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Fields with directed partial orders ☆

Niels Schwartz [a,*], YiChuan Yang [b]

[a] Fakultät für Informatik und Mathematik, Universität Passau, Postfach 2540, 94030 Passau, Germany
[b] School of Mathematics and System Sciences, LMIB of the Ministry of Education, Beihang University, Xue Yuan Road 37, Haidian District, 100191 Beijing, PR China

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ABSTRACT

We show that almost all fields of characteristic 0 carry a directed partial order. Especially, the field of complex numbers $\mathbb{C}$ can be made into a directed field, which answers an open question in [Y. Yang, On the existence of directed rings and algebras with negative squares, J. Algebra 295 (2006) 452–457].

1. Introduction

Let K be a commutative field with characteristic 0. We study the question whether K carries a directed partial order. Partial orders on K correspond to *positive cones*, i.e., to subsets $K^+ \subset K$ such that $K^+ \cap (-K^+) = \{0\}$, $K^+ + K^+ \subseteq K^+$ and $K^+ \cdot K^+ \subseteq K^+$. The corresponding order relation is given by: $x \leqslant y$ if and only if $y - x \in K^+$. It satisfies the monotonicity laws $\forall x, y, z$: $x \leqslant y \Rightarrow x + z \leqslant y + z$ and $\forall x, y, z$: $x \leqslant y$ & $0 \leqslant z \Rightarrow x \cdot z \leqslant y \cdot z$. The partial order is total if $K^+ \cup (-K^+) = K$ and directed if $K^+ - K^+ = K$. Lattice orders of fields are another class of partial orders that have received

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* Corresponding author.
E-mail addresses: Schwartz@fim.uni-passau.de (N. Schwartz), ycyang@buaa.edu.cn (Y. Yang).

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a considerable amount of attention. Every total order is a lattice order, and every lattice order is directed.

A field is said to be real if it carries a total order. Birkhoff and Pierce [2, p. 68], raised the question of whether the field $\mathbb{C}$ of complex numbers can be lattice ordered. More broadly, one can ask for the class of fields that carry some lattice order. Every real field is contained in this class since total orders are lattice orders. In [6, p. 186, Lemma 7 and p. 189, Theorem 8], it is shown that non-real algebraic number fields (finite or infinite over the field of rational numbers $\mathbb{Q}$) do not allow a lattice order. But otherwise the problem remains unsolved – it is not known whether there are any non-real fields that have a lattice order. Birkhoff and Pierce noted that $\mathbb{C}$ cannot be made into a lattice-ordered algebra over the totally ordered field $\mathbb{R}$.

Directed partial orders are more general than lattice orders. Therefore it is conceivable that there are non-real fields with a directed partial order. This is indeed the case, as has been shown in [7, Corollary 2.3]. Given a real field F with a non-archimedean total order, Yang constructs a directed partial order on the field $F(i)$. He then asks whether the field of complex numbers has a directed partial order [7, Question (2.4)]. Again, one may ask more broadly for the class of fields that can be endowed with a directed partial order.

In this paper, we show that almost all fields of characteristic 0 carry a directed partial order. (The only exceptions are the non-real algebraic number fields. For these we do not know the answer.) Especially, the field $\mathbb{C}$ of complex numbers can be made into a non-archimedean directed field, which answers the question in [7] whether $\mathbb{C}$ can be made into a directed field. (Note that DeMarr and Steger [3], have shown that $\mathbb{C}$ cannot be made into a directed algebra over the reals.)

2. Directed partial orders on polynomial rings

Let K be a field with a directed partial order K^+ and suppose that K contains a subfield K_0 such that $K_0^+ = K_0 \cap K^+$ is a non-archimedean total order. In this section we build on ideas in [7] to construct directed partial orders on the univariate polynomial ring $K[X]$.

