous function $I:M\to X$ such that for any $x\in M$, there is an $\epsilon_x>0$ such that $F(x,(0,\epsilon_x))\cap M=\emptyset$ and $\pi(x,(0,\epsilon_x))\cap M=\emptyset$. Notice that the points of M are isolated in every trajectory of the semidynamical system (X,π). The set M is called the *impulsive set*, the function I is called the *impulsive function*, and we write $N=I(M)$, $x^+=I(x)$ for every $x\in M$. We also define

$$M^+(x)=(\pi^+(x)\cap M)\backslash\{x\}.$$

Let $(X,\pi;M,I)$ be an impulsive semidynamical system and let $x\in X$. The *impulsive trajectory* of x in $(X,\pi;M,I)$ is a function $\widetilde{\pi}_x$ defined on a subset $[0,T(x))$ of $\mathbb{R}^+$ ($T(x)$ may be ∞) to X inductively as follows: Set $x=x_0=x_0^+$. If $M^+(x_0)=\emptyset$, then $\widetilde{\pi}_x(t)=\pi(x,t)$ for all $t\in\mathbb{R}^+$. If $M^+(x_0)\ne\emptyset$, there exists a positive $s_0\in\mathbb{R}^+$ such that $\pi(x_0,s_0)=x_1\in M$ and $\pi(x_0,t)\notin M$ for all $0<t<s_0$. We define $\widetilde{\pi}_x$ on $[0,s_0]$ by

$$\widetilde{\pi}_x(t)=\begin{cases}\pi(x,t), & 0\le t<s_0,\\ x_1^+, & t=s_0.\end{cases}$$

To complete the definition of $\widetilde{\pi}_x$ in this case, we continue the above process starting at x_1^+. Thus, either $M^+\left(x_1^+\right)=\emptyset$, and we define $\widetilde{\pi}_x(t)=\pi\left(x_1^+,t-s_0\right)$ for $t\ge s_0$ and $T(x)=\infty$, or else $M^+\left(x_1^+\right)\ne\emptyset$, which implies that there is an $s_1>0$ as before, and we define

$$\widetilde{\pi}_x(t)=\begin{cases}\pi\left(x_1^+,t-s_0\right), & s_0\le t<s_0+s_1,\\ x_2^+, & t=s_0+s_1,\end{cases}$$

where $x_2=\pi\left(x_1^+,s_1\right)\in M$ and $\pi\left(x_1^+,t\right)\notin M$ for any t with $0<t<s_1$. This process either ends after a finite number of steps, if $M^+\left(x_n^+\right)=\emptyset$ for some value of n, or $M^+\left(x_n^+\right)\ne\emptyset$, $n=1,2,\ldots$, and the process continues infinitely. This gives rise to either a finite or infinite sequence of points x_n in X, and with each x_n^+ is associated a real number s_n, where $\pi\left(x_n^+,s_n\right)=x_{n+1}\in M$ and $\pi\left(x_n^+,t\right)\notin M$ for $0<t<s_n$. The interval of

definition of $\widetilde{\pi}_x$ is clearly $[0, T(x))$, where $T(x) = \sum_{n=0}^{\infty} s_n$. This completes the definition of the trajectory $\widetilde{\pi}_x$.

Analogous to the nonimpulsive case, an impulsive semidynamical system $(X, \pi; M, I)$ satisfies the following two properties:

(i) $\widetilde{\pi}(x, 0) = x$ for any $x \in X$ and
(ii) $\widetilde{\pi}(\widetilde{\pi}(x, s), t) = \widetilde{\pi}(x, s+t)$, for any $x \in X$, $s, t \in [0, T(x))$ such that $s+t \in [0, T(x))$.

Next, we define a function Φ from X into the extended positive reals $\mathbb{R}^+ \cup \{\infty\}$ as follows: Let $x \in X$. If $M^+(x) = \emptyset$, we set $\Phi(x) = \infty$; otherwise, $M^+(x) \neq \emptyset$ and we set $\Phi(x) = s$, where $\pi(x, t) \notin M$ for any $t \in (0, s)$ but $\pi(x, s) \in M$ (i.e., $\Phi(x)$ is the smallest positive time for which the positive trajectory of x meets M). For $x \in X$, we call $\Phi(x)$ the "*time without impulse*" of x and $\pi(x, \Phi(x))$ the *impulsive point* of x.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. We shall often use the following fact that for any $x \in X, 0 \leq t < T(x)$, there exists a $k = 0, 1, 2, \ldots$ such that $t = \sum_{i=0}^{k-1} \Phi\left(x_i^+\right) + t'$ and $\widetilde{\pi}(x, t) = \pi\left(x_k^+, t\prime\right)$, where $0 \leq t\prime < \Phi\left(x_k^+\right)$, $x_0^+ = x$ and $\sum_{i=0}^{-1} \Phi\left(x_i^+\right) = 0$.

Fix $x \in X$; one of the following properties holds:

(i) $M^+(x) = \emptyset$ and hence the trajectory of x has no discontinuities.
(ii) For some $n \geq 1$, each x_k^+ is defined for $k = 1, \ldots, n$ and $M^+\left(x_n^+\right) = \emptyset$. In this case, the trajectory of x has a finite number of discontinuities.
(iii) For any $k \geq 1$, x_k^+ is defined and $M^+\left(x_k^+\right) \neq \emptyset$. In this case, the trajectory of x has infinitely many discontinuities.

Clearly, if x satisfies (i) or (ii), then $T(x) = \infty$. If x satisfies (iii), then either $T(x) = \infty$ or $T(x) \in (0, \infty)$.

For brevity, sometimes we denote an impulsive semidynamical system $(X, \pi; M, I)$ by $(X, \widetilde{\pi})$. Clearly, impulsive semidynamical systems present a more complex structure than the nonimpulsive systems because of their irregularity. These systems present interesting and important phenomena such as "beating," "dying," "merging," "noncontinuation of solutions," etc. In recent years, the theory of such systems has been studied and developed intensively. For instance, in [3], Ciesielski investigated the continuity of Φ. The stability of invariant sets was studied by Ciesielski [4] and Kaul [7]. Some interesting results about the limit sets were established by Bonotto and Federson [2] and Ding [5]. For the elementary properties of impulsive dynamical systems, the reader is referred to books [1, 6, 8, 9].

