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HOMOCLINIC ORBITS FOR SUPERLINEAR HAMILTONIAN SYSTEMS WITHOUT AMBROSETTI-RABINOWITZ GROWTH CONDITION

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Abstract. In this paper we prove the existence of homoclinic orbits for the first order non-autonomous Hamiltonian system

$$\dot{z} = \mathcal{J}H_z(t,z),$$

where $H(t,z)$ depends periodically on t. We establish some existence results of the homoclinic orbits for weak superlinear cases. To this purpose, we apply a new linking theorem to provide bounded Palais-Samle sequences.

1. Introduction and main results.

In this paper we are interested in the existence of homoclinic orbits of the Hamiltonian system

$$\dot{z} = \mathcal{J}H_z(t,z), \tag{HS}$$

where $z = (p,q) \in \mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$, $\mathcal{J} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$, and $H \in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ is of the form

$$H(t,z) = \frac{1}{2}B(t)z\cdot z + R(t,z), \tag{1.1}$$

with $B(t) \in C(\mathbb{R},\mathbb{R}^{4N^2})$ being a $2N\times 2N$ symmetric matrix valued function, and $R \in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ is superlinear in z. Here by a homoclinic orbit of (HS) we mean a solution of the equation satisfying $z(t) \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$.

Establishing the existence of homoclinic orbits for system like (HS) is a classical problem. Up to the year of 1990, there is a few of isolated results. In very recent years, many authors devoted to the existence of homoclinic orbits for Hamiltonian systems via critical point theory. For example, see [9, 10, 11, 12, 13, 14] for the second order systems, and [2, 4, 5, 6, 15, 16, 17, 18, 19, 20, 22, 26] for the first order systems. Usually, for superlinear case, one needs the following condition due to Ambrosetti-Rabinowitz [3];

$$\exists\ \mu > 2,\ 0 < \mu R(t,z) \le R_z(t,z)z, \quad \forall z \neq 0. \tag{1.2}$$

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Generally speaking, the role of (1.2) is to ensure the boundedness of all (PS)(or (PS^*)) sequences for the corresponding functional. Without (1.2), it is very difficulty to get the boundedness of (PS)(or (PS^*)) sequences. The purpose of this paper is to apply some linking theorem to deal with superlinear Hamiltonian system (HS) when the nonlinearity R doesn't satisfy the condition (1.2). Let $A := -(\mathcal{J}\frac{d}{dt} + B(t))$ be the self-adjoint operator acting in $L^2(\mathbb{R}, \mathbb{R}^{2N})$ and $\sigma(A)$ denote the spectrum of A. As we all know, the information of $\sigma(A)$ are very important in finding the homoclinic orbits. For example, if 0 is an essential spectrum of the operator A, then the operator A can not lead the behavior of the equation at 0, which brings difficulty in the usual variational arguments. So in the early results [16, 18, 19, 20, 22, 26], the authors assume

$(\mathcal{J})$ $B(t) \equiv \tilde{B}$ is independent of t such that $sp(\mathcal{J}\tilde{B}) \cap i\mathbb{R} = \emptyset$,

where $sp(\mathcal{J}\tilde{B})$ denotes the set of all eigenvalues of $\mathcal{J}\tilde{B}$. Clearly, the condition $(\mathcal{J})$ implies that there exists $\beta > 0$ such that $\sigma(A) \cap (-\beta, \beta) = \emptyset$. That is, 0 is not a spectrum of A, which is important for variational arguments. Recently, the above condition $(\mathcal{J})$ is weakened by Ding and Willem [2], and they allow 0 to be an essential spectrum of A. Assume

$(\mathcal{J}_1)$ $B(t)$ depends periodically on t with period 1, and there is $\alpha > 0$ such that $\sigma(A) \cap (0, \alpha) = \emptyset$.

Under the superlinear condition (1.2) and some additional conditions, the paper [2] proved that the system (HS) has at least one homoclinic orbit. We underline that, under the condition $(\mathcal{J}_1)$, 0 may be an essential spectrum of A, which brings difficulty in such case. To overcome this difficulty, they proved an embedding theorem as a compensation. Later, under the superlinear condition (1.2), Ding and Girardi [6] also considered the case when 0 may be an essential spectrum of A. The authors proved that the system (HS) has infinitely many homoclinic orbits provided $R(t, z)$ is even in z. In [2, 6], the condition (1.2) is important for them to get the boundedness of the (PS)-sequence. When zero is a continuous spectrum of the operator A, in this paper we shall prove a similar results as in [2] under some weaker conditions than (1.2). As far as we know, there were no results of existence of homoclinic orbit in this case. In order to state the main results, we assume that $R(t, z)$ satisfies the following conditions:

$(\mathcal{J}_2)$ $R(t, z) \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is 1-periodic in t; there exist positive constants c_1, c_2 and $\nu > 2$ such that

$$c_1|z|^\nu \le R_z(t, z)z \le c_2|z|^\nu, \quad \forall\ (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}.$$

$(\mathcal{J}_3)$ $R_z(t, z)z - 2R(t, z) > 0$ for all $t \in \mathbb{R}$ and $z \in \mathbb{R}^{2N} \setminus \{0\}$.

$(\mathcal{J}_4)$ There exists $\mu_0 > 2$ such that

$$\liminf_{z \to 0} \frac{R_z(t, z)z}{R(t, z)} \ge \mu_0,$$

uniformly for t$\in \mathbb{R}$.

$(\mathcal{J}_5)$ There exists $c_0 > 0$ such that

$$\liminf_{|z| \to \infty} \frac{R_z(t, z)z - 2R(t, z)}{|z|^\beta} \ge c_0,$$

uniformly for $t \in \mathbb{R}$, where $\nu > \beta > \nu^* := \frac{\nu(\nu-2)}{\nu-1}$.

Remark 1. In [27, 28], the conditions $(\mathcal{J}_2)-(\mathcal{J}_5)$ have been used to weaken the Ambrosetti-Rabinowitz superlinear growth condition (1.2) for Schrödinger equation.

Remark 2. It is easy to check that the classical Ambrosetti-Rabinowitz superlinear growth condition (1.2) implies $(\mathcal{J}_3)-(\mathcal{J}_5)$. But the converse proposition is not correct. Here we given the following example, and this example was first given in [27, 28].

Example 1. $R(t,z)=a(t)(|z|^\nu+(\nu-2)|z|^{\nu-\varepsilon}\sin^2(\frac{|z|^\varepsilon}{\varepsilon}))$, $\nu>2$, $0<\varepsilon<\min\{\nu-2,\nu-\nu^*\}$, where $a(t)>0$ and is 1-periodic in t.

Clearly, $(\mathcal{J}_3)-(\mathcal{J}_5)$ hold with $\mu_0=\nu$ and $\beta=\nu-\varepsilon$. However, similar to Remark 1.2 of [27]. Let $z_n=(\varepsilon(n\pi+\frac{3\pi}{4}))^{\frac{1}{\varepsilon}}L_{2N}$, where $L_{2N}=(1,0,\cdots,0)$. Then for any $\gamma>2$, one has

$$\begin{aligned}R_z(t,z_n)z_n-\gamma R(t,z_n)&=a(t)[(\nu-\gamma)|z_n|^\nu+(\nu-2)(\nu-\varepsilon-\gamma)|z_n|^{\nu-\varepsilon}\sin^2(\frac{|z_n|^\varepsilon}{\varepsilon})\\&\quad+(\nu-2)|z_n|^\nu\sin 2(\frac{|z_n|^\varepsilon}{\varepsilon})]\\&=a(t)|z_n|^\nu[2-\gamma+\frac{(\nu-2)(\nu-\varepsilon-\gamma)\sin^2(\frac{|z_n|^\varepsilon}{\varepsilon})}{|z_n|^\varepsilon}]\\&\to-\infty\quad\text{as}\quad n\to\infty.\end{aligned}$$

That is, the condition (1.2) can not be satisfied for $\gamma>2$.

Now we state the main result of this paper.

Theorem 1.1. *Let $(\mathcal{J}_1)$-$(\mathcal{J}_5)$ be satisfied. Then (HS) has at least one homoclinic orbit.*

Remark 3. If there exists $\alpha>0$ such that $(-\alpha,0)\cap\sigma(A)=\emptyset$ and $\bar{R}(t,z):=-R(t,z)$ satisfies the assumptions $(\mathcal{J}_2)-(\mathcal{J}_5)$, then the same conclusion of Theorem 1.1 remains valid.

