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Archimedean copula estimation and model selection via l_1-norm symmetric distribution

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ABSTRACT

Based on the relationship between Archimedean copulas and l_1-norm symmetric distributions, we propose a method to not only estimate the copula parameter but also select the copula model through the observation data in this paper. The strong consistency of the estimator is proved, and a Radial Information Criteria (RIC) is provided to select the appropriate Archimedean copula model fitting the data best. It can be extended to the multivariate cases conveniently because the selection is achieved by using the one-dimensional radial distribution to capture the dependence structure for multivariate data. The Monte Carlo simulation experiments illustrate that the proposed approach works well in parameter estimation and model selection for both bivariate and multivariate cases. An application in modelling the dependence structure of real stock indices is carried out with good performance as well.

1. Introduction

A copula is a joint distribution function of standard uniform random variables (Nelsen, 2006). For a continuous random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with marginal distribution functions $F_1(x_1), \ldots, F_d(x_d)$, by the famous Sklar's theorem (Sklar, 1959), there is a unique copula function C such that the cumulative distribution function

$$F(x_1, \ldots, x_d) = C\big(F_1(x_1), \ldots, F_d(x_d)\big).$$

Recently, copula theory has been raising the interest in many practical applications, such as survival analysis, actuarial science, hydrology etc. Especially in finance, notably in the problems of asset pricing and actuarial risk management, appropriate models for dependence between risks are obviously important. Fortunately, the copula function characterizes the dependence structures among the random variables so that it is convenient for modelling different types of dependence continuously from negative dependence through independence up to positive dependence. Therefore, it becomes an important tool in financial research (e.g. Embrechts et al. (1997), Cherubini et al. (2004), Denuit et al. (2005), McNeil et al. (2005) and Mendes and Souza (2004)).

A versatile subclass of copulas, called Archimedean copulas which is introduced in Kimberling (1974), is an important class of multivariate dependence model with attractive stochastic properties. It involves a non-parametric component $\varphi(\cdot)$, called generator, which is a univariate function and completely describes the dependency structure of the entire d-dimensional vector $\mathbf{X}$. This brings the essential simplification to inference for Archimedean copula. In practice, the Archimedean models are commonly used, especially in survival analysis (e.g. Oakes (1989), Faraggi and Korn (1996) and Klugman and Parsa (1999)).

However, in practice, we do not know from which copula family the data come in advance, thus the essential is how to select an appropriate copula family and estimate the parameters to fit the data best. In fact, the model selection in copulas is not easy owing to their complexities, especially for multivariate cases. Genest and Rivest (1993) provided a graphical method to select Archimedean copula based on the Kendall process

$$\mathbf{K}_n(t) = \sqrt{n}\big(K_n(t) - K(\theta_n, t)\big),$$

by using the distribution function $K(\cdot)$ of random variable $C(U_1, U_2)$, where $K_n(\cdot)$ and $K(\theta_n, \cdot)$ represent the non-parametric and parametric estimates of $K(\cdot)$ respectively, and U_1 and U_2 are uniformly distributed random variables on [0, 1]. A goodness of fit test based on the Kendall process was proposed in Genest et al. (2006), which requires a "good" estimator of the parameter θ firstly. In Chen and Fan (2005), the pseudo-likelihood ratio test was proposed to select the appropriate parameterized copula. A Bayesian method to select the most probable copula family among a given set was suggested in Huard et al. (2006), which treats the copula parameters as nuisance variables based on a priori Kendall's τ. However, when the closed-form relation between Kendall's τ and the copula parameter is unavailable, taking the family of Joe copula for example, it does not work; and it needs the prior information which may be inaccessible in applications.

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For the Archimedean copulas, an open question was proposed in Nelsen (2005): *If an Archimedean copula is appropriate for a given data set, are there statistical procedures for choosing a particular family?* In Wang and Wells (2000), model selection procedures for bivariate survival models for censored data generated by the Archimedean copula family were proposed. In route to constructing the selection methodology, it developed estimates for some time-dependent association measures, and then estimate the copula parameter from these measures.

In recent research, it is shown that the class of d-dimensional Archimedean copulas exactly coincides with the class of survival copulas of d-dimensional l_1-norm symmetric distributions that place no point mass at the origin McNeil and Neŝlehová (2009). This is a very important and useful discovery because it establishes a direction to analysis multi-dimensional distributions focusing on the corresponding l_1-norm symmetric distributions.

In this paper, under the some assumption that the d-dimensional data can be suitably modelled by a d-dimensional Archimedean copula on the d-unit cube, we decompose the multi-dimensional random variables following l_1-norm symmetric distribution into two independent parts, i.e., the radial part and the uniform simplex part. Then, we utilize the uniform simplex parts to do the estimation and the radial parts to set up a criteria for realizing the model selection. In result, we achieve the twofold purpose. First, a unified method is presented which could not only estimate the copula parameter but also select the copula model among Archimedean copula families for bivariate cases. Second, it is extended to multivariate Archimedean copula parameter estimation and model selection. This is a meaningful generation because in practice, such as risk management, we may interested in the dependence structure among more than two risks. To the best of our knowledge, there is no efficient model selection method for multivariate Archimedean copulas at present.

This paper is organized as follows. In Section 2, we present some basic concepts and main results about Archimedean copulas, Archimedean generators, and l_1-norm symmetric distributions. In Section 3, we introduce an optimal parameter estimator in terms of minimizing the Cramér–von Mises distance and a Radial Information Criteria (RIC) to realize the model selection in Archimedean copulas. We present some Numerical examples including Monte Carlo simulation experiments to illustrate the efficiency of the proposed approach in Section 4, and Section 5 provides a conclusion.

2. Archimedean copulas and the characterizations

2.1. Archimedean copulas and Archimedean generators

Following Ling (1965) and McNeil and Neŝlehová (2009), the related concepts considered in this paper are given as follows.