Let $(X, \widetilde{\pi})$ be an impulsive semidynamical system. Analogous to the case in semidynamical systems, we denote $\widetilde{\pi}(x, t)$ by $x * t$. For $A \subset X$, $B \subset \mathbb{R}^+$, set

$$A * B = \{x * t : x \in A, t \in B\}.$$

Thus, the set $\widetilde{C}^+(x) = \{x\} * \mathbb{R}^+ = \left\{\widetilde{\pi}_x(t) : t \in \mathbb{R}^+\right\}$ is called the *positive orbit* through x in $(X, \widetilde{\pi})$ and its closure is denoted by $\widetilde{K}^+(x)$. Fix $x \in X$, $\widetilde{L}^+(x) = \{y \in X : x * t_n \to y$ for some $t_n \to T(x)$and $t_n < T(x)\}$ is the *positive limit set* of x, equivalently $\widetilde{L}^+(x) = \cap\left\{\widetilde{K}^+(x * t) : t \in \mathbb{R}^+\right\}$. The *positive prolongation* $\widetilde{D}^+(x)$ is defined by $\widetilde{D}^+(x) = \{y \in X : x_n * t_n \to y$ for some $\{x_n\} \subset X$ with $x_n \to x$ and $\{t_n\} \subset [0, T(x))\}$.

Clearly, the operators $\widetilde{K}^+$, $\widetilde{L}^+$, and $\widetilde{D}^+$ can be considered as the maps from X to 2^X, where 2^X is the set of all subsets of X. In [10], Saperstone and Nishihama investigated both the upper semicontinuity and lower semicontinuity of those limit set maps in a given semidynamical system $\left(X, \pi, \mathbb{R}^{+}\right)$ defined on a metric space. It was proved in [10] that the semicontinuities are closely related to the stabilities. Also, Saperstone and Nishihama generalized an important result prompted by Boyarsky in [11], whose work concerned a characterization of the limit sets of probability measures arising from diffusion processes. In this paper, we go on the studies about semicontinuity of limit set maps in an impulsive semidynamical system. In fact, we prove that if the limit sets are stable (eventually stable or eventually weakly stable) then the corresponding limit set maps are upper semicontinuous or lower semicontinuous, respectively (see Theorems 3.5 and 3.6). Also, we show that the semicontiuity of limit set maps implies the minimality of corresponding limit sets (see Theorems 4.6 and 4.7). Finally, we give some sufficient conditions to guarantee that limit sets are minimal.

2 Definitions and Notations

Throughout this paper, there exists a semidynamical system (X, π) and an impulsive semidynamical system $(X, \widetilde{\pi})$ defined on a metric space (X, ρ). For any $r > 0$ and $A \subset X$, let $N_r(A) = \{x : \rho(x, A) < r\}$ denote the r-neighborhood of A, where the $\rho(x, A)$ is the distance from x to A. By $\mathcal{A}(A, \delta, \epsilon)$, we denote the annulus $\{x \in X : \delta < \rho(x, A) < \epsilon\}$. Let $\mathscr{K}$ denote the collection of nonempty compact subsets of X and $h : \mathscr{K} \times \mathscr{K} \to \mathbb{R}^{+}$ be the *Hausdorff* metric corresponding to ρ. That is, if $A, B \in \mathscr{K}$, $h(A, B) = \max\{\sup_{x\in A} \rho(x, B), \sup_{y\in B} \rho(A, y)\}$.

Definition 2.1 A map $f : X \to \mathscr{K}$ is called *upper semicontinuous* (USC) at x if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(y) \subset N_\epsilon(f(x))$ for all $y \in N_\delta(x)$. A map $f : X \to \mathscr{K}$ is called *lower semicontinuous* (LSC) at x if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(x) \subset N_\epsilon(f(y))$ for all $y \in N_\delta(x)$.

Equivalently, f is USC at x if and only if for any sequence $x_n \to x$,

$$\sup\{\rho(y, f(x)) : y \in f(x_n)\} \to 0 \text{ as } n \to \infty,$$

and f is LSC at x if and only if for any sequence $x_n \to x$,

$$\sup\{\rho(y, f(x_n)) : y \in f(x)\} \to 0 \text{ as } n \to \infty.$$

From the definition of Hausdorff metric on $\mathscr{K}$, one can easily show that a map f from X to $\mathscr{K}$ is continuous at x if and only if f is both USC and LSC at x.

From the point of an impulsive semidynamical system, the trajectories that are of interest are those with an infinite number of discontinuities and with $[0, \infty)$ as the interval of definition. We call them infinite trajectories.

Definition 2.2 Let $(X, \widetilde{\pi})$ be an impulsive semidynamical system. If $\left\{x_n^{+}\right\}$ is infinite and $\sum\limits_{i=0}^{\infty} s_i = \infty$, then we call $\widetilde{\pi}_x$ an *infinite impulsive trajectory* and set $\widehat{N} = \{x \in N : \widetilde{\pi}_x$ is an infinite impulsive trajectoty$\}$.

We define the function $t_n : \widehat{N} \longrightarrow \mathbb{R}^{+}$ as follows:

$$t_n(x) = \sum_{i=0}^{n-1} \Phi\left(x_i^{+}\right),\ n \geq 1,\ \text{and } t_0(x) = 0.$$

Throughout this paper, we assume that the following additional hypotheses hold:

(i) $N \bigcap M = \emptyset$, namely for any $x \in X$, $x_n^+ \notin M$, $n = 1, 2, 3, \ldots$,
(ii) Φ is continuous on $X \setminus M$,
(iii) For any $x \in X$, if $\Phi\left(x_n^+\right) < \infty$, for $n = 0, 1, 2, \ldots$, then $x \in \widehat{N}$,
(iv) $\widetilde{D}^+(x)$ is compact for every $x \in X$.

Obviously, under assumption (iii), $T(x) = \infty$ for any $x \in X$.