Throughout the paper we shall denote by $c>0$ various positive constants which may vary from lines to lines and are not essential to the problem.

2. Embedding theorem.

In order to establish a variational setting for the system (HS), in this section we shall study the spectrum of a Hamiltonian operator.

Recall that $A:=-(\mathcal{J}\frac{d}{dt}+B(t))$ is a self-adjoint operator in $L^2(\mathbb{R},\mathbb{R}^{2N})$ with domain $\mathcal{D}(A)=H^1(\mathbb{R},\mathbb{R}^{2N})$. Let $\sigma_d(A)$ and $\sigma_{ess}(A)$ be, respectively, the discrete spectrum of A and the essential spectrum of A. By Proposition 2.2 of [2], 0 is at most a continuous spectrum of A, so we only need to consider the case $0\in\sigma_{ess}(A)$. Let $|\cdot|_q$ denote the usual L^q-norm, and $(\cdot,\cdot)_2$ be the usual L^2-inner product. Set $\mathcal{H}:=L^2$.

Let $\{E(\lambda):\lambda\in\mathbb{R}\}$ be the spectral family of A. We have $A=U|A|$, called the polar decomposition, where $U=I-E(0)-E(-0)$. Clearly, $\mathcal{H}$ has orthogonal decomposition

$$\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^-,$$

where $\mathcal{H}^\pm=\{z\in\mathcal{H};Uz=\pm z\}$. For each $z\in\mathcal{H}$, we will write $z=z^-+z^+$, where $z^\pm\in\mathcal{H}^\pm$.

Let E be the completion space of $\mathcal{D}(|A|^{\frac{1}{2}})$ under the norm

$$\|z\|_E = ||A|^{\frac{1}{2}} z|_2.$$

E is a Hilbert space with the inner product

$$(z_1, z_2)_E := (|A|^{\frac{1}{2}} z_1, |A|^{\frac{1}{2}} z_2)_2.$$

By Lemma 7.2 in Appendix, we have that for all $z \in \mathcal{D}(|A|^{\frac{1}{2}})$,

$$c_1\|z\|_{H^{\frac{1}{2}}} \leq \|z\|_E + a|z|_2 \leq c_2\|z\|_{H^{\frac{1}{2}}} + 2a|z|_2, \tag{2.1}$$

where $c_1, c_2 > 0$ and $a > 4\sup_{t\in\mathbb{R}} |B(t)|$.

Let $E^+ := \mathcal{H}^+ \cap \mathcal{D}(|A|^{\frac{1}{2}})$. Since the spectrum of A on E^+ is bounded away from 0, thus we have

$$\|u\|_E^2 = (Au, u)_2 = \int_\alpha^{+\infty} \lambda d(E(\lambda)u, u)_2 \geq \alpha|u|_2^2, \quad \forall u \in E^+.$$

Together with (2.1), it follows that E^+ is a closed set and

$$\|\cdot\|_E \sim \|\cdot\|_{H^{\frac{1}{2}}} \quad \text{on} \quad E^+, \tag{2.2}$$

where the notation " $\sim$ " denotes the equivalence. Then E has an orthogonal decomposition

$$E = E^+ \oplus E^-,$$

with

$$E^- \supseteq \mathcal{H}^- \cap \mathcal{D}(|A|^{\frac{1}{2}}). \tag{2.3}$$

However, since 0 may belong to a spectrum of A, then $\|\cdot\|_E$ may not be equivalent to $H^{\frac{1}{2}}$-norm on E^-. Therefore, in the following we use the spectrum family of A to sperate $\sigma(A) \cap (-\infty, 0]$ into two segments. That is, for any $\varepsilon > 0$, set

$$\mathcal{H}_\varepsilon^- := E(-\varepsilon)\mathcal{H},$$

and $E_\varepsilon^- = \mathcal{H}_\varepsilon^- \cap \mathcal{D}(|A|^{\frac{1}{2}}) = \mathcal{H}_\varepsilon^- \cap E^-$. Let $\hat{\mathcal{H}}_\varepsilon^- := \mathcal{H}^- \cap (cl_{\mathcal{H}}(\cup_{\lambda<-\varepsilon} E(\lambda)\mathcal{H}))^\perp$, where $cl_{\mathcal{H}}(B)$ denotes the closure of the set B in $\mathcal{H}$. Similarly to E^+, since the spectrum of A restrict to E_ε^- is bounded away from 0. Thus,

$$\|\cdot\|_E \sim \|\cdot\|_{H^{\frac{1}{2}}} \quad \text{on} \quad E_\varepsilon^-. \tag{2.4}$$

However, $\hat{\mathcal{H}}_\varepsilon^-$ is not complete with respect to the norm $\|\cdot\|_E$, thus it is reasonable to introduce a new norm. Define

$$\|z\|_\nu = (||A|^{\frac{1}{2}} z|_2^2 + |z|_\nu^2)^{\frac{1}{2}}. \tag{2.5}$$

Let $E_{\varepsilon,\nu}^-$ be the completion of $\hat{\mathcal{H}}_\varepsilon^-$ under the norm $\|\cdot\|_\nu$.

Now let E_ν^- denote the completion of $\mathcal{D}(A) \cap \mathcal{H}^-$ with respect to the norm $\|\cdot\|_\nu$. Since $H^{\frac{1}{2}}$ is continuously embedded in L^p for any $p \in [2, \infty)$, by (2.4), E_ε^- is a closed subspace of E_ν^-. Moreover, noting that $E_{\varepsilon,\nu}^- \subset E^-$, it is orthogonal to E_ε^- with respect to $(\cdot,\cdot)_E$, we have

$$E_\nu^- = E_\varepsilon^- \oplus E_{\varepsilon,\nu}^-. \tag{2.6}$$

Lemma 2.1. *$E_{\varepsilon,\nu}^- \subset H_{loc}^1(\mathbb{R})$ and is embedded compactly in L_{loc}^∞, and continuously in L^p for all $\nu \leq p \leq +\infty$.*

Proof. The proof was actually given in [2]. We state it here for reader's convenience. By the spectral theory of self-adjoint operators, $\hat{\mathcal{H}}_\varepsilon^- \subset \mathcal{D}(A) = H^1$. Let $\{z_n\} \subset \hat{\mathcal{H}}_\varepsilon^-$ be Cauchy sequence with respect to $\|\cdot\|_\nu$. Then

$$\begin{aligned}|A(z_n - z_m)|_2^2 &= \int_{-\varepsilon}^0 \lambda^2 d|E(\lambda)(z_n - z_m)|_2^2 \\ &\le -\varepsilon \int_{-\varepsilon}^0 \lambda d|E(\lambda)(z_n - z_m)|_2^2 \\ &= \varepsilon||A|^{\frac{1}{2}}(z_n - z_m)|_2^2 \to 0,\end{aligned} \tag{2.7}$$

as $n, m \to \infty$. For any finite interval $I \subset \mathbb{R}$, one has

$$\int_I |z_n - z_m|^2 dt \le |I|^{1-\frac{2}{\nu}} |z_n - z_m|_\nu^2 \to 0.$$

Together with (2.7), we have

$$\begin{aligned}\int_I |\dot{z}_n - \dot{z}_m|^2 dt &= \int_I |A(z_n - z_m) + B(t)(z_n - z_m)|^2 dt \\ &\le 2|A(z_n - z_m)|_2^2 + 2\int_I |B(t)(z_n - z_m)|^2 dt \to 0,\end{aligned}$$

as $n, m \to \infty$. Therefore the limit z of $\{z_n\}$ with respect to $\|\cdot\|_\nu$ belongs to $H^1_{loc}(\mathbb{R})$. Moreover, since $H^1(I)$ is compactly embedded in $L^\infty(I)$ for any finite interval I, one sees that $E^-_{\varepsilon,\nu}$ is compactly embedded in $L^\infty(I)$.