The fields $K_0 \subseteq K$ are fixed in the entire section. The presence of the totally ordered subfield implies that $1 \in K^+$. Suppose that $x, y \in K^+$. We write $x \ll y$ to indicate that $n \cdot x \leqslant y$ for all n in the set $\mathbb{N}$ of natural numbers. Moreover, $x \equiv y$ means that there are $1 \leqslant m, n \in \mathbb{N}$ with $x \leqslant m \cdot y$ and $y \leqslant n \cdot x$. Note that these relations are defined only for positive elements – whenever we write a relation $x \ll y$ or $x \equiv y$ then $x, y \in K^+$. We record the following basic rules about the relations $\ll$ and $\equiv$. The simple proofs are omitted.

Lemma 2.1.

(a) *If $0 \leqslant t \leqslant x \ll y \leqslant z$ then $t \ll z$.*
(b) *If $x \ll y$ and $z \ll t$ then $x + z \ll y + t$.*
(c) *If $0 < z$ then $x \ll y$ implies $x \cdot z \ll y \cdot z$.*
(d) *If $x_1, \ldots, x_k \ll y$ then $x_1 + \cdots + x_k \ll y$.*
(e) *If $x \ll k \cdot y$ for some $1 \leqslant k \in \mathbb{N}$ then $x \ll y$.*
(f) *If $t \equiv x$, $x \ll y$ and $y \equiv z$ then $t \ll z$.*
(g) *If $t \in K$, $\pm t \leqslant x$, $x \ll y$, then $y - t \equiv y$.*
(h) *If $0 < z$ then $x \equiv y$ implies $x \cdot z \equiv y \cdot z$.*
(i) *If $x \equiv y$ and if $1 \leqslant k, l \in \mathbb{N}$ then $k \cdot x \equiv l \cdot y$.*
(j) *If $x \equiv y$ and $0 \leqslant t \leqslant x$ then $x + t \equiv y$.*

The hypothesis that K_0 is a non-archimedean totally ordered subfield is used in the following way: Given an element $z \in K^+$ there is always an element $u \in K^+$ such that $z \ll u$. This is trivial if $z = 0$. Otherwise, pick an element $v \in K_0^+$ such that $1 \ll v$, multiply this inequality with $z > 0$, use Lemma 2.1(c), and set $u = v \cdot z$.

We construct subsets of the polynomial ring $K[X]$ that will turn out to be directed partial orders, see Theorem 2.3. Recall that the set $K[X]^+ = \{\sum_{i=0}^{k} a_i \cdot X^i \mid \forall i\colon\ 0 \leqslant a_i\}$ is a partial order for the ring $K[X]$.

Construction 2.2. Suppose that $\sigma \in K$, $1 \leqslant \sigma$. We define $K[X]^+_\sigma$ to be the set of polynomials $P = \sum_{i=0}^k a_i \cdot X^i \in K[X]^+$ that satisfy the following condition:

- If $0 < a_j$, $j \geqslant 1$, then $\sigma \cdot a_j \ll a_{j-1}$.

The definition implies that the sequence of coefficients of a polynomial $P \in K[X]^+_\sigma$ decreases strictly until it reaches the value 0 and stays 0 ever after.

There are many different choices for the parameter σ. Different values for σ may lead to the same partial order. But in general different values yield different partial orders. So we have a large reservoir of partial orders on the polynomial ring. If we need a partial order that has some special properties then, varying the parameter σ, we have many partial orders from which we can try to pick a suitable one. We shall apply this method in Section 3.

Sometimes we can avoid the consideration of special cases if we extend the sequence of coefficients of the polynomial $P = \sum_{i=0}^k a_i \cdot X^i$ in both directions by setting $a_i = 0$ for $i < 0$ and $k < i$.