It is clear that $\widetilde{\pi}_x$ is not a continuous function. However, the following lemma holds:

Lemma 2.1 [7]. *Suppose that $\{x_n\}$ is a sequence in X and convergent to $x \in X \setminus M$. Then, for any $t \in [0, \infty)$, there exists a sequence of real numbers ϵ_n, $\epsilon_n \to 0$, such that $\widetilde{\pi}(x_n, t + \epsilon_n) \to \widetilde{\pi}(x, t)$. In particular, if $\widetilde{\pi}(x, t) = x_1^+$, then $\widetilde{\pi}(x_n, t + \epsilon_n)$ can be chosen to be $x_{n,1}^+$.*

Definition 2.3 [7]. A subset C of X is said to be *positively invariant* in $(X, \widetilde{\pi})$ if for any $x \in C \setminus M$, $\widetilde{C}^+(x) \subset C$.

Obviously, $\widetilde{C}^+(x)$ is positively invariant set for any $x \in X$.

Lemma 2.2 [7]. *Suppose $C \subset X$ is positively invariant in $(X, \widetilde{\pi})$, then $\overline{C}$ is positively invariant in $(X, \widetilde{\pi})$.*

Proof Suppose $x \in \overline{C} \setminus M$ and $t \in [0, \infty)$. Then, there exists a sequence $\{x_n\}$ in C with $x_n \to x$. Without loss of generality, we assume that $x_n \notin M$. Hence, by Lemma 2.1, $\widetilde{\pi}(x_n, t + \epsilon_n) \to \widetilde{\pi}(x, t)$ for some sequence $\epsilon_n \to 0$. Since C is positively invariant and $x_n \in C \setminus M$, so $\widetilde{\pi}(x_n, t + \epsilon_n) \in C$; consequently, $\widetilde{\pi}(x, t) \in \overline{C}$. This completes the proof.

Analogous to the case of semidynamical systems, we can show that $\widetilde{D}^+(x) = \bigcap\{\overline{N_r(x) * \mathbb{R}^+} : r > 0\}$. Thus, the set $\widetilde{D}^+(x)$ is positively invariant. □

3 The Semicontinuity of Map $\widetilde{K}^+$

Theorem 3.2 generalizes a result in [10], and its proof is not a simple analogy of that in semidynamical systems. In order to prove Theorem 3.2, we need the following lemma:

Lemma 3.1 *Let $(X, \widetilde{\pi})$ be an impulsive semidynamical system and $x \in X$. If $\widetilde{D}^+(x) = \bigcap\{\overline{N_r(x) * \mathbb{R}^+} : r > 0\}$ has a compact neighborhood, then for any $\epsilon > 0$, there is a $\delta > 0$ such that $\overline{N_\delta(x) * \mathbb{R}^+} \subset N_\epsilon(\widetilde{D}^+(x))$.*

Proof Suppose to the contrary that there exist an $\epsilon > 0$ and sequences $\{x_n\}$, $\{t_n\}$ with $x_n \to x$, $t_n \in \mathbb{R}^+$ such that $x_n * t_n \notin \overline{N_\epsilon(\widetilde{D}^+(x))}$. Without loss of generality, we assume that $\overline{N_\epsilon(\widetilde{D}^+(x))}$ is compact, since $\widetilde{D}^+(x)$ has a compact neighborhood and $\widetilde{D}^+(x)$ is compact.

If there are infinitely many n such that $t_n < \Phi(x_n)$, we choose a subsequence $\{n_k\}$ such that $t_{n_k} < \Phi(x_{n_k})$ for each k. Then, $x_{n_k} * t_{n_k} = x_{n_k} t_{n_k}$, and for every k, there is an $s_{n_k} \in [0, t_{n_k}]$ such that $x_{n_k} * s_{n_k} \in \partial \overline{N_\epsilon(\widetilde{D}^+(x))}$. By the compactness of $\partial \overline{N_\epsilon(\widetilde{D}^+(x))}$, we can assume that $x_{n_k} * s_{n_k} \to y \in \widetilde{D}^+(x)$. This is a contradiction since $\partial \overline{N_\epsilon(\widetilde{D}^+(x))} \bigcap \widetilde{D}^+(x) = \emptyset$.

If there are only finitely many n such that $t_n < \Phi(x_n)$ holds, we may assume that $\Phi(x_n) \le t_n$ for all $n \in N$. For brevity, we set $\Delta_n = \left\{t : \widetilde{\pi}(x_n, t) \in N_\epsilon(\widetilde{D}^+(x))\right\}$ and $\tau_n = \inf\left\{t : \widetilde{\pi}(x_n, t) \notin N_\epsilon(\widetilde{D}^+(x))\right\}$. Notice that $x_n * \tau_n \notin N_\epsilon\left(\widetilde{D}^+(x)\right)$ since $\widetilde{\pi}(x_n, \cdot)$ is right-continuous.

Now, take a $\delta \in (0, \epsilon)$. We show that there is an $n_0 \in N$ such that for any $n \ge n_0$, $p_n \in x_n * \Delta_n$, there is a $q_n \in \widetilde{D}^+(x)$ such that $\rho(p_n, q_n) < \delta$. Otherwise, there are sequences $n_k \to +\infty$ and $u_{n_k} \in \Delta_{n_k}$ such that $\rho(\widetilde{\pi}\left(x_{n_k}, u_{n_k}\right), \widetilde{D}^+(x)) \ge \delta$. This means that $\widetilde{\pi}\left(x_{n_k}, u_{n_k}\right) \in \overline{\mathcal{A}\left(\widetilde{D}^+(x), \delta, \epsilon\right)}$. Since $\overline{\mathcal{A}(\widetilde{D}^+(x), \delta, \epsilon)}$ is compact, we assume that $\widetilde{\pi}\left(x_{n_k}, u_{n_k}\right) \to y \in \widetilde{D}^+(x)$. It is a contradiction.