By (2.7), $\{Az_n\}$ is a Cauchy sequence in L^2. Hence $Az_n \to w$ in L^2. Since $Az_n \to Az$ in L^2_{loc}, $w = Az$, i.e. $Az \in L^2$. Note that for any finite interval $I \subset \mathbb{R}$

$$\begin{aligned}\int_I |\dot{z}|^2 dt = \int_I |Az + Bz|^2 dt &\le 2\int_I (|Az|^2 + |Bz|^2) dt. \\ &\le c(\int_I |Az|^2 + |I|^{1-\frac{2}{\nu}} (\int_I |z|^\nu)^{\frac{2}{\nu}}).\end{aligned} \tag{2.8}$$

Obviously, we have

$$z(\tau) = z(t) + \int_t^\tau \dot{z}(s) ds, \quad \text{for } \tau \in \mathbb{R}.$$

Integrating from $\tau - \frac{1}{2}$ to $\tau + \frac{1}{2}$ in the above equality, one has

$$|z(\tau)| \le (\int_{\tau-\frac{1}{2}}^{\tau+\frac{1}{2}} |z|^\nu dt)^{\frac{1}{\nu}} + \int_{\tau-\frac{1}{2}}^{\tau+\frac{1}{2}} |\dot{z}|^2 dt)^{\frac{1}{2}}. \tag{2.9}$$

Since $z \in \mathcal{H}$ and $Az \in \mathcal{H}$, (2.8) and (2.9) show that

$$|z(\tau)| \to 0 \quad \text{as} \quad |\tau| \to \infty.$$

That is, $z \in L^\infty$. Therefore $z \in L^\nu \cap L^\infty$ and so $z \in L^p$ for any $p \ge \nu$. Replacing z by $z_n - z$ in (2.8) and (2.9) one sees that $E^-_{\varepsilon,\nu}$ is continuously embedded in L^∞ and so is in L^p for any $p \ge \nu$. □

Let E_ν denote the completion of the set $\mathcal{D}(A)$ under the norm $\|\cdot\|_\nu$. It follows from (2.2), (2.4), (2.6) and Lemma 2.1 that E^-_ν and E^+ are closed set. Moreover, since $E_\nu \subset E$, and using the decomposition of E, it is easy to check that $E^-_\nu \cap E^+ = \{0\}$, and so

$$E_\nu = E^-_\nu \oplus E^+. \tag{2.10}$$

We now come to the following embedding theorem.

Theorem 2.2. *Suppose* $(\mathcal{J}_1)$ *is satisfied, and* E_ν *is defined in* (2.10). *Then* E_ν *is embedded continuously in* L^p *for all* $p \geq \nu$ *and compactly in* L^q_{loc} *for any* $q \geq 2$.

Proof. By (2.2), (2.4), (2.10) and Lemma 2.1, one can easily get the desired conclusion. ☐

3. **Linking theorem on Banach space.** In this part, we shall state an abstract critical theorem, which is first established in Hilbert space by Szulkin and Zou [4]. Recently, Willem and Zou [27] generalized it to Banach space. See also [7, 25] for the earlier results on that direction.

Let $(\Xi, \|\cdot\|)$ denote a reflexive Banach space with the direct sum decomposition $\Xi = X \oplus Y$. For $z \in \Xi$, we write $z = x + y$, where $x \in X$ and $y \in Y$. Assume X has a Schauder basis $\{e_1, e_2, \cdots\}$. We define $\|\cdot\|_\tau : \Xi \to [0, \infty)$ by

$$\|z\|_\tau := \max\{\|y\|, \sum_{i=1}^{\infty} \frac{1}{2^i} |c_j(x)|\},$$

for $z = \sum_{i=1}^{\infty} c_j(x) e_j + y \in \Xi$. Then $\|\cdot\|_\tau$ is a norm on Ξ. Below the topology generated by $\|\cdot\|_\tau$ will be called the τ-topology. Clearly, for $z = x + y \in \Xi$

$$\|y\| \leq \|z\|_\tau \leq \|z\|.$$

Therefore the τ-topology is weaker than the original one: any sequence $\{z_n\} \subset \Xi$ such that $z_n \to z$(in Ξ) converges to the τ-topology($z_n \to z$ in τ). Moreover, for any bounded sequence $\{z_n\}$,

$$z_n \to z \quad \text{in} \quad \tau \Longleftrightarrow x_n \rightharpoonup x \quad \text{and} \quad y_n \to y.$$

Recall from [7] that a homotopy $h = I - g$: $[0,1] \times A \mapsto \Xi$, where $A \subset \Xi$, is called admissible if:

(i) h is τ-continuous, i.e. $h(v_n, z_n) \xrightarrow{\tau} h(v, z)$ whenever $v_n \to v$ and $z_n \xrightarrow{\tau} z$;

(ii) g is τ-locally finite-dimensional, i.e., for each $(v, z) \in [0,1] \times A$, there exists a neighborhood $\mathbb{U}$ of (v, z) in the product topology of [0,1] and (Ξ, τ) such that $g(\mathbb{U} \cap ([0,1] \times A))$ is contained in a finite-dimensional subspace of Ξ.

Let A be a closed subset of Ξ. We say that a map $G : A \times [0,1]$ is an admissible homotopy if it is τ-continuous and for each $(u,t) \in A \times [0,1]$, there is a neighborhood $W_{(z,t)}$ in the product topology of $(\Xi, \tau) \times [0,1]$ such that the set $\{v - G(v,s) : (v,s) \in W_{(z,t)} \cap (A \times [0,1])\}$ is contained in a finite-dimensional subspace of Ξ. Observe that admissible map is continuous. On bounded subsets of Ξ the τ-topology coincides with the product topology of X_{weak} and Y_{strong}. We call the vector V: $N \to \Xi$ is τ-locally τ-Lipschitzian, where N is τ-open, if for any $z \in N$, there is a τ-neighborhood U such that $\|V(z_1) - V(z_2)\|_\tau \leq L_z \|z_1 - z_2\|_\tau$ for all $z_1, z_2 \in U$ and some $L_z \geq 0$.

Lemma 3.1. *[7, Proposition 2.2] Let* $V : \mathcal{O} \to \Xi$ *be a vector field, where* $\mathcal{O}$ *is a* τ- *open set. Assume that* V *is* τ*-locally* τ*-Lipschitzian and locally Lipschitzian, and for each* $z \in \mathcal{O}$*, there exists a* τ- *neighborhood* W_z *which is mapped by* V *into a finite-dimensional subspace of* Ξ*. Let* $F \subset \mathcal{O}$ *be a closed set. Assume that the solution of the Cauchy problem*

$$\frac{d}{dt}\eta = V(\eta), \quad \eta(z, 0) = z \in F,$$

$\eta(z,t)$ exists on [0,1] for each $z \in F$. Then the map $\eta : F \times [0,1] \to \Xi$ is an admissible homotopy.

Let $\Phi_\lambda \in C^1(\Xi, \mathbb{R})$, $R > r > 0$, and $z_0 \in Y$ with $\|z_0\| = 1$ be given and define

$$M := \{z = x + \rho z_0 : \|z\| \le R, \rho \ge 0\},\ N := \{z \in Y : \|z\| = r\},$$

$$\Lambda := \{h \in C([0,1] \times M, \Xi) : h \text{ is admissible, } h(0,z) = z \text{ and}$$
$$s \to (\Phi_\lambda(h(s,z))) \text{ is non-increasing}\},$$

And for any $a, k \in \mathbb{R}$ and $a < k$, $\Phi_{\lambda a} = \{z \in \Xi; \Phi_\lambda(z) \ge a\}$, $\Phi_\lambda^k = \{z \in \Xi; \Phi_\lambda(z) \le k\}$, $\Phi_{\lambda a}^k = \Phi_{\lambda a} \cap \Phi_\lambda^k$. Φ_λ is said to be τ-upper semi-continuous if $\Phi_{\lambda a}$ is τ-closed, and Φ'_λ is said τ-weak sequentially continuous in $\Phi_{\lambda a}^k$ if $\Phi'_\lambda(z_n) \rightharpoonup \Phi'_\lambda(z)$ when $z_n \to z$ in τ in $\Phi_{\lambda a}^k$. So the following theorem holds.