Theorem 2.3. *The set $K[X]^+_\sigma$ is the positive cone of a directed partial order on the polynomial ring $K[X]$, and (K, K^+) is a partially ordered subfield of $(K[X], K[X]^+_\sigma)$. In particular $(K[X], K[X]^+_\sigma)$ is a partially ordered algebra over the partially ordered field (K, K^+).*

Proof. It follows immediately from $K[X]^+_\sigma \subseteq K[X]^+$ that $K[X]^+_\sigma \cap (-K[X]^+_\sigma) = \{0\}$ and that $K^+ = K \cap K[X]^+_\sigma$. Moreover, the construction shows that $K[X]^+_\sigma + K[X]^+_\sigma \subseteq K[X]^+_\sigma$ and $K^+ \cdot K[X]^+_\sigma \subseteq K[X]^+_\sigma$. Thus, $(K[X], K[X]^+_\sigma)$ is a partially ordered vector space over the partially ordered field (K, K^+).

Claim. $K[X]^+_\sigma \cdot K[X]^+_\sigma \subseteq K[X]^+_\sigma$.

Proof. Suppose that $P = \sum_{i=0}^k a_i \cdot X^i$, $Q = \sum_{j=0}^l b_j \cdot X^j \in K[X]^+_\sigma$. We want to show that $P \cdot Q \in K[X]^+_\sigma$. If $P = 0$ or $Q = 0$ then there is nothing to prove. So suppose that $P \cdot Q \neq 0$. We may assume that $a_k \neq 0$ and $b_l \neq 0$ and that $k \leqslant l$. We write $P \cdot Q = \sum_{r=0}^{k+l} c_r \cdot X^r$. From $c_r = \sum_{i+j=r} a_i \cdot b_j$ it is clear that $c_r \geqslant 0$. It is only necessary to check the growth condition of Construction 2.2. There are three cases to consider:

Case 1. *$l < r \leqslant k + l$.*

For each $i = r - l, \ldots, k$ we know by hypothesis that $\sigma \cdot a_i \ll a_{i-1}$. With Lemma 2.1(a), (b) and (c) one concludes:

$$\begin{aligned} \sigma \cdot c_r &= \sum_{i=r-l}^{k} \sigma \cdot a_i \cdot b_{r-i} \ll \sum_{i=r-l}^{k} a_{i-1} \cdot b_{r-i} + a_k \cdot b_{r-1-k} \\ &= \sum_{i=r-1-l}^{k} a_i \cdot b_{r-1-i} = c_{r-1}. \end{aligned}$$

Case 2. *$k < r \leqslant l$.*

For each $i = 0, \ldots, k$ we know by hypothesis that $\sigma \cdot b_{r-i} \ll b_{r-i-1}$. Thus, Lemma 2.1(b) and (c) implies

$$\sigma \cdot c_r = \sum_{i=0}^{k} a_i \cdot \sigma \cdot b_{r-i} \ll \sum_{i=0}^{k} a_i \cdot b_{r-i-1} = c_{r-1}.$$

Case 3. $1 \leqslant r \leqslant k$.

For each $j = 0, \dots, r-1$ we know by hypothesis that $\sigma \cdot b_{j+1} \ll b_j$, and $\sigma \cdot a_r \ll a_{r-1}$. Now Lemma 2.1(b) and (c) yields

$$\begin{aligned}\sigma \cdot c_r = \sum_{i+j=r} \sigma \cdot a_i \cdot b_j &= \sum_{i+j=r, 1 \leqslant j} a_i \cdot \sigma \cdot b_j + \sigma \cdot a_r \cdot b_0 \\ &\ll \sum_{i+j=r, 1 \leqslant j} a_i \cdot b_{j-1} + a_{r-1} \cdot b_0 \\ &\leqslant 2 \cdot \sum_{i+j=r-1} a_i \cdot b_j = 2 \cdot c_{r-1},\end{aligned}$$

and we conclude that $\sigma \cdot c_r \ll c_{r-1}$, using Lemma 2.1(a) and (e).