Definition 1. A non-increasing and continuous function $\varphi : [0,\infty) \to [0,1]$ which satisfies the conditions $\varphi(0) = 1$, $\lim_{x\to\infty}\varphi(x) = 0$ and is strictly decreasing on $[0, \varphi^{-1}(0))$ is called an Archimedean generator, where $\varphi^{-1}(t) = \inf\{x : \varphi(x) = t\}$. If $\varphi^{-1}(0) = \infty$, we say that φ is a strict generator. A d-dimensional copula C is called Archimedean if it has the representation

$$C(u_1,\ldots,u_d) = \varphi\big(\varphi^{-1}(u_1) + \cdots + \varphi^{-1}(u_d)\big), \quad 0 \le u_i \le 1. \quad (1)$$

Definition 2. A real function f is called d-monotone on (a, b), where $a, b \in \mathbb{R}$ and $d \ge 2$, if

(a) f is differentiable on (a, b) up to order $d-2$, and the derivatives satisfy

$$(-1)^k f^{(k)}(x) \ge 0, \quad x \in (a,b) \quad (2)$$

for $k = 0, 1, \ldots, d-2$;

(b) $(-1)^{d-2} f^{(d-2)}(x)$ is non-increasing and convex on (a, b).

Moreover, a real function f is called completely monotone on (a, b) if it has derivatives of all orders on (a, b) and inequality (2) holds for all integers k.

The function f is called d-monotone (completely monotone) on a closed interval $[a, b]$ if it is continuous on $[a, b]$ and if f restricted to (a, b) is d-monotone (completely monotone). The following lemma gives the necessary and sufficient condition that an Archimedean generator induces a d-dimensional copula by means of (1), which was proved in McNeil and Neŝlehová (2009, Theorem 2).

Lemma 1. *Let φ be an Archimedean generator, then the function C given by (1) is a d-dimensional copula if and only if φ is d-monotone on* $[0,\infty)$.

Because a variety of different generators can provide different dependence structures, the dependence properties of an Archimedean copula reduce to analytical properties of its generator φ. Therefore, the problem of the model selection among Archimedean copulas becomes that of the selection of the generators in nature. Some main types of Archimedean copula generators are listed as follows, the corresponding Archimedean copulas can be constructed by (1).

(1) Clayton copula with generator

$$\varphi_\theta(x) = (1+\theta x)_+^{-1/\theta}, \quad \theta \neq 0, \text{ where } a_+ = \max\{a, 0\}.$$

(2) Gumbel copula with generator

$$\varphi_\theta(x) = \exp(-x^{1/\theta}), \quad \theta \ge 1.$$

(3) Frank copula with generator

$$\varphi_\theta(x) = -\frac{1}{\theta}\ln\left(\frac{\exp(x)+\exp(-\theta)-1}{\exp(x)}\right), \quad \theta \neq 0.$$

(4) Ali–Mikhail–Haq (AMH) copula with generator

$$\varphi_\theta(x) = \frac{1-\theta}{\exp(x)-\theta}, \quad \theta \in [-1, 1).$$

(5) Joe copula with generator

$$\varphi_\theta(x) = 1 - (1-\exp(-x))^{1/\theta}, \quad \theta \ge 1.$$

It can be verified that the Clayton generator is strict when $\theta > 0$, and d-monotone for $d \ge 2$ if and only if $\theta \ge -1/(d-1)$. The others listed above are strict and completely monotone.

A common feature of the first three systems listed above is that they include a special case of the independence distribution, i.e., $C_\varphi(x, y) = xy$, as well as the so-called upper Fréchet bound (Nelsen, 2006). More discussions and properties of these copulas can be found in Clayton (1978); Frank (1979) and Gumbel (1960). Since the Clayton, Gumbel and Frank copulas can model different tail dependence structures, most of the previous research in Archimedean copula model selection mainly consider above three one-parameter families (e.g. Genest and Rivest (1993), Wang and Wells (2000)). In order to show the general adaptability of the method presented below, we consider additionally another two families of Archimedean copulas, i.e. AMH copulas and Joe copulas.

2.2. Archimedean copulas and l_1-norm symmetric distributions

In order to build up the relationship of Archimedean copulas and l_1-norm symmetric distributions, we need the following lemmas.

Lemma 2. *For a continuous random vector $\mathbf{X} = (X_1,\ldots,X_d)$ with marginal survival functions $\bar F_1(x_1),\ldots,\bar F_d(x_d)$, where $\bar F_i(x_i) = P\{X_i > x_i\}$, let $\bar H$ be the survival function of $\mathbf{X}$, i.e.*

$$\bar{H}(x_1,\dots,x_d)=P\{X_1>x_1,\dots,X_d>x_d\},$$

then there is a unique copula function C, which is referred to as the survival copula of $\mathbf{X}$ *such that*

$$\bar{H}(x_1,\dots,x_d)=C\big(\bar{F}_1(x_1),\dots,\bar{F}_d(x_d)\big).$$

Lemma 2 is the restate of the original result by Sklar (1959) in terms of survival functions. The well-known Bernstein–Widder theorem (see Widder, 1946) indicates that an Archimedean generator is completely monotone on $[0,\infty)$ precisely when it is a Laplace transform of a non-negative random variable. However, Archimedean generator that is not completely monotone is able to appear as a survival copula of random vectors following l_1-norm symmetric distributions. This kind of distributions, introduced by Fang and Fang (1988), is formally defined as follows:

Definition 3. A random vector $\mathbf{X}$ on $\mathbb{R}_+^d=[0,\infty)^d$ follows an l_1-norm symmetric distribution if there exists a non-negative random variable R independent of S_d, where S_d is a random vector distributed uniformly on the unit simplex

$$\mathcal{S}_d=\{x\in\mathbb{R}_+^d:\|x\|_1=1\},$$

so that $\mathbf{X}$ permits the stochastic representation

$$X\overset{d}{=}RS_d.$$

The random variable R is referred to as the radial part of $\mathbf{X}$ and its distribution is called as the radial distribution.

The relationship between Archimedean copulas and l_1-norm symmetric distributions is the main result in McNeil and Neŝlehová (2009, Theorem 3(ii)), which is described by the following lemma:

Lemma 3. *Let U be distributed according to the d-dimensional Archimedean copula C with generator* φ*. Then* $(\varphi^{-1}(U_1),\dots,\varphi^{-1}(U_d))$ *has an* l_1*-norm symmetric distribution with survival copula C and the radial distribution function* F_R *satisfying*

$$F_R(x)=1-\sum_{k=0}^{d-2}\frac{(-1)^k x^k\varphi^{(k)}(x)}{k!}-\frac{(-1)^{d-1}x^{d-1}\varphi_+^{d-1}(x)}{(d-1)!},\qquad(3)$$

where $x\in[0,\infty)$.