Theorem 3.2. *[27, Theorem 2.2] Let $\Xi = X \oplus Y$ be a reflexive Banach space. A functional $\Phi_\lambda \in C^1(\Xi, \mathbb{R})$ has the form*

$$\Phi_\lambda(z) := A(z) - \lambda B(z),\ 1 \le \lambda \le 2.$$

We suppose that

(i) $B(z) \ge 0$;

(ii) $A(z) \to \infty$ or $B(z) \to \infty$ as $\|z\| \to \infty$;

(iii) Φ_λ is τ-upper semi-continuous, and Φ'_λ is τ-weakly sequentially continuous on $\Phi_{\lambda a}^k$ for $a, k \in \mathbb{R}$ and $a < k$;

(iv) There exist $R > r > 0$, $b > 0$ and $z_0 \in Y$, $\|z_0\| = 1$, such that $\Phi_\lambda|_N \ge b > 0 \ge \sup_{\partial M} \Phi_\lambda$ and $d := \sup_{z \in M} \Phi_\lambda(z) < \infty$ for all $\lambda \in [1,2]$.

Then for almost every $\lambda \in [1,2]$, there exists a bounded sequence $\{z_n\}$ such that $\Phi'_\lambda(z_n) \to 0$ and $\Phi_\lambda(z_n) \to v_\lambda$, where

$$d \ge v_\lambda := \inf_{h \in \Lambda} \sup_{z \in M} \Phi_\lambda(h(1,z)) \ge b > 0.$$

The proof of this theorem was given in [27], so we omit its details.

4. **Properties of the functional.** From now on, we consider the system (HS) on Banach space E_ν defined in Section 2. Set $\Xi := E_\nu = E_\nu^- \oplus E^+$, where $Y = E^+$, $X = E_\nu^-$. It is not difficult to check that $\|\cdot\|_\nu$ is uniformly convex, so E_ν is a reflexive Banach space.

Let

$$\Psi(z) = \int_{\mathbb{R}} R(t,z)dt.$$

By assumptions and Theorem 2.2, $\Psi(z) \in C^1(E_\nu, \mathbb{R})$ and

$$\Psi'(z)v = \int_{\mathbb{R}} R_z(t, z(t))v(t)dt, \forall z, v \in E_\nu.$$

Define

$$\Phi_\lambda(z) := \frac{1}{2}\|z^+\|_E^2 - \frac{\lambda}{2}\|z^-\|_E^2 - \lambda\Psi(z), 1 \le \lambda \le 2,$$

for $z = z^- + z^+ \in E_\nu$. Then $\Phi_\lambda \in C^1(E_\nu, \mathbb{R})$. In order to obtain the boundedness of the $(PS)_c$-sequence, we shall apply Theorem 3.2 to the functional Φ_λ. So we first study the properties for the functional Φ_λ. If $\lambda = 1$, we have

$$\Phi(z) := \Phi_1(z) = \frac{1}{2}\|z^+\|_E^2 - \frac{1}{2}\|z^-\|_E^2 - \Psi(z).$$

Thus, for $\psi \in C_0^\infty(\mathbb{R})$

$$\Phi'(z)\psi = \int_{\mathbb{R}} (-\mathcal{J}\dot{z} - Bz - R_z(t,z), \psi)dt. \tag{4.1}$$

It follows that critical points of Φ are solutions of (HS). Moreover, if z is a solution of (HS), by Theorem 2.2, $R_z(t,z) \in L^s(\mathbb{R}, \mathbb{R}^{2N})$ for $s \in [2,\infty)$. A standard argument shows that z is also a homoclinic orbit of (HS)(see [2]).

Lemma 4.1. *Assume $(\mathcal{J}_1) - (\mathcal{J}_5)$ hold. If $z(t) \neq 0$ is a critical point of $\Phi(z)$ in E_ν, then z is a homoclinic orbit of (HS).*

Lemma 4.2. *There exists a positive constant $r > 0$ such that*

$$b := \inf_{z\in N} \Phi_\lambda(z) > 0,$$

where $N := \{z \in E^+ : \|z\|_\nu = r\}$.

Proof. For all $z \in E^+$, by Theorem 2.2 and $(\mathcal{J}_2)$, we have

$$\begin{aligned}\Phi_\lambda(z) &= \frac{1}{2}\|z\|_E^2 - \lambda\int_{\mathbb{R}} R(t,z)dt \geq \frac{1}{2}\|z\|_E^2 - c\lambda|z|_\nu^\nu \\ &\geq \frac{1}{2}\|z\|_E^2 - c\|z\|_E^\nu.\end{aligned}$$

Now the desired result follows. □

Lemma 4.3. *Let $z_0 \in E^+$ and $\|z_0\|_\nu = 1$, $\partial M = \{z = z^- + \rho z_0; \|z\|_\nu = R$ and $\rho \geq 0$ or $\|z\|_\nu \leq R$ and $\rho = 0\}$. Then there exist $R > r > 0$ such that*
(i) $\Phi_\lambda|_{\partial M} \leq 0$;
(ii) $d := \sup_{z\in M} \Phi_\lambda(z) < \infty$,
where $r > 0$ is given in Lemma 4.2.

Proof. Noting that $\Phi_\lambda(z) \leq \Phi(z)$ for any $z \in E$ and $\lambda \in [1,2]$, it suffices to prove that $\Phi|_{\partial M} \leq 0$. Since $R(t,z) \geq 0$, by $(\mathcal{J}_2)$, for $z = z^- + \rho z_0$, we have

$$\Phi(z) \leq \frac{\rho^2}{2}\|z_0\|_E^2 - \frac{1}{2}\|z^-\|_E^2 - c_1\int_{\mathbb{R}} |z^- + \rho z_0|^\nu dt.$$

There exists a continuous projection from the closure of $E_\nu^- \oplus \mathbb{R}z_0$ in L^ν to $\mathbb{R}z_0$(see [7]), thus, $|\rho z|_\nu \leq c|z^- + \rho z_0|_\nu$ for some $c > 0$. Hence

$$\Phi(z) \leq \frac{\rho^2}{2} - \frac{1}{2}\|z^-\|_E^2 - c\rho^\nu.$$

It follows that $\Phi(z^- + \rho z_0) \to -\infty$ as $\|z^- + \rho z_0\|_\nu \to \infty$. Since $\Phi \leq 0$ on E_ν^- , the conclusion (i) hods for R sufficiently large. Moreover, since the set M is bounded, it follows $d < \infty$. □

5. **Existence of homoclinic orbit.** In this section, we will establish Theorem 1.1. Recall that the functional

$$\Phi_\lambda = \frac{1}{2}\|z^+\|_E^2 - \lambda(\frac{1}{2}\|z^-\|_E^2 + \int_{\mathbb{R}} R(t,z)dt),\ z = z^- + z^+ \in E_\nu$$

defined in Section 4. By $(\mathcal{J}_2)$, it is easy to check that $A(z) \to \infty$ or $B(z) \to \infty$ if $\|z\|_\nu^2 = \|z^+\|_E^2 + \|z^-\|_E^2 + |z|_\nu^2 \to \infty$ and $B(z) \geq 0$. Together with Lemmas 4.2-4.3, we know that the conditions $(i), (ii)$ and (iv) of Theorem 3.2 are satisfied. So we have the following lemma.

Lemma 5.1. *For almost every $\lambda \in [1,2]$, there exists a bounded sequence $\{z_n\}$ ($\|z_n\|_\nu \le C''$) such that $\Phi'_\lambda(z_n) \to 0$ and $\Phi_\lambda(z_n) \to v_\lambda$, where $v_\lambda \in [b,d]$.*

Proof. Let $a \in \mathbb{R}$. Assume that $z_m \in \Phi_{\lambda a}$ with $z_m \to z$ in τ. Then $a \le \frac{1}{2}\|z_m^+\|_E^2 - (\frac{\lambda}{2}\|z_m^-\|_E^2 + \lambda\Psi(z_m))$. Since $z_m^+ \to z^+$, we have $\|z_m^+\|_E$ is bounded. Moreover, it follows from $\frac{\lambda}{2}\|z_m^-\|_E^2 \le \frac{1}{2}\|z_m^+\|_E^2 - a$ that $\|z_m^-\|_E$ is bounded. By $(\mathcal{J}_2)$, one has

$$c|z_m|_\nu^\nu + a \le a + \frac{\lambda}{2}\|z_m^-\|_E^2 + \lambda\Psi(z_m) \le \frac{\lambda}{2}\|z_m^+\|_E^2.$$

Thus, $|z_m|_\nu$ is bounded and so $\|z_m\|_\nu$. Therefore, $z_m \rightharpoonup z$ which implies $z_m \to z$ in L^p_{loc}(p$\ge$ 2) and along a subsequence $z_m(t) \to z(t)$ a.e. $t \in \mathbb{R}$. Consequently, by the weakly semi-continuous of norm and Fatou's lemma we get $a \le \Phi_\lambda(z)$. Now let $z_m \to z$ in τ(in $\Phi^k_{\lambda a}$). Similar to above arguments, we know that $\|z_m\|_\nu$ is bounded, and so $z_m \rightharpoonup z$ in E_ν. Then $z_m \to z$ in L^p_{loc} and $R_z(t,z_m) \to R_z(t,z)$ in $L^{p/(p-1)}_{loc}$. Hence $\Phi'_\lambda(z_m)\psi \to \Phi'_\lambda(z)\psi$ for $\psi \in E_\nu$. So the condition (iii) of Theorem 3.2 is satisfied. By Theorem 3.2, it follows that the results of this lemma holds. □

In order to prove that $\Phi(z)$ has non-zero critical point, we need the following definition.