Claim. *$K[X]_\sigma^+$ is directed.*

Proof. We show that for each $P = \sum_{i=0}^k a_i \cdot X^i \in K[X]$ there is a polynomial $Q \in K[X]_\sigma^+$ with $Q - P \in K[X]_\sigma^+$. Since K^+ is a directed partial order there is an element $b \in K$ such that $\pm a_0, \pm a_1, \dots, \pm a_k, 1 \leqslant b$. We define the coefficients of the polynomial Q recursively: To start with we set $c_k = 2 \cdot b$. Suppose that $c_k, \dots, c_{k-j}$ have been defined, $j < k$. Then one picks c_{k-j-1} such that $\sigma \cdot c_{k-j} \ll c_{k-j-1}$. We set $Q = \sum_{i=0}^k c_i \cdot X^i$.

It is immediately clear that $Q \in K[X]_\sigma^+$. It remains to show that $Q - P \in K[X]_\sigma^+$. First observe that $c_i - a_i \geqslant c_k - b = b > 0$ for each i. We must check the growth condition: Suppose that $1 \leqslant i \leqslant k$. The construction and Lemma 2.1(a) imply $\pm a_i < 2 \cdot b = c_k \leqslant \sigma \cdot c_k \ll c_{k-1} \leqslant \sigma \cdot c_{k-1} \ll \cdots \ll c_i$. It follows (by Lemma 2.1(g)) that $c_i - a_i \equiv c_i$ if $i < k$. Moreover, $3 \cdot c_k \geqslant 2 \cdot c_k - 2 \cdot a_k \geqslant c_k$ implies $c_k - a_k \equiv c_k$ (by Lemma 2.1(h) and (i)). We conclude that $\sigma \cdot (c_i - a_i) \equiv \sigma \cdot c_i \ll c_{i-1} \equiv c_{i-1} - a_{i-1}$ (by Lemma 2.1(g), (h)), and the proof is finished. □

Proposition 2.4. *For $n \in \mathbb{N}$, let $K[X]_n$ be the vector space of polynomials with degree $\leqslant n$. Then $K[X]_n$ is a convex and directed subspace of $(K[X], K[X]_\sigma^+)$. The partially ordered factor space $K[X]_{n+1}/K[X]_n$ is isomorphic to (K, K^+).*

Proof. Suppose that $0 \leqslant Q \leqslant P$ with $Q = \sum_{i \in \mathbb{N}} b_i \cdot X^i \in K[X]$ and $P = \sum_{i \in \mathbb{N}} a_i \cdot X^i \in K[X]_n$. It follows from $K[X]_\sigma^+ \subseteq K[X]^+$ that $0 \leqslant b_i \leqslant a_i$ for each $i \in \mathbb{N}$, and convexity has been proved.

The partial order $K[X]_n \cap K[X]_\sigma^+$ of $K[X]_n$ is directed, cf. the proof of Theorem 2.3.

We define a map $\gamma_{n+1} : K[X]_{n+1} \to K$ by setting $\gamma_{n+1}(\sum_{i=0}^{n+1} a_i \cdot X^i) = a_{n+1}$. This is a homomorphism of vector spaces, and it induces an isomorphism $\overline{\gamma_{n+1}} : K[X]_{n+1}/K[X]_n \to K$ of vector spaces. The image of the partial order $K[X]_{n+1} \cap K[X]_\sigma^+$ in $K[X]_{n+1}/K[X]_n$ is a partial order since $K[X]_n \subseteq K[X]_{n+1}$ is convex [4, p. 31]. The homomorphism $\overline{\gamma_{n+1}}$ is clearly monotonic. In order to show that the partially ordered vector spaces are isomorphic it suffices to prove that for each $a \in K^+$ there is a polynomial $P = \sum_{i=0}^{n+1} a_i \cdot X^i \in K[X]_\sigma^+$ with $a_{n+1} = a$. Pick an element $v \in K_0$, $1 \ll v$. The polynomial with coefficients $a_i = (\sigma \cdot v)^{n+1-i} \cdot a$ is suitable. □