For any $n \ge n_0$, we have $x_n * [0, \tau_n) \subset N_\delta\left(\widetilde{D}^+(x)\right)$ since $[0, \tau_n) \subset \Delta_n$. And from the discussion above, we also have $x_n * \tau_n \notin N_\epsilon\left(\widetilde{D}^+(x)\right)$, which means that $\widetilde{\pi}(x_n, \cdot)$ is not continuous at τ_n; hence, $x_n * \tau_n = I(y_n)$ for some $y_n \in M$. Moreover, $y_n \in \overline{N_\delta\left(\widetilde{D}^+(x)\right)}$ since $\widetilde{\pi}(x_n, t) \to y_n$ as $t \nearrow \tau_n$. According to the compactness of $\overline{N_\delta\left(\widetilde{D}^+(x)\right)}$, we assume that $y_n \to y \in \overline{N_\delta\left(\widetilde{D}^+(x)\right)}$. From the continuity of I, we get $I(y_n) \to I(y)$; this means that $I(y) \notin \widetilde{D}^+(x))$ since $I(y_n) \notin N_\epsilon\left(\widetilde{D}^+(x)\right)$ for every n. On the other hand, we have

$$I(y) = \lim_{n\to\infty} I(y_n) = \lim_{n\to\infty} x_n * \tau_n = \lim_{n\to\infty} \widetilde{\pi}(x_n, \tau_n) \in \widetilde{D}^+(x).$$

This is a contradiction. Thus, the proof is completed. □

Theorem 3.2 *If $\widetilde{K}^+$ is USC at x, then $\widetilde{K}^+(x) = \widetilde{D}^+(x)$. The converse is true provided that $\widetilde{K}^+(x)$ has a compact neighborhood.*

Proof Suppose $\widetilde{K}^+$ is USC at x. Since $\widetilde{K}^+(x) \subset \widetilde{D}^+(x)$ always holds, we need only to show that $\widetilde{D}^+(x) \subset \widetilde{K}^+(x)$. Let $y \in \widetilde{D}^+(x)$. By the definition of $\widetilde{D}^+(x)$, there are sequences $x_n \to x$, $t_n \in \mathbb{R}^+$ such that $x_n * t_n \to y$ as $n \to \infty$. As each $x_n * t_n \in \widetilde{K}^+(x_n)$ and $\widetilde{K}^+$ is USC at x, we have

$$\rho(x_n * t_n, \widetilde{K}^+(x)) \le \sup\left\{\rho(y, \widetilde{K}^+(x)) : y \in \widetilde{K}^+(x_n)\right\} \to 0.$$

Since $\widetilde{K}^+(x)$ is closed, it follows that $y \in \widetilde{K}^+(x)$. Hence, $\widetilde{D}^+(x) \subset \widetilde{K}^+(x)$ and so $\widetilde{D}^+(x) = \widetilde{K}^+(x)$.

Conversely, suppose that $\widetilde{K}^+(x) = \widetilde{D}^+(x)$ and $\widetilde{K}^+(x)$ has a compact neighborhood. According to Lemma 3.1, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\overline{N_\delta(x) * \mathbb{R}^+} \subset N_\epsilon\left(\widetilde{K}^+(x)\right)$.

Thus, for any $y \in N_\delta(x)$, we have

$$\widetilde{K}^+(y)) \subset \overline{N_\delta(x) * \mathbb{R}^+} \subset N_\epsilon\left(\widetilde{K}^+(x)\right).$$

Consequently, $\widetilde{K}^+$ is USC at x.

In order to show that $\widetilde{K}^+$ is always LSC on $X \setminus M$, we need the following new characterization of $\widetilde{K}^+(x)$ and $\widetilde{L}^+(x)$: □

Definition 3.1 [10]. Let $x \in X$ and set

$$\widetilde{Q}^+(x) = \left\{y \in X \mid \forall x_n \to x,\ \exists t_n \in \mathbb{R}^+ \text{such that } x_n * t_n \to y\right\},$$
$$\widetilde{W}^+(x) = \{y \in X \mid \forall x_n \to x,\ \exists t_n \to +\infty \text{ such that } x_n * t_n \to y\}.$$

Theorem 3.3 $\widetilde{Q}^+(x) = \widetilde{K}^+(x)$ *and* $\widetilde{W}^+(x) = \widetilde{L}^+(x)$ *for each* $x \in X \setminus M$.

Proof Suppose $x \in X \setminus M$. It is easy to see that $\widetilde{Q}^+(x) \subset \widetilde{K}^+(x)$ and $\widetilde{W}^+(x) \subset \widetilde{L}^+(x)$ by picking the sequence $x_n = x$. We only show the reverse inclusion. First, we show that $x * t \in \widetilde{Q}^+(x)$ for any $t \in \mathbb{R}^+$. In fact, let $x_n \to x$; according to Lemma 2.1, there is a sequence of real numbers $\epsilon_n, \epsilon_n \to 0$ such that $x_n * (t + \epsilon_n) \to x * t$. Then, let $t_n = t + \epsilon_n$ for every n. It follows that $x_n * t_n = x_n * (t + \epsilon_n) \to x * t$. Thus, $\widetilde{C}^+(x) \subset \widetilde{Q}^+(x) \subset \widetilde{K}^+(x)$. Furthermore, it is evident from the definition of $\widetilde{Q}^+(x)$ that $\widetilde{Q}^+(x)$ is closed. Consequently, $\widetilde{Q}^+(x) = \widetilde{K}^+(x)$.

To prove that $\widetilde{L}^+(x) \subset \widetilde{W}^+(x)$, it is sufficient to show that $\widetilde{W}^+(x) \supset \bigcap \{\widetilde{Q}^+(x * n) : n \in N^+\}$. Suppose $y \in \widetilde{Q}^+(x * n)$ for every $n \in N$ and $x_k \to x$. Then, for each n, there is a sequence $\epsilon_k^n \to 0$ such that $x_k * (n + \epsilon_k^n) \to x * n$ as $k \to \infty$ according to Lemma 2.1. For each n, there exists a $t_k^n \in \mathbb{R}^+$ such that $(x_k * (n + \epsilon_k^n)) * t_k^n = x_k * (n + \epsilon_k^n + t_k^n) \to y$ as $k \to \infty$. For each n, choose k_n such that $\rho((x_k * (n + \epsilon_k^n)) * t_k^n, y) < 1/n$ whenever $k \geq k_n$. Without loss of generality, we can assume that $k_1 < k_2 < \ldots$. We now construct a sequence $t_k \to \infty$ such that $x_k * t_k \to y$. For $1 \leq k < k_2$, set $t_k = 1 + \epsilon_k^1 + t_k^1$. For $k_n \leq k < k_{n+1} (n \geq 2)$, set $t_k = (n + \epsilon_k^n + t_k^n)$. Then, we have $t_k \to \infty$ and $x_k * t_k \to y$ as $k \to \infty$. Thus, $y \in \widetilde{W}^+(x)$ and the proof is completed. □