Definition 5.2. Let $\{z_n\} \subset E_\nu$ be a bounded sequence. Then, up to a subsequence, either

(1) there exist $\gamma > 0$, $R > 0$ and $y_n \in \mathbb{R}$ such that $\lim_{n\to\infty}\int_{y_n-R}^{y_n+R}|z_n|^2dt \ge \gamma > 0$, or

(2) $\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-R}^{y+R}|z_n|^2dt = 0$ for all $0 < R < \infty$.

In the cases (1) and (2), we say that $\{z_n\}$ is non-vanishing and vanishing, respectively. The definitions are introduced in [4]

Since the Palais-Samle conditions is not satisfied due to the unboundedness of the domain $\mathbb{R}$, then we need the following lemma as a compact compensation. This lemma is first established by P. L. Lions [24].

Lemma 5.3. *Let $a > 0$ and $\{z_n\} \subset H^{\frac{1}{2}}$ be bounded. If*

$$(**) \qquad \sup_{y\in\mathbb{R}}\int_{B(y,a)}|z_n|^2 \to 0,\ n \to \infty,$$

where $B(y,a)$ is the interval $(y-a, y+a)$, then $z_n \to 0$ in $L^t(\mathbb{R})$ for $2 < t < \infty$. Particularly, if $\{z_n\} \subset E^+$ is bounded and satisfies $(**)$, then $z_n \to 0$ in $L^t(\mathbb{R})$ for $2 < t < \infty$.

Proof. Usually, this lemma is stated for $z_n \subset H^1$(see [21, 24]). However, a simple modification of the argument of Lemma 1.21 in [21] shows that the conclusion remains valid in $H^{\frac{1}{2}}$. Moreover, since the norms $\|\cdot\|_\nu$ and $\|\cdot\|_{H^{\frac{1}{2}}}$ are equivalent in E^+, then the second conclusion follows. □

Lemma 5.4. *Let $\lambda \in [1,2]$ be fixed. If a bounded sequence $\{z_n\} \subset E_\nu$ satisfies*

$$\lim_{n\to\infty}\Phi_\lambda(z_n) \in [b,d] \quad and \quad \lim_{n\to\infty}\Phi'_\lambda(z_n) = 0,$$

then there exist $k_n \in \mathbb{Z}$ such that, up to be a subsequence, $u_n := z_n(t+k_n)$ satisfies

$$u_n \rightharpoonup u_\lambda \ne 0, \quad 0 < \Phi_\lambda(u_\lambda) \le v_\lambda \quad and \quad \Phi'_\lambda(u_\lambda) = 0.$$

Proof. Since $\sup\|z_n\|_\nu < +\infty$, then $\sup\|z_n^+\|_\nu < +\infty$, where $z_n = z_n^+ + z_n^-$, $z_n^- \in E_\nu^-$ and $z_n^+ \in E^+$. If $\{z_n^+\}$ is vanishing, i.e.,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{B(y,a)}|z_n^+|^2dt = 0,$$

by Lemma 5.3, one has $z_n^+ \to 0$ in $L^t(\mathbb{R})$ for $2 < t < \infty$. Therefore, by $(\mathcal{J}_2)$, Theorem 2.2 and Hölder inequality, one sees that

$$|\int_{\mathbb{R}} R_z(t,z_n)z_n^+dt| \le c\int_{\mathbb{R}}|z_n|^{\nu-1}|z_n^+|dt \le c|z_n^+|_\nu|z_n|_\nu^{\nu-1} \to 0. \tag{5.1}$$

Since $\Phi'_\lambda(z_n) \to 0$ and $\Phi'_\lambda(z_n)z_n^+ = \|z_n^+\|_E^2 - \lambda\int_{\mathbb{R}} R_z(t,z_n)z_n^+dt$, we know that $\|z_n^+\|_E \to 0$ and

$$\Phi_\lambda(z_n) \le \|z_n^+\|_E \to 0,$$

a contradiction. Thus, $\{z_n^+\}$ is non-vanishing. That is, there exist $\gamma > 0$, $\iota > 0$ and $\hat{y}_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty}\int_{\hat{y}_n-\iota}^{\hat{y}_n+\iota}|z_n^+|^2dt \ge \gamma > 0.$$

Hence we can find $k_n \in \mathbb{Z}$ such that, setting $u_n := z_n(t+k_n)$,

$$\lim_{n\to\infty}\int_{-\iota-1}^{\iota+1}|u_n^+|^2dt \ge \gamma > 0, \tag{5.2}$$

where $u_n^\pm = z_n^\pm(t+k_n)$. Since $\|z_n\|_\nu = \|u_n\|_\nu$, then $\{u_n\}$ is still bounded,

$$\lim_{n\to\infty}\Phi_\lambda(u_n) = v_\lambda \in [b,d] \quad \text{and} \quad \lim_{n\to\infty}\Phi'_\lambda(u_n) = 0. \tag{5.3}$$

Therefore, up to a subsequence, $u_n \rightharpoonup u_\lambda$ and $u_n(t) \to u_\lambda(t)$ a.e. $t \in \mathbb{R}$, for some $u_\lambda \in E_\nu$. Since $u_n \to u_\lambda$ in $L^2_{loc}(\mathbb{R},\mathbb{R}^{2N})$, it follows from (5.2) that $u_\lambda \ne 0$. Recall that Ψ' is weakly sequentially continuous. Therefore $\Phi'_\lambda(u_n) \rightharpoonup \Phi'_\lambda(u_\lambda)$ and by (5.3), $\Phi'_\lambda(u_\lambda) = 0$.

Finally, by $(\mathcal{J}_3)$ and Fatou's lemma,

$$\begin{aligned} v_\lambda &= \lim_{n\to\infty}(\Phi_\lambda(u_n) - \frac{1}{2}\Phi'_\lambda(u_n)u_n) \\ &= \lim_{n\to\infty}\lambda\int_{\mathbb{R}}(\frac{1}{2}R_z(t,u_n)u_n - R(t,u_n))dt \\ &\ge \lambda\int_{\mathbb{R}}(\frac{1}{2}R_z(t,u_\lambda)u_\lambda - R(t,u_\lambda))dt = \Phi_\lambda(u_\lambda) > 0. \end{aligned}$$

□

Lemma 5.5. *There exists a sequence $\{\lambda_n\} \subset [1,2]$ and $\{z_n\} \subset E_\nu \setminus \{0\}$ such that*

$$\lambda_n \to 1, \quad 0 < \Phi_{\lambda_n}(z_n) \le d \quad \textit{and} \quad \Phi'_{\lambda_n}(z_n) = 0.$$

Proof. This is a straightforward consequence of Lemmas 5.1 and 5.4. □

Lemma 5.6. *The sequence $\{z_n\}$ obtained in Lemma 5.5 is bounded in E_ν.*

Proof. We modify an arguments of Lemma 4.26 in [28]. For $z_n \in E_\nu$, set $z_n = z_n^- + z_n^+$, where $z_n^- \in E_\nu^-$, $z_n^+ \in E^+$. Since $\Phi'_{\lambda_n}(z_n)z_n = 0$, by $(\mathcal{J}_2)$,

$$\|z_n^+\|_E^2 - \lambda_n\|z_n^-\|_E^2 = \lambda_n\int_{\mathbb{R}} R_z(t,z_n)z_ndt \ge c|z_n|_\nu^\nu. \tag{5.4}$$

Therefore, $\|z_n^-\|_E^2 \leq \|z_n^+\|_E^2$, $|z_n|_\nu^\nu \leq c\|z_n^+\|_E^2$ and $|z_n|_\nu \leq c\|z_n^+\|_E^{2/\nu}$. It suffices to prove the boundedness of $\|z_n^+\|_E^2$.