Remark 2.5. Lattice orders are directed partial orders. Therefore one may ask whether the partial order $K[X]_\sigma^+$ is even a lattice order. We claim that this is not the case: For the proof we view the polynomial ring as a vector space. Assume that the partial order is a lattice order. Then the convex and directed subspaces $K[X]_n \subset K[X]$, $n \in \mathbb{N}$, are *l*-ideals [1, Definition 2.3.4], hence are lattice-ordered as well. If $n = 0$ then $K[X]_n = K$, and this is a lattice-ordered vector space if and only if the field K is lattice ordered. Now consider the case $n = 1$. Let the polynomial $a_0 + a_1 \cdot X$ be the supremum of 0 and X. Then $1 \leqslant a_1 \leqslant \sigma \cdot a_1 \ll a_0$. The polynomial $\frac{1}{2} \cdot a_0 + a_1 \cdot X$ is larger than both 0 and X, as well.

It follows that $a_0 + a_1 \cdot X \leqslant \frac{1}{2} \cdot a_0 + a_1 \cdot X$ with respect to $K[X]^+_\sigma$. But this is clearly false, and we have shown that none of the partially ordered vector space $K[X]_n$, $n > 0$, is lattice ordered.

3. Convex ideals

We continue with the fields $K_0 \subseteq K$ of Section 2. It is our plan to produce finite algebraic extensions $K \subseteq L$ with a directed partial order by forming factor rings $K[X]/(P)$, where P is some monic irreducible polynomial. Let $\pi : K[X] \to K[X]/(P)$ be the canonical homomorphism. Suppose that $K[X]^+_\sigma$ is one of the partial orders of Section 2 and that the ideal $(P) \subset K[X]$ is convex. Then $\pi(K[X]^+_\sigma)$ is a partial order of $K[X]/(P)$ [4, p. 31], and the partial order is automatically directed.

We show that, given a monic polynomial $P \in K[X]$ with positive degree, there is a parameter $\sigma \in K$, $1 \leqslant \sigma$ such that the ideal (P) is convex for the partial order $K[X]^+_\sigma$.

First we characterize the convex ideals:

Proposition 3.1. *A proper ideal $(P) \subset K[X]$ is convex with respect to the partial order $K[X]^+_\sigma$ if and only if it is trivially ordered, i.e., $(P) \cap K[X]^+_\sigma = \{0\}$.*

Proof. One direction of the equivalence is obvious since trivially ordered ideals are always convex. Conversely, suppose that the proper ideal (P) is convex and is not trivially ordered. The assumption implies that $P \neq 0$, hence $\deg(P) > 0$. There is some polynomial Q such that $Q \cdot P \in K[X]^+_\sigma$, $Q \cdot P \neq 0$. Let c be the leading coefficient of $Q \cdot P$. It follows immediately from the definition of $K[X]^+_\sigma$ that $0 < c \leqslant Q \cdot P$. Convexity yields $c \in (P)$, which shows that the ideal (P) is not proper, a contradiction. $\square$

The following theorem is the main result of this section. We shall use it to show that every proper ideal of $K[X]$ is convex for $K[X]^+_\sigma$ if the parameter σ is chosen appropriately.

Theorem 3.2. *Suppose that $P = \sum_{i=0}^k a_i \cdot X^i \in K[X]$, $a_k = 1$ and $1 \leqslant k$ and that $\sigma \geqslant \pm a_0, \dots, \pm a_{k-1}, 1$. If $Q \cdot P + R \in K[X]^+_\sigma$ with $\deg(R) \leqslant k-1$ then $Q, R \in K[X]^+_\sigma$, and $\deg(R) = k-1$ if $Q \neq 0$.*

Proof. We point out that the condition $\sigma \geqslant a_{k-1}$ implies that $P \notin K[X]^+_\sigma \cup (-K[X]^+_\sigma)$.