Theorem 3.4 *The map $\widetilde{K}^+$ is LSC on $X \setminus M$.*

Proof Let $x \in X \setminus M$. Suppose that $\widetilde{K}^+$ is not LSC at x. Then, there exist an $\epsilon > 0$ and a sequence $x_n \to x$ such that $\sup\{\rho(y, \widetilde{K}^+(x_n)) : y \in \widetilde{K}^+(x)\} \geq \epsilon$ for each n. From the compactness of $\widetilde{K}^+(x)$, there exists a sequence $y_n \in \widetilde{K}^+(x)$ such that $\rho(y_n, \widetilde{K}^+(x_n)) \geq \epsilon$. We may assume that $y_n \to y \in \widetilde{K}^+(x) = \widetilde{Q}^+(x)$ for some y. By the definition of $\widetilde{Q}^+(x)$, there is a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $x_n * t_n \to y$. Consequently, $\rho(x_n * t_n, y_n) \to 0$. As $x_n * t_n \in \widetilde{K}^+(x_n)$ for each n, we have $\rho(y_n, \widetilde{K}^+(x_n)) \leq \rho(y_n, x_n * t_n) \to 0$. It is a contradiction. Hence, $\widetilde{K}^+$ is LSC at x. □

Corollary 3.5 *If $\widetilde{K}^+(x)$ is compact for some $x \in X \setminus M$, then the following statements are equivalent:*

(i) $\widetilde{K}^+$ *is USC at x.*
(ii) $\widetilde{K}^+$ *is continuous at x.*

Definition 3.2 [7]. Let $(X, \widetilde{\pi})$ be an impulsive semidynamical system. A subset C of X is said to be *stable* in $(X, \widetilde{\pi})$ if every neighborhood of C contains a positively invariant neighborhood of C. Equivalently, C is said to be *stable* if for each neighborhood U of C, there is a neighborhood V of C such that $\widetilde{C}^+(x) \subset U$ for every $x \in V \setminus M$. $\widetilde{\pi}$ is said to be *stable* if each $\widetilde{K}^+(x)$ is positively stable.

Note that the stability defined in Definition 3.2 is a positive stability. Since we are dealing with the system only in positive direction, we have dispensed with the modifier "positive" for convenience.

The following theorem shows that the upper semicontinuity of $\widetilde{K}^+$ is related to the stability of $\widetilde{K}^+(x)$:

Theorem 3.6 *Let $(X, \widetilde{\pi})$ be an impulsive semidynamical system and $x \in X$. If $\widetilde{K}^+$ is USC on $\widetilde{K}^+(x) \setminus M$, then $\widetilde{K}^+(x)$ is stable.*

Proof Suppose $\widetilde{K}^+$ is USC on $\widetilde{K}^+(x) \setminus M$. Let U be a neighborhood of $\widetilde{K}^+(x)$. Since $\widetilde{K}^+(x)$ is compact, we may choose an $\epsilon > 0$ such that $N_\epsilon\left(\widetilde{K}^+(x)\right) \subset U$. Then, for any $z \in \widetilde{K}^+(x) \setminus M$, there is a $\delta_z > 0$ such that $\widetilde{K}^+(y) \subset N_{\epsilon/2}\left(\widetilde{K}^+(z)\right)$ whenever $y \in N_{\delta_z}(z)$. On the other hand, $\widetilde{K}^+(z) \subset \widetilde{K}^+(x)$ for every $z \in \widetilde{K}^+(x) \setminus M$ since $\widetilde{K}^+(x)$ is a positively invariant and closed set. Set $V = \bigcup\left\{N_{\delta_z}(z) : z \in \widetilde{K}^+(x) \setminus M\right\}$. By the definition of an impulsive semidynamical system, we know that $int\left(M \bigcap \widetilde{L}^+(x)\right) = \emptyset$. That is, $int\left(\widetilde{K}^+(x) \bigcap M\right) = \emptyset$. Thus, V is an open neighborhood of $\widetilde{K}^+(x)$. It follows that

$$\widetilde{C}^+(y) \subset \bigcup\left\{N_{\epsilon/2}\left(\widetilde{K}^+(z)\right) : z \in \widetilde{K}^+(x) \setminus M\right\} \subset N_\epsilon\left(\widetilde{K}^+(x)\right) \subset U \text{ for all } y \in V \setminus M.$$

Thus, $\widetilde{K}^+(x)$ is stable. □

Theorem 3.7 *Let $x \in X \setminus M$. If $\widetilde{K}^+(x)$ is stable, then $\widetilde{K}^+$ is USC at x.*

Proof Suppose $\widetilde{K}^+(x)$ is stable for some $x \in X \setminus M$. For any $\epsilon > 0$, there exists a $\delta > 0$ with $N_\delta(x) \cap M = \emptyset$ such that $\widetilde{C}^+(y) \subset N_{\epsilon/2}\left(\widetilde{K}^+(x)\right)$ for all $y \in N_\delta\left(\widetilde{K}^+(x)\right) \setminus M$. In particular, if $y \in N_\delta(x) \subset N_\delta\left(\widetilde{K}^+(x)\right) \setminus M$, then

$$\widetilde{K}^+(y) \subset \overline{N_{\epsilon/2}\left(\widetilde{K}^+(x)\right)} \subset N_\epsilon\left(\widetilde{K}^+(x)\right).$$

Consequently, $\widetilde{K}^+$ is USC at x. □

Theorem 3.8 *Let $x \in X \setminus M$. If $\widetilde{C}^+(x)$ is stable, then $\widetilde{K}^+$ and $\widetilde{D}^+$ are USC at x.*

Proof Suppose $\widetilde{C}^+(x)$ is stable. For a fixed $\epsilon > 0$, there is a neighborhood U of $\widetilde{C}^+(x)$ such that $(U \setminus M) * \mathbb{R}^+ \subset N_{\epsilon/2}\left(\widetilde{C}^+(x)\right)$. Since $x \notin M$ and U is also a neighborhood of x, we may choose a $\delta > 0$ such that $N_\delta(x) \cap M = \emptyset$ and $N_\delta(x) \subset U$, i.e., $N_\delta(x) \subset U \setminus M$. Consequently, we have

$$N_\delta(x) * \mathbb{R}^+ \subset N_{\epsilon/2}\left(\widetilde{C}^+(x)\right) \subset N_{\epsilon/2}\left(\widetilde{K}^+(x)\right) \subset N_{\epsilon/2}\left(\widetilde{D}^+(x)\right).$$