By $(\mathcal{J}_4)$ and $(\mathcal{J}_5)$, let $\varepsilon_0 > 0$ such that $\mu_0 - \varepsilon_0 > 2$, then there exist $R_1 \geq R_0 > 0$ such that

$$R_z(t,z)z \geq (\mu_0 - \varepsilon_0)R(t,z), \forall t \in \mathbb{R}, |z| \leq R_0, \tag{5.5}$$

and

$$R_z(t,z)z - 2R(t,z) \geq c_0|z|^\beta, \forall t \in \mathbb{R}, |z| \geq R_1.$$

Furthermore, by $(\mathcal{J}_3)$, we can choose $\epsilon > 0$ small enough such that

$$R_z(t,z)z - 2R(t,z) \geq \epsilon|z|^\beta,\ \forall t \in \mathbb{R},\ |z| \geq R_0. \tag{5.6}$$

Since $\Phi_{\lambda_n}(z_n) \leq d$ and $\Phi'_{\lambda_n}(z_n) = 0$, we have

$$\begin{aligned} d \geq \Phi_{\lambda_n}(z_n) - \frac{1}{\mu_0 - \varepsilon_0}\Phi'_{\lambda_n}(z_n)z_n &= (\frac{1}{2} - \frac{1}{\mu_0 - \varepsilon_0})(\|z_n^+\|_E^2 - \lambda_n\|z_n^-\|_E^2) \\ &\quad + \lambda_n\int_{\mathbb{R}}(\frac{1}{\mu_0 - \varepsilon_0}R_z(t,z_n)z_n - R(t,z_n))dt. \end{aligned}$$

Hence, by (5.5) and $(\mathcal{J}_2) - (\mathcal{J}_3)$, we get that

$$\begin{aligned} \|z_n^+\|_E^2 - \lambda_n\|z_n^-\|_E^2 &\leq c + c\int_{\mathbb{R}}(R(t,z_n) - \frac{1}{\mu_0 - \varepsilon_0}R_z(t,z_n)z_n)dt \\ &= c + c(\int_{|z_n|\geq R_0} + \int_{|z_n|\leq R_0})(R(t,z_n) - \frac{1}{\mu_0 - \varepsilon_0}R_z(t,z_n)z_n)dt \\ &\leq c + c\int_{|z_n|\geq R_0}(R(t,z_n) - \frac{1}{\mu_0 - \varepsilon_0}R_z(t,z_n)z_n)dt \\ &\leq c + c(\frac{1}{2} - \frac{1}{\mu_0 - \varepsilon_0})\int_{|z_n|\geq R_0} R_z(t,z_n)z_n dt \\ &\leq c + c\int_{|z_n|\geq R_0}|z_n|^\nu dt. \end{aligned} \tag{5.7}$$

Moreover, $\Phi_{\lambda_n}(z_n) - \frac{1}{2}\Phi'_{\lambda_n}(z_n)z_n \leq d$, $(\mathcal{J}_3)$ and (5.6) imply that

$$c \geq \int_{\mathbb{R}}(\frac{1}{2}R_z(t,z_n)z_n - R(t,z_n)) \geq \frac{\epsilon}{2}\int_{|z_n|\geq R_0}|z_n|^\beta dt. \tag{5.8}$$

Choose $t \in (\frac{\nu-2}{\beta(\nu-1)}, \frac{1}{\nu}) \subset (0,1)$. Since $\nu\frac{\nu-2}{\nu-1} = \nu^* < \beta < \nu$, then, by (5.8), Hölder inequality and Theorem 2.2, we have

$$\begin{aligned} \int_{|z_n|\geq R_0}|z_n|^\nu dt &= \int_{|z_n|\geq R_0}|z_n|^{\beta t\nu}|z_n|^{(1-\beta t)\nu}dt \\ &\leq (\int_{|z_n|\geq R_0}|z_n|^\beta dt)^{t\nu}(\int_{|z_n|\geq R_0}|z_n|^{\frac{(1-t\beta)\nu}{1-t\nu}}dt)^{1-t\nu} \\ &\leq c|z_n|_{p^*}^{(1-t\beta)\nu} \leq c\|z_n\|_\nu^{(1-t\beta)\nu} \\ &\leq c(\|z_n^+\|_E + \|z_n^-\|_E + |z_n|_\nu)^{(1-t\beta)\nu} \\ &\leq c\|z_n^+\|_E^{(1-t\beta)\nu} + c\|z_n^+\|_E^{2(1-t\beta)}, \end{aligned} \tag{5.9}$$

where $p^* = \frac{(1-t\beta)\nu}{1-t\nu} > \nu$. Consequently, (5.4), (5.7) and (5.8) imply that

$$c\int_{\mathbb{R}} |z_n|^\nu dt \le \|z_n^+\|_E^2 - \lambda_n \|z_n^-\|_E^2 \le c + c\int_{|z_n|\ge R_0} |z_n|^\nu dt$$
$$\le c + c\|z_n^+\|_E^{(1-t\beta)\nu} + c\|z_n^+\|_E^{2(1-t\beta)}.$$

That is, $|z_n|_\nu \le c + c\|z_n^+\|_E^{(1-t\beta)} + c\|z_n^+\|_E^{\frac{2}{\nu}(1-t\beta)}$. On the other hand, $\Phi'_{\lambda_n}(z_n)z_n^+ = 0$ and $(\mathcal{J}_2)$ imply that

$$\|z_n^+\|_E^2 = \int_{\mathbb{R}} R_z(t,z_n)z_n^+ dt \le c\int_{\mathbb{R}} |z_n|^{\nu-1}|z_n^+|dt$$
$$\le c|z_n|_\nu^{\nu-1}|z_n^+|_\nu \le c(c + c\|z_n^+\|_E^{1-t\beta} + c\|z_n^+\|_E^{\frac{2}{\nu}(1-t\beta)})^{\nu-1}\|z_n^+\|_E$$
$$\le c\|z_n^+\|_E + c\|z_n^+\|_E^{(1-t\beta)(\nu-1)+1} + c\|z_n^+\|_E^{\frac{2(\nu-1)}{\nu}(1-t\beta)+1}.$$

Since $(1-t\beta)(\nu-1)+1 < 2$, we have that $\|z_n^+\|_E < +\infty$. □

Lemma 5.7. *The sequence $\{z_n\}$ obtained in Lemma 5.5 is non-vanishing.*

Proof. It suffices to show that $\|z_n^+\|_E \ge c_1 > 0$. Indeed, if $\{z_n\}$ is vanishing, since $\Phi'_{\lambda_n}(z_n)z_n^+ = 0$, similar to (5.1) of Lemma 5.4, we have

$$\|z_n^+\|_E^2 = \lambda_n \int_{\mathbb{R}} R_z(t,z_n)z_n^+ dt \to 0,$$

as $n \to \infty$, a contradiction. In the following we shall show that $\|z_n^+\|_E \ge c_1 > 0$. Since $\Phi'_{\lambda_n}(z_n)z_n = 0$, we have

$$\|z_n^-\|_E^2 \le c\|z_n^+\|_E^2, \quad |z_n|_\nu^\nu \le c\|z_n^+\|_E^2.$$

Therefore, by $(\mathcal{J}_2)$ and Hölder inequality,

$$|\int_{\mathbb{R}} R_z(t,z_n)z_n^+ dt| \le c\int_{\mathbb{R}} |z_n|^{\nu-1}|z_n^+|dt$$
$$\le c|z_n|_\nu^{\nu-1}|z_n^+|_\nu \le c\|z_n^+\|_E^{\frac{2(\nu-1)}{\nu}}\|z_n^+\|_E$$
$$= c\|z_n^+\|_E^{\frac{2(\nu-1)}{\nu}+1}.$$