From Lemma 3, it is clear that once the random vector U follows the distribution of an Archimedean copula C with generator φ, there must exist a corresponding l_1-norm symmetric distribution for the random vector $(\varphi^{-1}(U_1),\dots,\varphi^{-1}(U_d))$. Since the problem we consider in this paper is "If an Archimedean copula is appropriate for a given data set, are there statistical procedures for choosing a particular family?", which guarantees that there exists a corresponding l_1-norm symmetric distribution being appropriate. On the other side, if a survival copula of random vector is not following any l_1-norm symmetric distribution, it is impossible being an Archimedean copula, which is not in the consideration of this paper.

Let $\varphi(x)$ be a d-monotone function on $[0,\infty)$ and F_R be the radial distribution function associated with φ by Eq. (3). If S_d is uniformly distributed on $\mathcal{S}_d$ and $R\sim F_R$ is a random variable independent of S_d, then the distribution of RS_d is just the l_1-norm symmetric distribution associated with φ, and $\varphi(RS_d)$ is distributed according to the d-dimensional Archimedean copula C with generator φ. The relationship discussed above can be concluded in the following lemma (see McNeil and Neŝlehová, 2009, Proposition 11):

Lemma 4. *Let U be distributed according to the d-dimensional Archimedean copula C with generator* φ_θ*, where* θ *is the parameter of the generator. Then*

$$\sum_{i=1}^{d}\varphi_\theta^{-1}(U_i)\quad\text{and}\quad\left(\frac{\varphi_\theta^{-1}(U_1)}{\sum_{i=1}^{d}\varphi_\theta^{-1}(U_i)},\dots,\frac{\varphi_\theta^{-1}(U_d)}{\sum_{i=1}^{d}\varphi_\theta^{-1}(U_i)}\right)$$

are independent. Moreover, the random variable

$$V_\theta(U_1,\dots,U_d)=\left(1-\frac{\varphi_\theta^{-1}(U_j)}{\sum_{i=1}^{d}\varphi_\theta^{-1}(U_i)}\right)^{d-1}$$

is uniformly distributed on $[0,1]$ *for any* $j=1,\dots,d$.

Lemma 4 is actually a decomposition of random vector U which follows an Archimedean copula C into two independent parts, the first part follows the radial distribution F_R and the second part is in fact is a random vector distributed uniformly on the unit simplex. An important application of this result is constructing a method of stochastic simulation for Archimedean copulas (see McNeil and Neŝlehová, 2009, pp. 24–25).

For the Archimedean copula model, an important application of it in insurance and finance is the frailty models which have been introduced by Lancaster (1979) and Vaupel et al. (1979) and have been popularized by Oakes (1989). The basic idea is using an unobserved random variable to introduce dependence between survival times, and finally the corresponding copula is a special case of an Archimedean copula where the generator φ is the inverse of a Laplace transform.

In the follows, we concentrate on the model estimation and selection for a given data set, the radial part is a one-dimensional function which reflects different dependence structure, so it is appropriate being used to set up an criteria for model selection; and the uniform simplex part is completely the same for any Archimedean copula, which could be used to estimate the copula parameter.

3. Archimedean copula estimation and selection in terms of radial information criteria

Definition 4. An Archimedean copula generator $\varphi_\theta(x)$ is called compositionally monotonic if for a constant $k>0$ the composition function $\varphi_\theta\big(k\times\varphi_\theta^{-1}(y)\big)$ is uniformly monotonic on the parameter θ, and strictly monotonic for $y\in(0,M)$ where $M=\inf\{y:\varphi_\theta(k\times\varphi_\theta^{-1}(y))=0\}$.

Remark 1. If the generator φ_θ is strict, then $M=1$. Otherwise, such as the Clayton family when $\theta<0$, $M=\varphi_\theta(-(k\theta)^{-1})$.

Proposition 1. *The Archimedean copula generators of Clayton, Gumbel, Frank, AMH and Joe are compositionally monotonic.*

Proof. We denote $G_\theta(x)=\frac{\partial\varphi_\theta(x)}{\partial\theta}$, and $H_\theta(y)=\frac{\partial\varphi_\theta^{-1}(y)}{\partial\theta}$.

(1) For Clayton family:

$$G_\theta(x)=\begin{cases}(1+\theta x)^{-1/\theta}\cdot\big(\ln(1+\theta x)-\theta x/(1+\theta x)\big)/\theta^2,\\ \quad 1+\theta x>0,\\ 0,\quad 1+\theta x\le 0.\end{cases}$$

Because $\frac{1}{t}+\ln(t)\ge 1$ when $t>0$ and the equality achieves at $t=1$, $\ln(1+\theta x)-\theta x/(1+\theta x)$ keeps positive if $1+\theta x>0$. Thus, $\varphi_\theta(x)$ is strictly monotonic on θ for $x\in(0,\infty)$ if $\theta>0$ and for $x\in(0,-1/\theta)$ if $\theta<0$. In addition,

$$H_\theta(y)=\big(1-y^{-\theta}(\theta\ln(y)+1)\big)/\theta^2.$$

Then $H_\theta(y) > 0$ ($y \in (0,1)$) because $\frac{\partial H_\theta(y)}{\partial y} = y^{-1-\theta} \cdot \ln(y) < 0$ and $H_\theta(1) = 0$ for $y \in (0,1)$. It implies that $\varphi_\theta^{-1}(y)$ is strictly monotonic on θ for fixed $y \in (0,1)$. Finally, the composition function $\varphi_\theta\big(k \times \varphi_\theta^{-1}(y)\big)$ is a strictly monotonic function on θ for $y \in (0,1)$ if $\theta > 0$ and for $y \in (0, \varphi_\theta(-(k\theta)^{-1}))$ if $\theta < 0$.

(2) For Gumbel family: From

$$\varphi_\theta\big(k \times \varphi_\theta^{-1}(y)\big) = \exp\big(\ln(y) \cdot k^{1/\theta}\big),$$

and the fact that $k^{1/\theta}$ is a strictly monotonic function of θ, we know the function $\varphi_\theta\big(k \times \varphi_\theta^{-1}(y)\big)$ is strictly monotonic on θ for $y \in (0,1)$.