The claim is trivial if $Q = 0$. Therefore we suppose now that $Q \neq 0$. One writes $Q = \sum_{j=0}^l b_j \cdot X^j$ with $b_l \neq 0$, $Q \cdot P = \sum_{r=0}^{k+l} c_r \cdot X^r$ and $R = \sum_{i=0}^{k-1} d_i \cdot X^i$. First one shows (using downward induction) that the coefficients of Q satisfy the condition of Construction 2.2:

To start with, note that $b_l = a_k \cdot b_l = c_{k+l} > 0$, hence $b_l \equiv c_{k+l}$. If $\deg(Q) = l = 0$, then we have shown $Q \in K[X]^+_\sigma$. If $l \geqslant 1$, then

$$0 < \sigma \cdot b_l = \sigma \cdot c_{k+l} \ll c_{k+l-1} = b_{l-1} + a_{k-1} \cdot b_l \leqslant b_{l-1} + \sigma \cdot b_l,$$

which implies that $\sigma \cdot b_l \ll b_{l-1}$ and $b_{l-1} \equiv c_{k+l-1}$ (using Lemma 2.1(d), (g)).

Now suppose that the coefficients $b_l, \dots, b_{l-j}$ (with $j < l$ and $l \geqslant 1$) are positive and satisfy the growth requirements and that $b_l \equiv c_{k+l}, \dots, b_{l-j} \equiv c_{k+l-j}$. We show that b_{l-j-1} is positive, $\sigma \cdot b_{l-j} \ll b_{l-j-1}$ and $b_{l-j-1} \equiv c_{k+l-j-1}$.

Suppose that $1 \leqslant s \leqslant j+1 \leqslant l$. Since $\pm a_{k-s} \leqslant \sigma$ it follows that $\pm a_{k-s} \cdot b_{l-j-1+s} \leqslant \sigma \cdot b_{l-j-1+s}$. The induction assumption implies that $\sigma \cdot b_{l-j-1+s} \equiv \sigma \cdot c_{k+l-j-1+s}$ (by Lemma 2.1(h)). Using Lemma 2.1(a), it follows from the growth condition for $b_l, \dots, b_{l-j}$ that $\sigma \cdot b_l \ll \sigma \cdot b_{l-1} \ll \cdots \ll \sigma \cdot b_{l-j} \equiv \sigma \cdot c_{k+l-j} \ll c_{k+l-j-1}$. (For the last '$\ll$', note that $k+l-j-1 \geqslant k$, hence c_{k+l-j} and $c_{k+l-j-1}$ are coefficients of $Q \cdot P + R$.) One concludes that $b_{l-j-1} = c_{k+l-j-1} - a_{k-1} \cdot b_{l-j} - \cdots - a_{k-j-1} \cdot b_l \geqslant 0$, and it follows from Lemma 2.1(g) and from the inequality

$$\pm a_{k-1} \cdot b_{l-j} \pm \cdots \pm a_{k-j-1} \cdot b_l \leqslant \sigma \cdot (b_{l-j} + \cdots + b_l) \ll c_{k+l-j-1}$$

that $b_{l-j-1} \equiv c_{k+l-j-1}$. Together with $\sigma \cdot b_{l-j} \equiv \sigma \cdot c_{k+l-j} \ll c_{k+l-j-1}$ this implies $\sigma \cdot b_{l-j} \ll b_{l-j-1}$, and the growth condition is satisfied, see Lemma 2.1(f). Thus, $Q \in K[X]_\sigma^+$.