Since for each $y \in N_\delta(x)$, there exists a $\delta_1 > 0$ such that $N_{\delta_1}(y) \subset N_\delta(x)$. Hence, we also have

$$N_{\delta_1}(y) * \mathbb{R}^+ \subset N_{\epsilon/2}\left(\widetilde{C}^+(x)\right) \subset N_{\epsilon/2}\left(\widetilde{K}^+(x)\right) \subset N_{\epsilon/2}\left(\widetilde{D}^+(x)\right).$$

Since $\widetilde{D}^+(y) = \bigcap\left\{\overline{N_r(y) * \mathbb{R}^+} : r > 0\right\}$, we have $\widetilde{D}^+(y) \subset \overline{N_{\delta_1}(y) * \mathbb{R}^+}$. Therefore,

$$\widetilde{K}^+(y) \subset \widetilde{D}^+(y) \subset \overline{N_{\epsilon/2}\left(\widetilde{K}^+(x)\right)} \subset N_\epsilon\left(\widetilde{K}^+(x)\right) \subset N_\epsilon\left(\widetilde{D}^+(x)\right) \text{ for all } y \in N_\delta(x).$$

Consequently, $\widetilde{K}^+$ and $\widetilde{D}^+$ are USC at x. □

4 The Semicontinuity of Map $\widetilde{L}^+$

Now, we present a result about the continuity of $\widetilde{\pi}$ at a nonimpulsive time.

Lemma 4.1 *Let $(X, \widetilde{\pi})$ be an impulsive semidynamical system. Then, for each $x \in X \setminus M$, $\epsilon > 0$ and $t \neq t_n(x)$, $n = 1, 2, 3, \ldots$, where $t_n(x) = \sum_{i=0}^{n-1} \Phi\left(x_i^+\right)$, there exists a $\delta = \delta(\epsilon, x, t) > 0$ such that $N_\delta(x) * t \subset N_\epsilon(x * t)$.*

Proof Suppose it is not the case; then, there exist an $\epsilon > 0$ and a sequence $x_n \to x$ such that $x_n * t \notin N_\epsilon(x * t)$. Since $t \neq t_n(x)$ for all $n \in N$, we may assume that $t_k(x) < t < t_{k+1}(x)$ for some $k \in N$. By the continuity of $\Phi(x)$, we have $t_k(x)$ is continuous on $X \setminus M$ for each k. Therefore, $t_k(x_n) \to t_k(x)$ as $n \to \infty$ for every $k \in N$. Without loss of generality, we assume that $t_k(x_n) < t < t_{k+1}(x_n)$ holds for every $n \in N$. Since for each k, $x_{n,k}^+ \to x_k^+$ as $n \to \infty$, we have

$$x_n * t = \pi\left(x_{n,k}^+, t - t_k(x_n)\right) \to \pi\left(x_k^+, t - t_k(x)\right) = x * t \in N_\epsilon(x * t).$$

This is impossible. Consequently, the proof is completed. □

Definition 4.1 A subset C of X is said to be *positively minimal* if $C = \widetilde{K}^+(x)$ for each $x \in C \setminus M$. Or, equivalently, C is said to be *positively minimal* if C is closed and positively invariant, but none of its nonempty proper subsets has these two properties.

Lemma 4.2 *Suppose a subset $C \subset X$; then, C is positively minimal if and only if $C = \widetilde{L}^+(x)$ for each $x \in C \setminus M$.*

The proof is trivial, so it is omitted.

Definition 4.2 [10]. A set $C \subset X$ is called *eventually stable* if for every neighborhood U of C, there exists a neighborhood V of C such that if $y \in V \setminus M$, there is a $\tau = \tau(y) > 0$ such that $y * [\tau, \infty) \subset U$. If the τ above does not depend upon $y \in V \setminus M$, then C is called *uniformly eventually stable*.

Theorem 4.3 *Suppose $x \in X$. If the map $\widetilde{L}^+$ is USC on $\widetilde{L}^+(x)\setminus M$, then $\widetilde{L}^+(x)$ is positively minimal.*

Proof Suppose $\widetilde{L}^+$ is USC on $\widetilde{L}^+(x) \setminus M$. Let $y \in \widetilde{L}^+(x) \setminus M$. We need only to show $\widetilde{L}^+(x) \subset \widetilde{L}^+(y)$. Since $y \in \widetilde{L}^+(x)$, there is a sequence $t_n \to \infty$ such that $x * t_n \to y$. Let $z \in \widetilde{L}^+(x)$. Then, $z \in \widetilde{L}^+(x * t_n) = \widetilde{L}^+(x)$ for each n. Since

$$\rho\left(z, \widetilde{L}^+(y)\right) \leq \sup\left\{\rho\left(w, \widetilde{L}^+(y)\right) : w \in \widetilde{L}^+\left(x * t_n\right)\right\} \to 0 \text{ as } n \to \infty,$$

it follows that $z \in \widetilde{L}^+(y)$. Consequently, $\widetilde{L}^+(x) \subset \widetilde{L}^+(y)$ and $\widetilde{L}^+(x)$ are positively minimal. □

Theorem 4.4 *Suppose $x \in X$. If the map $\widetilde{L}^+$ is USC on $\widetilde{L}^+(x)\setminus M$, then $\widetilde{L}^+(x)$ is eventually stable.*

Proof Suppose $\widetilde{L}^+$ is USC on $\widetilde{L}^+(x) \setminus M$. Let $\epsilon > 0$ and $y \in \widetilde{L}^+(x) \setminus M$; there exists a $\delta_y > 0$ such that $\widetilde{L}^+(z) \subset N_\epsilon\left(\widetilde{L}^+(y)\right)$ for $z \in N_{\delta_y}(y)$. According to Theorem 4.3, we have that $\widetilde{L}^+(x)$ is positively minimal. Hence, $\widetilde{L}^+(x) = \widetilde{L}^+(y)$ for every $y \in \widetilde{L}^+(x) \setminus M$. Thus, $\widetilde{L}^+(z) \subset N_\epsilon\left(\widetilde{L}^+(x)\right)$ for each $z \in N_{\delta_y}(y)$. Set $V = \bigcup\left\{N_{\delta_y}(y) : y \in \widetilde{L}^+(x) \setminus M\right\}$. Similar to the case in Theorem 3.6, we have that V is an open neighborhood of $\widetilde{L}^+(x)$. Then, $\widetilde{L}^+(z) \subset N_\epsilon\left(\widetilde{L}^+(x)\right)$ for each $z \in V \setminus M$. This shows that $\widetilde{L}^+(x)$ is eventually stable. □