(3) For Frank family: From

$$G_\theta(x) = \ln\left(\frac{\exp(x) + \exp(-\theta) - 1}{\exp(x)}\right)^{1/\theta^2} + \frac{\exp(-\theta)}{\theta\big(\exp(x) + \exp(-\theta) - 1\big)},$$

we have $G_\theta(0) = 0$ and $\lim_{x\to\infty} G_\theta(x) = 0$. Because

$$\frac{\partial G_\theta(x)}{\partial x} = \frac{(1 - (1+\theta)\exp(-\theta)) \cdot \exp(x) - (1 - \exp(-\theta))^2}{\theta^2(\exp(x) + \exp(-\theta) - 1)^2}$$

has only one zero point on $(0, \infty)$ and

$$\left.\frac{\partial G_\theta(x)}{\partial x}\right|_{x=0} = \frac{(1-\theta)\exp(\theta) - 1}{\theta^2} < 0,$$

$G_\theta(x)$ keeps negative on $(0,\infty)$. Therefore, $\varphi_\theta(x)$ is strictly monotonic on θ. In addition,

$$H_\theta(y) = \frac{y}{1 - \exp(\theta y)} - \frac{1}{1 - \exp(\theta)} < 0, \quad y \in (0,1)$$

because $H_\theta(1) = 0$, and

$$\frac{\partial H_\theta(y)}{\partial y} = \frac{1 - (1 - \theta y)\exp(\theta y)}{\big(1 - \exp(\theta y)\big)^2}$$

is positive when $y \in (0,1)$. It implies that $\varphi_\theta^{-1}(y)$ is strictly monotonic on θ for fixed $y \in (0,1)$. In result, the composition function $\varphi_\theta\big(k \times \varphi_\theta^{-1}(y)\big)$ is strictly monotonic on θ for $y \in (0,1)$.

(4) For AMH family: Noticing that

$$G_\theta(x) = \frac{1 - \exp(x)}{\big(\exp(x) - \theta\big)^2} < 0, \quad x \in (0, +\infty),$$

$$H_\theta(y) = \frac{y - 1}{\theta \times (y-1) + 1} < 0, \quad y \in (0,1),$$

we know that $\varphi_\theta\big(k \times \varphi_\theta^{-1}(y)\big)$ is strictly monotonic on θ when $y \in (0,1)$.

(5) For Joe family: From

$$G_\theta(x) = \big(1 - \exp(-x)\big)^{1/\theta} \cdot \ln\big(1 - \exp(-x)\big)/\theta^2 < 0, \quad x \in (0, +\infty),$$

$$H_\theta(y) = \frac{(1-y)^\theta \cdot \ln(1-y)}{1 - (1-y)^\theta} < 0, \quad y \in (0,1),$$

we know that $\varphi_\theta\big(k \times \varphi_\theta^{-1}(y)\big)$ is also strictly monotonic on θ when $y \in (0,1)$. □

3.1. Bivariate Archimedean copula estimation and selection

Suppose that there is an i.i.d. sample (X_{1t}, X_{2t}) $(t = 1, \dots, n)$ from a joint distribution with the marginals F_1 and F_2. The K competing models in a finite set are $\mathcal{M}_1, \dots, \mathcal{M}_K$ with corresponding generator families $\varphi_\theta^1, \dots, \varphi_\theta^K$. In the next, we introduce a general method to estimate and select the dependence structure among the set of competing models, i.e., find an appropriate Archimedean copula generator φ_θ such that joint distribution function of the variables can be expressed as:

$$\begin{aligned} F(x_1, x_2) &= C_{\varphi_\theta}(F_1(x_1), F_2(x_2)) \\ &= \varphi_\theta\left(\varphi_\theta^{-1}(F_1(x_1)) + \varphi_\theta^{-1}(F_2(x_2))\right). \end{aligned}$$

In the next, we denote the random variable $F_i(X_i)$ by U_i.

By Lemma 4, the random variable

$$V_\theta(U_1, U_2) = 1 - \frac{\varphi_\theta^{-1}(U_2)}{\sum_{i=1}^{2} \varphi_\theta^{-1}(U_i)} = \frac{\varphi_\theta^{-1}(U_1)}{\varphi_\theta^{-1}(U_1) + \varphi_\theta^{-1}(U_2)} \tag{4}$$

should be uniformly distributed on $[0, 1]$. So the estimation procedure consists of estimating the parameter value $\hat{\theta}_n$ that minimizes the Cramér–von Mises distance:

$$\mathrm{d}(\theta) = n \int_0^1 \left(V_\theta^n(t) - U(t)\right)^2 \mathrm{d}U(t), \tag{5}$$

where $V_\theta^n(t)$ and $V_\theta(t)$ are respectively the empirical distribution function and distribution function of the random variable $V_\theta(U_1, U_2)$, and $U(t)$ is the uniform distribution function on $[0, 1]$. In the next, we discuss the consistent property of the estimate $\hat{\theta}_n$.

Theorem 1. *Let (U_1, U_2) distribute according to an Archimedean copula with generator φ_{θ_0} ($\theta_0 \in \Theta$). If the generator is compositionally monotonic, then $V_{\theta_1}(U_1, U_2)$ and $V_{\theta_2}(U_1, U_2)$ given by (4) for parameters θ_1 and θ_2 respectively follow the same distribution if and only if $\theta_1 = \theta_2$.*

Proof. The sufficiency is clear, so we only need to prove the necessity.

We assume $\theta_1 \neq \theta_2$, and without loss of generality, suppose $\theta_1 > \theta_2$. For any $c > 0$, we have

$$\begin{aligned} P_{\theta_0}\{V_\theta(U_1, U_2) \le c\} &= P_{\theta_0}\left\{\frac{\varphi_\theta^{-1}(U_1)}{\varphi_\theta^{-1}(U_1) + \varphi_\theta^{-1}(U_2)} \le c\right\} \\ &= P_{\theta_0}\left\{\frac{\varphi_\theta^{-1}(U_1)}{\varphi_\theta^{-1}(U_2)} \le \frac{c}{1-c}\right\} \\ &= \int_0^1 \int_0^{\varphi_\theta\left(\frac{1-c}{c}\varphi_\theta^{-1}(u_1)\right)} p(u_1, u_2)\mathrm{d}u_2\mathrm{d}u_1, \end{aligned}$$

where $p(u_1, u_2)$ denotes the density function of (U_1, U_2). Since the density function is continuous in $[0, 1] \times [0, 1]$, there exist one point $q = (u_1^0, u_2^0)$ and its neighborhood $O(q, r)$ such that $p(u_1, u_2) > \epsilon > 0$ for any $(u_1, u_2) \in O(q, r)$.