The coefficients $c_0, \dots, c_k$ satisfy the following inequalities: If $r \leqslant k$ then

$$\pm c_r = \pm(a_r \cdot b_0 + \cdots + a_0 \cdot b_r) \leqslant \sigma \cdot b_0 + \cdots + \sigma \cdot b_r \leqslant 2 \cdot \sigma \cdot b_0 \equiv \sigma \cdot c_k.$$

(Note that we have shown $b_0 \equiv c_k$, and use Lemma 2.1(d), (h).) Using induction again it is shown that the coefficients of R satisfy the requirements of Construction 2.2: It follows from $\sigma \cdot c_k \ll c_{k-1} + d_{k-1}$ and $c_{k-1} \leqslant 2 \cdot \sigma \cdot b_0 \equiv \sigma \cdot c_k$ that $d_{k-1} \equiv c_{k-1} + d_{k-1}$ (by Lemma 2.1(g)). In particular, one sees that $0 < \sigma \cdot c_k \ll d_{k-1}$, and $\deg(R) = k - 1$. If $k = 1$, the proof is finished. Now suppose that $k \geqslant 2$. Then $\sigma \cdot (c_{k-1} + d_{k-1}) \ll c_{k-2} + d_{k-2} \leqslant 2 \cdot \sigma \cdot b_0 + d_{k-2}$ yields $0 < \sigma \cdot d_{k-1} \ll d_{k-2}$ and $d_{k-2} \equiv c_{k-2} + d_{k-2}$. For the induction we assume that $d_{k-1}, \dots, d_{k-j}$ (with $j < k$ and $k \geqslant 2$) are positive and satisfy the growth condition and that $d_{k-s} \equiv c_{k-s} + d_{k-s}$ for $s = 1, \dots, j$. Now Lemma 2.1(h) yields $\sigma \cdot d_{k-j} \equiv \sigma \cdot (c_{k-j} + d_{k-j}) \ll c_{k-j-1} + d_{k-j-1}$. The inequalities

$$\pm c_{k-j-1} \leqslant 2 \cdot \sigma \cdot b_0 \equiv \sigma \cdot c_k \ll c_{k-1} + d_{k-1} \ll \cdots \ll c_{k-j-1} + d_{k-j-1}$$

imply that $d_{k-j-1} \geqslant 0$, $d_{k-j-1} \equiv c_{k-j-1} + d_{k-j-1}$ (by Lemma 2.1(g)) and $\sigma \cdot d_{k-j} \ll d_{k-j-1}$ (by Lemma 2.1(f)). $\square$

Corollary 3.3. *Suppose that $P = \sum_{i=0}^{k} a_i \cdot X^i \in K[X]$, $a_k = 1$ and $1 \leqslant k$. Pick an element $\sigma \geqslant \pm a_0, \dots, \pm a_{k-1}, 1$. Then the ideal $(P) \subset K[X]$ is trivially ordered, hence convex, with respect to $K[X]_\sigma^+$.*

Proof. We assume by way of contradiction that there is a polynomial $Q \in K[X]$, $Q \neq 0$, such that $Q \cdot P \in K[X]_\sigma^+$. We set $R = 0$ and apply Theorem 3.2 with the polynomial $Q \cdot P = Q \cdot P + R$ and arrive at the contradiction $\deg(R) = k - 1 \geqslant 0$. $\square$

The next result is extremely helpful for the intuitive understanding of the partial order of a factor ring $K[X]/(P)$, where (P) is convex with respect to $K[X]_\sigma^+$.

Corollary 3.4. *Suppose that $P = \sum_{i=0}^{k} a_i \cdot X^i \in K[X]$, $a_k = 1$ and $1 \leqslant k$, and that $\sigma \geqslant \pm a_0, \dots, \pm a_{k-1}, 1$. Let $\pi : K[X] \to K[X]/(P)$ be the canonical homomorphism onto the factor ring. We define $\pi_{k-1} : K[X]_{k-1} \to K[X]/(P)$ to be the restriction of π. Then π_{k-1} is an isomorphism of partially ordered vector spaces* (with respect to the partial orders $K[X]_{k-1} \cap K[X]_\sigma^+$ and $\pi(K[X]_\sigma^+)$).