Theorem 4.5 *Suppose $x \in X$. If $\widetilde{L}^+(x)$ is positively minimal and eventually stable, then $\widetilde{L}^+$ is USC on $\widetilde{L}^+(x) \setminus M$.*

Proof Suppose that $\widetilde{L}^+(x)$ is positively minimal and eventually stable. Let $y \in \widetilde{L}^+(x) \setminus M$. Then, for each $\epsilon > 0$, we choose a $\delta > 0$ with $N_\delta(y) \bigcap M = \emptyset$ such that for every

$z \in N_\delta\left(\widetilde{L}^+(x)\right) \setminus M$, there exists $\tau = \tau(z) > 0$ such that $\widetilde{C}^+(z * \tau) \subset N_{\epsilon/2}\left(\widetilde{L}^+(x)\right)$. Therefore, for every $z \in N_\delta(y) \subset N_\delta\left(\widetilde{L}^+(x)\right) \setminus M$, we have

$$\widetilde{L}^+(z) \subset N_{\epsilon/2}\left(\widetilde{L}^+(x)\right) \subset N_\epsilon\left(\widetilde{L}^+(x)\right) = N_\epsilon\left(\widetilde{L}^+(y)\right).$$

Consequently, $\widetilde{L}^+$ is USC at y. The proof is completed. □

Theorem 4.6 *Given $x \in X$. If the map $\widetilde{L}^+$ is USC on $\widetilde{L}^+(x) \setminus M$ and $\widetilde{L}^+(x)$ is eventually stable, then $\widetilde{L}^+$ is USC on $\widetilde{A}\left(\widetilde{L}^+(x)\right)$, where $\widetilde{A}\left(\widetilde{L}^+(x)\right) = \left\{y \in X : \widetilde{L}^+(y) \subset \widetilde{L}^+(x)\right\}$.*

Proof Note that $\widetilde{A}\left(\widetilde{L}^+(x)\right)$ is the region of attraction of $\widetilde{L}^+(x)$ (see [12]). Let $y \in \widetilde{A}\left(\widetilde{L}^+(x)\right)$; then, $\widetilde{L}^+(y) \subset \widetilde{L}^+(x)$. According to Theorem 4.3, $\widetilde{L}^+(x)$ is positively minimal; therefore, $\widetilde{L}^+(y) = \widetilde{L}^+(x)$. Thus, it will be sufficient to show that $\widetilde{L}^+$ is USC at $x \in \widetilde{A}\left(\widetilde{L}^+(x)\right)$. Since $\widetilde{L}^+(x)$ is eventually stable, for each $\epsilon > 0$, there is a $\delta_1 > 0$ such that for every $z \in N_{\delta_1}\left(\widetilde{L}^+(x)\right) \setminus M$, there exists $T = T(z) \geq 0$ such that $\widetilde{C}^+(z * T) \subset N_{\epsilon/2}\left(\widetilde{L}^+(x)\right)$. On the other hand, there exists a $\tau > 0$ such that $x * \tau \in N_{\delta_1}\left(\widetilde{L}^+(x)\right) \setminus M$, where $\tau \neq t_n(x)$, $n = 1, 2, 3, \ldots$. By Lemma 4.1, there exists a $\delta > 0$ such that $N_\delta(x) * \tau \subset N_{\delta_1}\left(\widetilde{L}^+(x)\right) \setminus M$. Thus, for each $y \in N_\delta(x)$, we have $\widetilde{L}^+(y) = \widetilde{L}^+(y * \tau) \subset \overline{N_{\epsilon/2}\left(\widetilde{L}^+(x)\right)} \subset N_\epsilon\left(\widetilde{L}^+(x)\right)$. Hence, $\widetilde{L}^+$ is USC at x. □

Theorem 4.7 *If $C \subset X$ is a nonempty, compact, and positively invariant set, then C contains a positively minimal set.*

Proof Let $C \subset X$ be a nonempty, compact, and positively invariant set. Then, the collection of all nonempty, closed, and positively invariant subsets of C is partially ordered by set inclusion. Now, let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a linearly ordered family of nonempty, closed, and positively invariant subsets of C. From the compactness of C, it follows that $\bigcap\{C_\lambda : \lambda \in \Lambda\}$ is a nonempty, closed, and positively invariant subset of C, and clearly, it is the lower bound of the linearly ordered family $\{C_\lambda\}_{\lambda \in \Lambda}$. Consequently, by an equivalent version of Zorn's lemma, we obtain that C contains a positively minimal set. □

Theorem 4.8 *Let $x \in X$. If $\widetilde{L}^+(y)$ is eventually stable for each $y \in \widetilde{L}^+(x) \setminus M$, then $\widetilde{L}^+(x)$ is positively minimal.*

Proof If $\widetilde{L}^+(x)$ is not positively minimal, then by Theorem 4.7, there exists a positively minimal set C with $\emptyset \neq C \subsetneq \widetilde{L}^+(x)$. Obviously, $x \notin C \setminus M$. Let $y \in \widetilde{L}^+(x) \setminus C$, $0 < \epsilon < \rho(y, C)$, and $z \in C \setminus M$ $\left(\subset \widetilde{L}^+(x) \setminus M\right)$. By assumption, $\widetilde{L}^+(z)$ is eventually stable; then, there exists a $\delta > 0$ such that for every $y\prime \in N_\delta\left(\widetilde{L}^+(z)\right) \setminus M$, there is a $T = T(y\prime) \geq 0$ such that $\widetilde{C}^+(y\prime * T) \subset N_{\epsilon/2}\left(\widetilde{L}^+(z)\right)$. Since $z \in C \setminus M \subset C = \widetilde{L}^+(z) \subset \widetilde{L}^+(x)$, there is a $\tau_1 \geq 0$ such that $x * \tau_1 \in N_\delta\left(\widetilde{L}^+(z)\right) \setminus M$, where $\tau_1 \neq t_n(x)$, $n = 1, 2, 3, \ldots$. Thus, there exists a $T_1 = T_1(x * \tau_1) \geq 0$ such that $(x * \tau_1) * [T_1, \infty) = x * [\tau_1 + T_1, \infty) \subset N_{\epsilon/2}\left(\widetilde{L}^+(z)\right)$. Hence, $\widetilde{L}^+(x) \subset N_{\epsilon/2}\left(\widetilde{L}^+(z)\right) \subset N_\epsilon\left(\widetilde{L}^+(z)\right)$. It leads to a contradiction of the fact $y \in \widetilde{L}^+(x)$ but $y \notin N_\epsilon\left(\widetilde{L}^+(z)\right) = N_\epsilon(C)$. Consequently, $\widetilde{L}^+(x)$ must be positively minimal. □