By Proposition 1, $\varphi_\theta(\frac{1-c}{c}\varphi_\theta^{-1}(u_1))$ is a strictly monotonic function on θ for any $u_1 \in (0, \inf\{y : \varphi_\theta(\frac{1-c}{c}\varphi_\theta^{-1}(y)) = 0\})$. We suppose that it is strictly monotonic increasing without loss of generality, so $\varphi_{\theta_1}\big(\frac{1-c}{c}\varphi_{\theta_1}^{-1}(u_1^0)\big) > \varphi_{\theta_2}\big(\frac{1-c}{c}\varphi_{\theta_2}^{-1}(u_1^0)\big)$ if $\varphi_{\theta_1}(\frac{1-c}{c}\varphi_{\theta_1}^{-1}(u_1^0)) > 0$.

Because $\varphi_{\theta_1}\big(\frac{1-c}{c}\varphi_{\theta_1}^{-1}(u_1^0)\big)$ is continuous on c, there exists a c_0 such that $\varphi_{\theta_1}\big(\frac{1-c_0}{c_0}\varphi_{\theta_1}^{-1}(u_1^0)\big) = u_2^0$. Therefore, the intersection of $O(q, r)$ and $\big\{(u_1, u_2) : 1 \ge u_1 \ge 0, \varphi_{\theta_1}\big(\frac{1-c}{c}\varphi_{\theta_1}^{-1}(u_1)\big) \ge u_2 \ge \varphi_{\theta_2}\big(\frac{1-c}{c}\varphi_{\theta_2}^{-1}(u_1)\big)\big\}$ must have a positive Lebesgue measure denoted as D. It implies

$$\begin{aligned} &P_{\theta_0}\{V_{\theta_1}(U_1, U_2) \le c_0\} - P_{\theta_0}\{V_{\theta_2}(U_1, U_2) \le c_0\} \\ &\quad = \int_0^1 \int_{\varphi_{\theta_2}\left(\frac{1-c_0}{c_0}\varphi_{\theta_2}^{-1}(u_1)\right)}^{\varphi_{\theta_1}\left(\frac{1-c_0}{c_0}\varphi_{\theta_1}^{-1}(u_1)\right)} p(u_1, u_2)\mathrm{d}u_2\mathrm{d}u_1 \ge D\epsilon > 0. \end{aligned}$$

This contradicts with that $V_{\theta_1}(U_1, U_2)$ and $V_{\theta_2}(U_1, U_2)$ follow the same distribution. So there must be $\theta_1 = \theta_2$. □

Lemma 5. *Let*

$D(\theta, t) = \{(u_1, u_2) : V_\theta(u_1, u_2) \le t, 0 \le u_i \le 1, i = 1, 2\}$,

then $\mathcal{C} = \{D(\theta, t) : \theta \in \Theta, t \in (0, 1)\}$ *is a class of convex sets.*

Proof. Note that

$$D(\theta, t) = \left\{(u_1, u_2) : 0 \le u_1 \le 1, 0 \le u_2 \le \varphi_\theta\left(\frac{1-t}{t}\varphi_\theta^{-1}(u_1)\right)\right\}.$$

Denote the function $g(x) = k \cdot \varphi_\theta^{-1}(x)$ with $k > 0$. Because the generator $\varphi_\theta(x)$ is 2-monotonic, $\varphi_\theta(x)$ is a convex and non-increasing function. Noticing that $\varphi_\theta(x)$ is strictly decreasing on $[0, \varphi_\theta^{-1}(0))$, we have

$$g''(x) = -k \cdot \varphi_\theta''(x)/(\varphi_\theta'(x))^2 \le 0, \quad x \in [0, 1],$$

so $g(x)$ is concave. As a result, the composition function $\varphi_\theta(\frac{1-t}{t}\varphi_\theta^{-1}(u_1))$ is concave for any $t \in (0, 1)$ and $\theta \in \Theta$.

Let (u_{11}, u_{12}) and (u_{21}, u_{22}) be arbitrary two points in $D(\theta, t)$. It is clear that $0 \le \lambda u_{11} + (1-\lambda)u_{21} \le 1$ for any $\lambda \in [0, 1]$, and

$$\begin{aligned}&\lambda u_{12} + (1-\lambda)u_{22}\\ &\quad\le \lambda\varphi_\theta\left(\frac{1-t}{t}\varphi_\theta^{-1}(u_{11})\right) + (1-\lambda)\varphi_\theta\left(\frac{1-t}{t}\varphi_\theta^{-1}(u_{21})\right)\\ &\quad\le \varphi_\theta\left(\frac{1-t}{t}\varphi_\theta^{-1}(\lambda u_{11} + (1-\lambda)u_{21})\right),\end{aligned}$$

so $\big(\lambda u_{11} + (1-\lambda)u_{21}, \lambda u_{12} + (1-\lambda)u_{22}\big) \in D(\theta, t)$. □

As in Rao (1962), a measure μ is said to be continuous (or non-atomic) if the μ-measure of each single point set is zero and a set A is said to be a continuity set for μ if A has a μ-null boundary. Then we have the following lemma:

Lemma 6. *Let* (U_{1i}, U_{2i}) $(i = 1, \ldots, n)$ *be an i.i.d. sample of* (U_1, U_2)*, which distributes according to an Archimedean copula with generator* $\varphi_{\theta_0}(\theta_0 \in \Theta)$*, then* $P_{\theta_0}\{\sup_{\theta\in\Theta, 0<t<1} |V_\theta^n(t) - V_\theta(t)| \to 0\} = 1$.