Proof. Obviously, the map π_{k-1} is an isomorphism of vector spaces and is order preserving. We must show that, given an element $F + (P) \in \pi(K[X]_\sigma^+)$, there is an element $R \in K[X]_{k-1} \cap K[X]_\sigma^+$ with $R + (P) = F + (P)$: We may assume that $F \in K[X]_\sigma^+$. Using polynomial division we write $F = Q \cdot P + R$ with $R \in K[X]_{k-1}$. Now Theorem 3.2 shows that $R \in K[X]_\sigma^+$, and the proof is finished. $\square$

4. Fields with a directed partial order

In this section we show that most fields of characteristic 0 carry a directed partial order. The main tool is Corollary 3.3.

Theorem 4.1. *Let $K_0 \subseteq K$ be fields as in Section 2. If $K \subseteq L$ is an algebraic extension then there is a directed partial order L^+ on L such that $L^+ \cap K = K^+$.*

Proof. First we deal with finite extensions. Suppose that $[L : K] < \infty$. Since the characteristic of the fields is 0 one can identify L with a factor ring of $K[X]$ modulo some monic irreducible polynomial P. We apply Corollary 3.3 to determine a directed partial order $K[X]_\sigma^+$ on the polynomial ring such that

$(P) \cap K[X]_\sigma^+ = \{0\}$. Then the canonical map $\pi : K[X] \to L = K[X]/(P)$ maps $K[X]_\sigma^+$ onto a directed partial order L^+ of L. This partial order clearly restricts to the partial order K^+ of K.

Now let $K \subseteq L$ be an arbitrary algebraic extension. By Zorn's Lemma there is a field M, $K \subseteq M \subseteq L$, that is maximal with the property that there is a directed partial order M^+ on M with $M^+ \cap K = K^+$. If $M = L$ then the proof is finished. If not, then any element $a \in L \setminus M$ yields a proper algebraic extension of M that is contained in L. By the case of finite extensions there is a directed partial order $M(a)^+$ on $M(a)$ that extends M^+. This contradicts the maximality of M, and the proof is finished. □

Corollary 4.2. *If L is any field that has transcendence degree at least 1 over $\mathbb{Q}$ then L carries a directed partial order.*

Proof. Let $T \subseteq L$ be a transcendence basis over $\mathbb{Q}$. The purely transcendental extension $\mathbb{Q}(T)$ of $\mathbb{Q}$ carries numerous non-archimedean total orders [5, p. 11, Satz 4 and p. 79, Satz 1]. By Theorem 4.1 any one of these can be extended to a directed partial order on the algebraic extension L of $\mathbb{Q}(T)$. □

We have shown that there are many non-real fields that have a directed partial order. Especially, the field of complex numbers can be made into a non-archimedean directed field, which answers the open question in [7] whether $\mathbb{C}$ can be made into a directed field.

It is an obvious question whether any of the directed partial orders we have constructed is even a lattice order. However, this is not the case. Again, let K be a directed partially ordered field as in Section 2, let $P \in K[X]$ be a monic irreducible polynomial, $\deg(P) = k \geqslant 2$, and let $1 \leqslant \sigma \in K$ be a parameter as in Corollary 3.4. Then the partially ordered vector spaces $(K[X]_{k-1}, K[X]_{k-1} \cap K[X]_\sigma^+)$ and $(K[X]/(P), \pi(K[X]_\sigma^+))$ are canonically isomorphic. The partial order $K[X]_{k-1} \cap K[X]_\sigma^+$ is not a lattice order, see Remark 2.5. Note that [8, Remark 2.4] shows that the directed partial orders constructed in [7] are not lattices by means of segments.

Our results do not apply to non-real algebraic number fields since they do not contain non-archimedean totally ordered subfields. It remains an open question whether non-real algebraic number fields carry a directed partial order.

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