It follows from $\Phi'_{\lambda_n}(z_n)z_n^+ = 0$ that

$$\|z_n^+\|_E^2 = \lambda_n \int_{\mathbb{R}} R_z(t,z_n)z_n^+ dt \le c\|z_n^+\|_E^{\frac{2(\nu-1)}{\nu}+1}.$$

Noting that $\frac{2(\nu-1)}{\nu} > 1$, then there exists $c_1 > 0$ such that $c_1 \le \|z_n^+\|_E$. □

Proof of Theorem 1.1. We have shown that there exist $\lambda_n \to 1$ and a bounded sequence $\{z_n\}$ such that $0 < \Phi_{\lambda_n}(z_n) \le d$ and $\Phi'_{\lambda_n}(z_n) = 0$. Therefore,

$$\Phi'(z_n) = \Phi'_{\lambda_n}(z_n) + (\lambda_n - 1)(z_n^- + \Psi'(z_n)) = (\lambda_n - 1)(z_n^- + \Psi'(z_n)) \to 0.$$

It follows from Lemma 5.7 that $\{z_n\}$ is non-vanishing. By Lemma 5.4, there exists $k_n \in \mathbb{Z}$ such that if $\bar{z}_n(t) := z_n(t+k_n)$, then $\bar{z}_n(t) \rightharpoonup \bar{z}(t) \ne 0$ and $\Phi'(\bar{z}) = 0$. This is the desired result. □

Corollary 1. *Let $H(t,z)$ be the form of (1.1). Assume that $A = -(\mathcal{J}\frac{d}{dt} + B(t))$ satisfies the conditions of Remark 3. Then (HS) has at least one homoclinic orbit.*

It follows from the Remark 3 and Theorem 1.1.

6. **Some examples of matrices.** In this section, we shall give some examples of matrix satisfying the condition $(\mathcal{J}_1)$. Therefore, some preliminary results is needed, which due to Ding and Willem in [2].

Definition 6.1. Let $S(t) \in C(\mathbb{R}; \mathbb{R}^{4N^2})$ be a symmetric matrix valued function, and let $F(t)$ be the fundamental matrix with $F(0) = I$ for the equation

$$\dot{x}(t) = \mathcal{J}S(t)x,$$

S(t) is said to have an exponential dichotomy if there is a projector P and positive constants K, ξ such that

$$\begin{cases} |F(t)PF^{-1}(s)| \le Ke^{-\xi(t-s)}, & \text{if } s \le t \\ |F(t)(I-P)F^{-1}(s)| \le Ke^{-\xi(s-t)}, & \text{if } s \ge t, \end{cases} \tag{6.1}$$

(see [1]).

Definition 6.2. A continuous symmetric matrix valued function B(t) will be called right (resp. left) dichotomic if there is $\bar{\delta} > 0$ such that $B_\delta(t) := B(t) + \delta$ has an exponential dichotomy for each $\delta \in (0, \bar{\delta}]$ (resp. $\delta \in [-\bar{\delta}, 0)$).

Lemma 6.3. *If $B(t)$ is right dichotomic, then it satisfies $(\mathcal{J}_1)$.*

Proof. The proof was actually given in [2]. We state it here for reader's convenience. Noting that

$$\dot{x} = \mathcal{J}B(t)x \Longleftrightarrow A_\varepsilon x + \varepsilon x = 0,$$

where

$$A_\varepsilon := -(\mathcal{J}\frac{d}{dt} + B_\varepsilon(t)) = A - \varepsilon.$$

By Lemma 7.1 in Appendix, for any $\varepsilon \in (0, \tilde{\varepsilon}]$, there are $a_\varepsilon < 0 < b_\varepsilon$, both a_ε and b_ε being in $\sigma(A_\varepsilon)$, such that $(a_\varepsilon, b_\varepsilon) \subset \rho(A_\varepsilon) := \mathbb{C} \setminus \sigma(A_\varepsilon)$. Let $\chi := \min\{\tilde{\varepsilon}, b_{\tilde{\varepsilon}}\}$. Then since $\varepsilon \in \sigma(A)$ if and only if $0 \in \sigma(A_\varepsilon)$, we see that $(0, \chi) \subset \rho(A)$. The desired conclusion follows. □

Remark 4. In the same way, one can check that if $B(t)$ is left dichotomic then there is $\alpha > 0$ such that $(-\alpha, 0) \subset \rho(A)$.

The following two matrix satisfy $(\mathcal{J}_1)$.

Example 1. Let

$$B_1(t) := -e^{|\sin(\pi t)|+1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Clearly, we have that

$$B_{1\delta}(t) := B_1(t) + \delta I_2 = \begin{pmatrix} \delta - e^{|\sin(\pi t)|+1} & -e^{|\sin(\pi t)|+1} \\ -e^{|\sin(\pi t)|+1} & \delta - e^{|\sin(\pi t)|+1} \end{pmatrix},$$

and

$$ev(\mathcal{J}B_{1\delta}) = \{\lambda_{\pm 1} = \cdot, \cdot, \cdot, = \lambda_{\pm N} = \pm((-\delta^2 + 2\delta e^{|\sin(\pi t)|+1})^{\frac{1}{2}}\},$$

where $ev(B_1)$ denotes the set of all eigenvalues of B_1. Therefore, B_1 is right dichotomic if $0 < \delta < 2e$.

Example 2. Let M(t) be 1-periodic continuous symmetric matrix valued function and

$$M_1 := \int_0^1 M(t)dt$$

be its mean value. Similar to example 2 of [2]. Set $M_{1\varepsilon} := M_1 + \varepsilon$ and let

$$\lambda_j(\varepsilon) = \alpha_j(\varepsilon) + i\gamma_j(\varepsilon),\ j = 1, \cdots, 2N$$

denote the eigenvalues of $\mathcal{J}M_{1\varepsilon}$. Assume that there is $\bar{\varepsilon} > 0$ such that

$$\begin{aligned}\alpha_1(\varepsilon) \le \cdots \le \alpha_N(\varepsilon) < 0 < \alpha_{N+1}(\varepsilon)\\ \le \cdots \le \alpha_{2N},\ \forall \varepsilon \in (0, \bar{\varepsilon}](\varepsilon \in [\bar{\varepsilon}, 0)).\end{aligned}$$

Then $M(t)$ is right(left) dichotomic. In particular, if

$$M_1 := diag(\lambda_1, \cdots, \lambda_{2N})$$

with $\lambda_1 \le \cdots \le \lambda_N(t) < 0 \le \lambda_{N+1} \le \cdots \le \lambda_{2N}$, then $M(t)$ is right dichotomic.

7. **Appendix.** Recalling that A$=-(\mathcal{J}\frac{d}{dt} + B(t))$ is a self-adjoint operator in $\mathcal{H}$. By $(\mathcal{J}_1)$, we have $\mathcal{D}(|A|^{\frac{1}{2}}) = H^{\frac{1}{2}}$, where $|A|^{\frac{1}{2}}$ denotes the square root of $|A|$. In this Appendix, we mainly refer to the paper [2]. For reader's convenience, some of the results, together with the proofs, will be provided here. Set $W^{1,s} := W^{1,s}(\mathbb{R}, \mathbb{R}^{2N})$ for $s \ge 1$, $H^1 := W^{1,2}$ and $H^{\frac{1}{2}} := H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2N})$. For a self-adjoint operator A in $\mathcal{H}$, we denote by $|A|$ its absolute value. Now we have

Lemma 7.1. *Suppose that $S(t)$ has an exponential dichotomy and $s \ge 1$. Then the following conclusions hold:*

(1) The operator

$$B_s : L^s \supset W^{1,s} \to L^s, u \mapsto -(\mathcal{J}\frac{d}{dt} + S(t))u,$$

has a bounded inverse B_s^{-1} satisfying with some $d = d(s, \sigma) > 0$

$$|B_s^{-1}z|_\sigma \le d|z|_s, \forall z \in L^s,$$

for all $\sigma \ge s$;

(2) $B := B_2$ is s self-adjoint, and there are $b > 0$, $b_1 > 0$, $b_2 > 0$ such that $\sigma(B) \cap [-b, b] = \varnothing$ and

$$b_1\|z\|_{H^1} \le |Bz|_2 \le b_2\|z\|_{H^1} \quad \text{for all} \quad z \in H^1;$$