Proof. Let μ be the joint distribution of (U_1, U_2) and μ_n be its empirical distribution. It is clear that μ is a non-atomic measure. By Lemma 5, for any $\theta \in \Theta$ and $t \in (0, 1)$, $D(\theta, t)$ is convex. Noticing that the boundary

$$\begin{aligned}\partial D(\theta, t) = {}&\{0 < u_1 \le 1, u_2 = 0\} \cup \{u_1 = 1, 0 \le u_2 \le 1\}\\ &\cup \left\{0 < u_1 \le 1, u_2 = \varphi_\theta\left(\frac{1-t}{t}\varphi_\theta^{-1}(u_1)\right)\right\},\end{aligned}$$

we have $\mu(\partial D(\theta, t)) = 0$, therefore $D(\theta, t)$ is a continuity set for μ. Let $I(\cdot)$ be the indicative function. Because

$$\begin{aligned}V_\theta^n(t) &= \frac{1}{n}\sum_{i=1}^n I\big(V_\theta(u_{1i}, u_{2i}) \le t\big)\\ &= \frac{1}{n}\sum_{i=1}^n I\big((u_{1i}, u_{2i}) \in D(\theta, t)\big)\\ &= \mu_n\big(D(\theta, t)\big),\\ V_\theta(t) &= \int_0^1\int_0^{\varphi_\theta\left(\frac{1-t}{t}\varphi_\theta^{-1}(u_1)\right)} p(u_1, u_2)\mathrm{d}u_2\mathrm{d}u_1\\ &= \int_0^1\int_0^1 I\big((u_1, u_2) \in D(\theta, t)\big) \cdot p(u_1, u_2)\mathrm{d}u_2\mathrm{d}u_1\\ &= \mu\big(D(\theta, t)\big),\end{aligned}$$

this lemma thus holds from Rao (1962, Theorem 7.1). □

Lemma 7. *Following the same conditions in Lemma 6, for any subset* $\Theta_0 \subseteq \Theta$*, denote* $a_n = \inf_{\theta\in\Theta_0}\int_0^1 \big(V_\theta^n(t) - t\big)^2\mathrm{d}t$*, then*

$$P_{\theta_0}\left\{\lim_{n\to+\infty} a_n = \inf_{\theta\in\Theta_0}\int_0^1 \big(V_\theta(t) - t\big)^2\mathrm{d}t\right\} = 1.$$

Proof. Following the Lebesgue dominated convergence theorem,

$$\int_0^1 \big(V_\theta(t) - t\big)^2 = \lim_{n\to+\infty}\int_0^1 \big(V_\theta^n(t) - t\big)^2\mathrm{d}t.$$

So it is clear that

$$P_{\theta_0}\left\{\lim_{n\to+\infty} a_n \le \inf_{\theta\in\Theta_0}\lim_{n\to+\infty}\int_0^1 \big(V_\theta^n(t) - t\big)^2\mathrm{d}t\right\} = 1.$$

By Lemma 6, for $\forall\epsilon > 0$, there exists an N such that for any $n > N$,

$$\begin{aligned}&P_{\theta_0}\left\{\sup_{\theta\in\Theta_0, 0<t<1} |V_\theta^n(t) - V_\theta(t)| \le \epsilon\right\}\\ &\quad\ge P_{\theta_0}\left\{\sup_{\theta\in\Theta, 0<t<1} |V_\theta^n(t) - V_\theta(t)| \le \epsilon\right\} = 1.\end{aligned}$$

Therefore,

$$\begin{aligned}&P_{\theta_0}\left\{\sup_{\theta\in\Theta_0}\left|\int_0^1 \big(V_\theta^n(t) - t\big)^2\mathrm{d}t - \int_0^1 \big(V_\theta(t) - t\big)^2\mathrm{d}t\right| < 2\epsilon\right\}\\ &\quad\ge P_{\theta_0}\left\{\sup_{\theta\in\Theta_0}\int_0^1 \left|\big(V_\theta^n(t) - t\big)^2 - \big(V_\theta(t) - t\big)^2\right|\mathrm{d}t < 2\epsilon\right\}\\ &\quad= P_{\theta_0}\left\{\sup_{\theta\in\Theta_0}\int_0^1 \left|\big(V_\theta^n(t) - t\big) + \big(V_\theta(t) - t\big)\right|\right.\\ &\qquad\left.\cdot\left|V_\theta^n(t) - V_\theta(t)\right|\mathrm{d}t < 2\epsilon\right\}\\ &\quad\ge P_{\theta_0}\left\{\sup_{\theta\in\Theta_0}\int_0^1 2\cdot\left|V_\theta^n(t) - V_\theta(t)\right|\mathrm{d}t < 2\epsilon\right\}\\ &\quad\ge P_{\theta_0}\left\{\sup_{\theta\in\Theta_0, 0<t<1}\left|V_\theta^n(t) - V_\theta(t)\right| < \epsilon\right\} = 1.\end{aligned}$$

Denote $A_\epsilon = \{(u_{1i}, u_{2i})$: there exists an N such that $a_n \ge \inf_{\theta\in\Theta_0}\int_0^1 \big(V_\theta(t) - t\big)^2\mathrm{d}t - 3\epsilon$ for $n > N, i = 1, 2, \ldots\}$. Then $B_\epsilon = \{(u_{1i}, u_{2i})$: there exist an N and $\theta_n \in \Theta_0$ such that $a_n \ge \int_0^1 \big(V_{\theta_n}(t) - t\big)^2\mathrm{d}t - 3\epsilon$ for $n > N, i = 1, 2, \ldots\} \subseteq A_\epsilon$. Because

$$\begin{aligned}&P_{\theta_0}\left\{\int_0^1 \big(V_{\theta_n}^n(t) - t\big)^2\mathrm{d}t \ge \int_0^1 \big(V_{\theta_n}(t) - t\big)^2\mathrm{d}t - 2\epsilon\right\}\\ &\quad\ge P_{\theta_0}\left\{\left|\int_0^1 \big(V_{\theta_n}^n(t) - t\big)^2\mathrm{d}t - \int_0^1 \big(V_{\theta_n}(t) - t\big)^2\mathrm{d}t\right| < 2\epsilon\right\} = 1,\end{aligned}$$

$C_\epsilon = \{(u_{1i}, u_{2i})$: there exist an N and $\theta_n \in \Theta_0$ such that $a_n \ge \int_0^1 \big(V_{\theta_n}^n(t) - t\big)^2\mathrm{d}t - \epsilon$ for $n > N, i = 1, 2, \ldots\} \subseteq B_\epsilon$. By the definition of a_n, we have $P_{\theta_0}\{C_\epsilon\} = 1$. Therefore, $P_{\theta_0}\{A_\epsilon\} = 1$, and then $P_{\theta_0}\big\{\lim_{n\to+\infty} a_n \ge \inf_{\theta\in\Theta_0}\int_0^1 \big(V_\theta(t) - t\big)^2\mathrm{d}t - 3\epsilon\big\} = 1$.