Definition 4.3 [10]. A subset $C \subset X$ is called *eventually weakly stable* if for every neighborhood U of C, there exists a neighborhood V of C such that for every $y \in V \setminus M$ there is a sequence $t_n \to \infty$ such that $y * t_n \in U$.

Theorem 4.9 *Let $x \in X$. For each $y \in \widetilde{L}^+(x) \setminus M$, $\widetilde{L}^+(y)$ is positively minimal and $\widetilde{L}^+(y) \bigcap M = \emptyset$. Then, the map $\widetilde{L}^+$ is LSC on $\widetilde{L}^+(x) \setminus M$ if and only if $\widetilde{L}^+(z)$ is eventually weakly stable for every $z \in \widetilde{L}^+(x) \setminus M$.*

Proof Suppose that $\widetilde{L}^+$ is LSC on $\widetilde{L}^+(x) \setminus M$. If $\widetilde{L}^+(z)$ is not eventually weakly stable for some $z \in \widetilde{L}^+(x) \setminus M$, then there exist an $\epsilon > 0$ and a sequence $\{y_n\} \subset X \setminus M$ with $y_n \to y \in \widetilde{L}^+(z) \setminus M$ such that $y_n * \tau$ is ultimately in $X \setminus N_\epsilon\left(\widetilde{L}^+(z)\right)$. Hence, $\widetilde{L}^+(y_n) \subset X \setminus N_\epsilon\left(\widetilde{L}^+(z)\right)$. It follows that

$$\sup\left\{\rho(\omega, \widetilde{L}^+(y_n)) : \omega \in \widetilde{L}^+(y)\right\} \geq \epsilon \quad \text{with } y_n \to y.$$

This is a contradiction, since $\widetilde{L}^+$ is LSC at $y \in \widetilde{L}^+(x) \setminus M$.

Conversely, suppose that $\widetilde{L}^+(z)$ is eventually weakly stable for every $z \in \widetilde{L}^+(x) \setminus M$. If $\widetilde{L}^+$ is not LSC at $y \in \widetilde{L}^+(x) \setminus M$, then there exist an $\epsilon > 0$ and $y_n \to y$ so that $\sup\left\{\rho\left(\omega, \widetilde{L}^+(y_n)\right) : \omega \in \widetilde{L}^+(y)\right\} \geq \epsilon$ for each $n \in N$. The compactness of $\widetilde{L}^+(y)$ implies that there exist an $N_1 > 0$ and a $v \in \widetilde{L}^+(y)$ such that $\rho\left(v, \widetilde{L}^+(y_n)\right) \geq \epsilon/2$ for $n \geq N_1$. Since $\widetilde{L}^+(y)$ is positively minimal, for each $z \in \widetilde{L}^+(y) \setminus M$, there is a $T_z = T(z) \geq 0$ such that $z * T_z \in N_{\epsilon/4}(v)$, where $T_z \neq t_n(z)$, $n = 1, 2, 3, \ldots$. By Lemma 4.1, there exists an $r_z > 0$ such that $N_{r_z}(z) * T_z \subset N_{\epsilon/4}(v)$. Set $V = \bigcup\left\{N_{r_z}(z) : z \in \widetilde{L}^+(y) \setminus M\right\}$ with $T_z \geq 0$, $r_z > 0$ such that $N_{r_z}(z) * T_z \subset N_{\epsilon/4}(v)$. Obviously, V is an open neighborhood of $\widetilde{L}^+(y)$. Choose a $\delta > 0$ such that

$$N_\delta(\widetilde{L}^+(y)) \subset V.$$

According to Theorem 3.3, there is a sequence $t_n \to \infty$ such that $y_n * t_n \to v$.

By the eventually weakly stability of $\widetilde{L}^+(y)$, there exists an $\alpha > 0$ such that for every $z\prime \in N_\alpha\left(\widetilde{L}^+(y)\right) \setminus M$, there are $t_{k\prime} \to \infty$ such that $z\prime * t_{k\prime} \in N_\delta(\widetilde{L}^+(y))$. Choose an $N_2 \geq N_1$ such that $y_n * t_n \in N_\alpha\left(\widetilde{L}^+(y)\right) \setminus M$ for every $n \geq N_2$. Fix an $n \geq N_2$. Then, there exist $t_k^n \to \infty$ such that $(y_n * t_n) * t_k^n \in N_\delta\left(\widetilde{L}^+(y)\right)$. The compactness of $\widetilde{K}^+(y_n)$ implies that $\widetilde{L}^+(y_n * t_n) \bigcap N_\delta\left(\widetilde{L}^+(y)\right) = \widetilde{L}^+(y_n) \bigcap N_\delta\left(\widetilde{L}^+(y)\right) \neq \emptyset$. Let $\omega_n \in \widetilde{L}^+(y_n) \bigcap N_\delta\left(\widetilde{L}^+(y)\right)$. Then, $\rho\left(\omega_n, \widetilde{L}^+(y)\right) < \delta$, and there exists a $z \in \widetilde{L}^+(y) \setminus M$ such that $\omega_n \in N_{r_z}(z)$ and $\omega_n * T_z \in N_{\epsilon/4}(v)$. Thus, $\widetilde{L}^+(y_n)) \bigcap N_{\epsilon/4}(v) \neq \emptyset$ for $n \geq N_2$. It follows that $\rho\left(v, \widetilde{L}^+(y_n)\right) < \epsilon/4$. This is a contradiction. Consequently, $\widetilde{L}^+$ is LSC at $y \in \widetilde{L}^+(x) \setminus M$. The proof is completed. □

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