(3) $\mathcal{D}(|B|^{\frac{1}{2}}) = H^{\frac{1}{2}}$, and there are d_1, $d_2 > 0$ such that

$$d_1\|z\|_{H^{\frac{1}{2}}} \le ||B|^{\frac{1}{2}}z|_2 \le d_2\|z\|_{H^{\frac{1}{2}}} \quad \text{for all} \quad z \in H^{\frac{1}{2}}.$$

Proof. For any $z \in L^s$, $s \ge 1$, there is a unique $u \in W^{1,s}$ satisfying

$$-(\mathcal{J}\frac{d}{dt} + S)u = z$$

given by

$$u(t) = \int_{-\infty}^t F(t)PF^{-1}(s)\mathcal{J}zds - \int_t^\infty F(t)(I-P)F^{-1}(s)\mathcal{J}zds.$$

Set

$$\lambda^+(s) = \lambda^-(-s) = \begin{cases} 1, & \text{if } s \ge 0,\\ 0, & \text{if } s < 0.\end{cases}$$

Then

$$\begin{aligned} u(t) &= \int_{\mathbb{R}} F(t)PF^{-1}(s)\lambda^+(t-s)\mathcal{J}zds \\ &\quad - \int_{\mathbb{R}} F(t)(I-P)F^{-1}(s)\lambda^-(t-s)\mathcal{J}zds \\ &= u_1(t)+u_2(t), \end{aligned}$$

and by Eq. (6.1)

$$|u_1(t)| \le K\int_{\mathbb{R}} e^{-\xi(t-s)}\lambda^+(t-s)|z|ds$$

and

$$|u_2(t)| \le K\int_{\mathbb{R}} e^{-\xi(s-t)}\lambda^-(t-s)|z|ds.$$

Setting $f^+(\tau)=e^{-\xi\tau}\lambda^+(\tau)$ and $f^-(\tau)=e^{\xi\tau}\lambda^-(\tau)$, one has

$$|u_1(t)| \le K(f^+ * |z|)(t) \quad \text{and} \quad |u_2(t)| \le K(f^- * |z|)(t),$$

where $*$ denotes the convolution. Observe that

$$\int_{\mathbb{R}} |f^+|^\sigma = \int_{\mathbb{R}} |f^-|^\sigma = \frac{1}{\xi\sigma} \quad \forall \sigma \ge 1 \quad \textit{and} \quad |f^\pm|_\infty = 1.$$

By the convolution inequality, for any $\vartheta \ge 1$ satisfying $\frac{1}{\vartheta}=\frac{1}{s}+\frac{1}{\sigma}-1$,

$$|u_j|_\vartheta \le K(\xi\sigma)^{-1/\sigma}|z|_s, \quad j=1,2$$

and for $\frac{1}{s}+\frac{1}{s'}=1$, $s>1$

$$|u_j|_\infty \le K(\xi s')^{-1/s'}|z|_s, \quad j=1,2.$$

and also

$$|u_j|_\infty \le K|z|_1, \quad \text{if} \quad s=1, \quad j=1,2.$$

Therefore,

$$|u|_\vartheta \le K(\xi\sigma)^{-1/\sigma}|z|_s, \quad \vartheta, s, \sigma \ge 1 \quad \text{and} \quad \frac{1}{\vartheta}=\frac{1}{s}+\frac{1}{\sigma}-1. \tag{7.1}$$

Now the conclusion (1) follows from Eq. (7.1).

It is easy to verify that $B=B_2$ is self-adjoint. Note that if there is a sequence of positive numbers $b_n \to 0$ such that $\sigma(B)\cap[-b_n,b_n]=\varnothing$, then there is a sequence $\{z_n\}\subset\mathcal{D}(A)$ with $|z_n|_2=1$ and $|Bz_n|_2\to 0$, contradicting (7.1). That is, $0\notin\sigma(B)$. The inequality of (2) is clear by (7.1).

We now verify (3). Let $\Gamma := -\frac{d^2}{dt^2}$. Then $\mathcal{D}(\Gamma)=H^2$. By an interpolation theory (see [2](page 764, line 15) or [23](see section 2.5.2))

$$(\mathcal{D}(\Gamma^0),\mathcal{D}(\Gamma))_{\theta,2}=(\mathcal{H},H^2)_{\theta,2}=H^{2\theta}, \quad 0<\theta<1.$$

On the other hand (see [2](page 764, line 17) or [23](see section 1.18.10))

$$(\mathcal{D}(\Gamma^0),\mathcal{D}(\Gamma))_{\theta,2}=\mathcal{D}(\Gamma^\theta).$$

Consequently,

$$\mathcal{D}(\Gamma^\theta)=H^{2\theta}$$

equipped with the norm

$$\|z\|^2_{\mathcal{D}(\Gamma^\theta)}=\int_0^\infty(1+\lambda^{2\theta})d|E_\lambda z|_2^2=|z|_2^2+|\Gamma^\theta z|_2^2,$$

where $\{E_\lambda; -\infty < \lambda < \infty\}$ is the spectral family of Γ. In particular, let $\theta = 1/4$,

$$H^{\frac{1}{2}} = \mathcal{D}(\Gamma^{\frac{1}{4}}), \quad \|u\|^2_{H^{\frac{1}{2}}} \le |z|_2^2 + |\Gamma^{1/4} z|_2^2.$$

Since $|\Gamma^{1/2} z|_2 = |\dot{z}|_2 \le c_1 |Bz|_2$ for $z \in H^1$ by the conclusion (2), one has $(\Gamma^{1/2} z, z)_2$ $\le c_2(|B|z, z)_2$(see Theorem 4.12 in [8]), and so $|\Gamma^{1/4} z|_2 \le c_2 ||B|^{1/2} z|_2$. Together with Eq. (7.1), it follows that the first inequality of (3) holds. Similarly, considering the operator $\tilde{\Gamma} := \frac{d^2}{dt^2} + 1$, one can check the second one of (3). □

Lemma 7.2. *Under the assumption of* $(\mathcal{J}_1)$, *we have*

$$c_1 \|z\|_{H^{1/2}} \le ||A|^{\frac{1}{2}} z|_2 + a|z|_2 \le c_2 \|z\|_{H^{1/2}} + 2a|z|_2, \quad \text{for } z \in H^{\frac{1}{2}},$$

where $c_i > 0$, *(i=1, 2) and* $a > \sup_{t\in\mathbb{R}} |B(t)|$.

Proof. Now we consider the matrix $B_a := B(t) + a\tilde{B}$, where $a > 0$, $B(t)$ satisfies $(\mathcal{J}_1)$ and $\tilde{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Clearly $a\mathcal{J}\tilde{B}$ has the eigenvalues $\lambda_1 = \cdots = \lambda_N = a$ and $\lambda_{N+1} = \cdots = \lambda_{2N} = -a$, and its fundamental matrix is $F_a = exp(at \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})$. Therefore $a\tilde{B}$ has an exponential dichotomy. By the roughness of the exponential dichotomy, for any

$$a > 4 \sup_{t\in\mathbb{R}} |B(t)|, \tag{7.2}$$

B_a also have an exponential dichotomy(see [1]). In Eq. (7.2), we fix an a. Consider the self-adjoint operator

$$A_a = -(\mathcal{J}\frac{d}{dt} + B_a) = A - a\tilde{B}.$$

Since for $z \in \mathcal{D}(A)$

$$|A_a z|_2 = |(A - a\tilde{B})z|_2 \le |Az|_2 + a|z|_2,$$

by Lemma 7.1,

$$\begin{aligned} c_1 \|z\|^2_{H^{1/2}} &\le (|A_a|z, z)_2 \le (|A|z, z)_2 + a|z|_2^2 \\ &\le c_2 \|z\|^2_{H^{1/2}} + a|z|_2^2. \end{aligned}$$

By Proposition III 8.12 of [8], we have

$$c_1 \|z\|_{H^{1/2}} \le ||A|^{\frac{1}{2}} z|_2 + a|z|_2 \le c_2 \|z\|_{H^{1/2}} + 2a|z|_2,$$

for all $z \in H^{\frac{1}{2}} = \mathcal{D}(|A|^{\frac{1}{2}})$, where $c_i > 0$, (i=1, 2). □

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