As ϵ can be arbitrarily small, we get

$$P_{\theta_0}\left\{\lim_{n\to+\infty} a_n \ge \inf_{\theta\in\Theta_0}\int_0^1 \big(V_\theta(t) - t\big)^2\mathrm{d}t\right\} = 1.$$

This lemma thus holds. □

Theorem 2. *Let* $(U_{1i}, U_{2i})(i = 1, \ldots, n)$ *be an i.i.d. sample from the two-dimensional Archimedean copula with generator* $\varphi_{\theta_0}(\theta_0 \in \Theta)$*, and* $\hat{\theta}_n$ *be the optimal estimate of parameter* θ_0 *in terms of minimizing the Cramér–von Mises distance given by (5). If* $\varphi_\theta(x)$ *is compositionally monotonic, then* $\lim_{n\to\infty}\hat{\theta}_n = \theta_0$ *almost surely.*

Proof. Let

$$\hat{\theta}_n = \arg\min_{\theta\in\Theta} n\int_0^1 \left(V_\theta^n(t) - U(t)\right)^2 \mathrm{d}U(t) = \arg\min_{\theta\in\Theta} \int_0^1 \left(V_\theta^n(t) - t\right)^2 \mathrm{d}t.$$

By Theorem 1, $\theta_0 = \arg\min_{\theta\in\Theta}\int_0^1 (V_\theta(t)-t)^2\mathrm{d}t$, so

$$P_{\theta_0}\left\{\int_0^1 \left(V_{\hat{\theta}_n}^n(t) - t\right)^2\mathrm{d}t \le \int_0^1 \left(V_{\theta_0}^n(t) - t\right)^2\mathrm{d}t\right\} = 1.$$

Denote $A_\epsilon = \{(u_{1i}, u_{2i})$: there exists an N such that $\hat{\theta}_n \in [\theta_0-\epsilon, \theta_0+\epsilon]$ for $n > N,\ i = 1, 2, \ldots\}$, so $A_\epsilon^c \subset C_\epsilon = \{(u_{1i}, u_{2i}) : \inf_{|\theta-\theta_0|\ge\epsilon}\int_0^1 \left(V_\theta^n(t) - t\right)^2\mathrm{d}t \le \int_0^1\left(V_{\theta_0}^n(t) - t\right)^2\mathrm{d}t$ for infinitely many $n,\ i = 1, 2, \ldots\}$. Then we have

$$\begin{aligned}
&P_{\theta_0}\left\{\lim_{n\to+\infty}\left(\inf_{|\theta-\theta_0|\ge\epsilon}\int_0^1\left(V_\theta^n(t)-t\right)^2\mathrm{d}t - \int_0^1\left(V_{\theta_0}^n(t)-t\right)^2\mathrm{d}t\right)\le 0\right\}\\
&\overset{a}{=} P_{\theta_0}\left\{\inf_{|\theta-\theta_0|\ge\epsilon}\left(\lim_{n\to+\infty}\int_0^1\left(V_\theta^n(t)-t\right)^2\mathrm{d}t - \int_0^1\left(V_{\theta_0}^n(t)-t\right)^2\mathrm{d}t\right)\le 0\right\}\\
&\overset{b}{=} P_{\theta_0}\left\{\inf_{|\theta-\theta_0|\ge\epsilon}\left(\int_0^1\left(V_\theta(t)-t\right)^2\mathrm{d}t - \int_0^1\left(V_{\theta_0}(t)-t\right)^2\mathrm{d}t\right)\le 0\right\}\\
&\overset{c}{=} P_{\theta_0}\left\{\inf_{|\theta-\theta_0|\ge\epsilon}\int_0^1\left(V_\theta(t)-t\right)^2\mathrm{d}t\le 0\right\}\\
&\overset{d}{=} P_{\theta_0}\left\{\min\left(\int_0^1\left(V_{\theta_0+\epsilon}(t)-t\right)^2\mathrm{d}t, \int_0^1\left(V_{\theta_0-\epsilon}(t)-t\right)^2\mathrm{d}t\right)\le 0\right\}\\
&\overset{e}{=} 0,
\end{aligned}$$

where $\overset{a}{=}$ follows from Lemma 7, $\overset{b}{=}$ follows from the Lebesgue dominated convergence theorem, $\overset{c}{=}$ follows from Lemma 4, $\overset{d}{=}$ follows from the definition of compositionally monotonic generator given in Definition 4, and $\overset{e}{=}$ follows from Theorem 1. So $P_{\theta_0}\{C_\epsilon\} = 0$, and then $P_{\theta_0}\{A_\epsilon\} = 1$, which implies $\lim_{n\to\infty}\hat{\theta}_n = \theta_0$ almost surely. □

Theorem 2 guarantees the strong consistency of the estimator. From Lemma 3, Archimedean copulas are characterized by the radial distribution of the radial part. If (U_1, U_2) follows an Archimedean copula with generator φ, we can express $(U_1, U_2) = \varphi(RS_d)$ with S_d distributed uniformly on the unit simplex $\mathcal{S}_d$. So it is clear that the dependence information of (U_1, U_2) is concentrated on the generator φ and the radial distribution of R. In addition, the radial distribution of R depends only on the φ. Therefore, we can reduce the selection of the multivariate Archimedean copula which fits data best to a one-dimensional problem, by the following four steps:

(1) Estimate the parameters for all competing models by minimizing the Cramér–von Mises distance, and denote them as $\hat{\theta}_n^1, \ldots, \hat{\theta}_n^K$;
(2) For $k = 1, \ldots, K$, calculate the estimated radial distribution functions $F_{\hat{\theta}_n^k}^k(x)$ of Archimedean copulas with generators $\varphi_{\hat{\theta}_n^k}^k$ using Eq. (3);
(3) For $k = 1, \ldots, K$, compute the observed experimental radial cumulate distribution functions $\hat{F}_{\hat{\theta}_n^k}^k(x)$ under the assumption with generators $\varphi_{\hat{\theta}_n^k}^k$;
(4) Determine the model from which the observed experimental radial distribution function $\hat{F}_{\hat{\theta}_n^k}^k(x)$ that is closest to the according estimated distribution function $F_{\hat{\theta}_n^k}^k(x)$.

From Eq. (3), we can compute the radial distributions of the l_1-norm symmetric distributions associated with the Archimedean copulas conveniently. As both of $\hat{F}_\theta^k(x)$ and $F_\theta^k(x)$ are distribution functions, we could select the "optimal" copula in terms of the Radial Information Criteria (RIC) to minimize the distance:

$$d_{\mathcal{M}_k}(\theta) = \|\hat{F}_\theta^k(x) - F_\theta^k(x)\|, \tag{6}$$

where the function $\|\cdot\|$ can be Kolmogorov distance, Hellinger distance or l_p-norm. Especially, for the l_2-norm, $d_{\mathcal{M}_k}(\theta)$ can be expressed as:

$$\sum_{t=1}^N (\hat{F}_\theta^k(x_t) - F_\theta^k(x_t))^2, \tag{7}$$

where x_t is the observed radial part, which can be calculated by $x_t = \sum_{i=1}^2 \varphi_\theta^{k-1}(u_{it})$.

If the actual joint distribution of data follows an Archimedean copula $C_\varphi(U_1, U_2)$, its experimental radial part of the l_1-norm symmetric distribution associated with this Archimedean copula should follow the estimated radial distribution. Therefore, the smaller $d_{\mathcal{M}_k}(\theta)$ is, the better the model $\mathcal{M}_k$ fits to the data.

3.2. Multivariate Archimedean copula estimation and selection

In this subsection, we extend the method of bivariate Archimedean copula estimation and selection to the multivariate cases.

Theorem 3. *Let $(U_1, \ldots, U_d)$ be distributed according to the d-dimensional Archimedean copula C with generator $\varphi_\theta(x)(\theta \in \Theta)$. Then, for any $i \ne j$, (U_i, U_j) is distributed according to the two-dimensional Archimedean copula with the same generator $\varphi_\theta(x)$.*

Proof. Since

$$C(u_1, \ldots, u_d) = \varphi_\theta(\varphi_\theta^{-1}(u_1) + \cdots + \varphi_\theta^{-1}(u_d)),$$

$$\begin{aligned}
P(U_i < u_i, U_j < u_j) &= \int_0^1 \cdots \int_0^{u_i} \cdots \int_0^{u_j} \cdots \int_0^1 \mathrm{d}C(U_1, \ldots, U_d)\\
&= C(1, \ldots, u_i, \ldots, u_j, \ldots, 1) - C(0, \ldots, 0)\\
&= \varphi_\theta(\varphi_\theta^{-1}(u_i) + \varphi_\theta^{-1}(u_j)),
\end{aligned}$$

this theorem thus holds. □

Theorem 3 is very useful because we can take advantage of the procedure of bivariate case conveniently for multivariate cases. Let $V_{\theta_{ij}}^n(t)$ be the empirical distribution function of $V_\theta(U_i, U_j)$, which is uniformly distributed on [0, 1] for any $0 \le i < j \le d$. So, the estimation is achieved when $\hat{\theta}_n$ brings the minimal Cramér–von Mises distance of

$$\mathrm{d}(\theta) = n\sum_{i<j}\int_0^1 (V_{\theta_{ij}}^n(t) - U(t))^2\mathrm{d}U(t). \tag{8}$$

The steps of model selection are same as those of bivariate cases to select the "optimal" copula to minimize the distance given by (6). The difference is that the radial distribution should be changed as multi-dimensional cases which could be calculated by Eq. (3). So far we have accomplished the whole Archimedean copula estimation and selection problem.

4. Numerical examples

In this section, some simulation experiments and a real data example are provided to show the efficiency of our proposed method.

Table 1
Sample average biases ($\times 100$) and variances ($\times 100$) of three estimators of the association parameter θ for Clayton's family with bivariate distributions, $\gamma = \ln(\theta + 1)$.

γ	$\hat{\theta}_n$		$\tilde{\theta}_n$		$\tilde{\theta}_n^*$	
	Bias	Var	Bias	Var	Bias	Var
0.1	5.65	2.43	1.915	2.35	4.09	1.54
0.2	2.93	3.74	0.99	2.93	3.81	2.37
0.3	2.78	4.83	1.08	3.57	4.58	3.04
0.4	3.05	6.13	1.13	4.59	5.88	4.03
0.5	4.21	7.31	3.25	5.87	7.53	4.84
0.6	3.74	8.51	1.28	6.94	6.37	5.86
0.7	5.57	9.90	3.26	8.81	7.98	7.26
0.8	5.38	11.95	3.07	10.51	7.81	9.07
0.9	5.64	14.12	2.97	13.71	7.29	10.37
1.0	6.31	16.83	5.37	16.05	7.38	13.40
1.1	6.24	19.66	4.94	19.78	6.92	16.10
1.2	6.73	24.07	6.15	24.39	7.32	20.62
1.3	7.63	30.51	8.89	31.98	6.79	25.08
1.4	9.15	37.23	11.37	39.04	6.84	30.89
1.5	3.50	42.54	8.69	42.56	−2.50	32.87
1.6	7.92	56.76	9.00	58.96	2.34	46.45
1.7	4.21	67.65	7.36	68.59	−2.06	53.52
1.8	5.12	84.70	13.59	91.04	−3.81	68.80
1.9	−5.81	99.67	10.25	106.94	−13.40	80.32
2.0	−5.02	116.41	17.25	126.02	−16.69	96.44

4.1. Two-dimensional simulation

In order to illustrate the performance of our method, we select the five Archimedean copula families, i.e. Clayton, Gumbel, Frank, AMH and Joe to form the competing model set. The data (x_{1t}, x_{2t}), $t = 1, \ldots, n$ with exponent marginal distributions are randomly generated from the five copula families of different sizes and correlations by the Matlab program provided in Perkins and Lane (2003). As in Genest et al. (1995) and the canonical maximum likelihood method in Durrleman et al. (2000), we take the marginal empirical distributions as the marginal distributions.

Then we use the minimal Cramér–von Mises distance (5) to get the estimation of the parameter for each Archimedean family. Followed Genest et al. (1995), Table 1 reports the sample biases and variances of $\hat{\theta}_n$, along with those of two competing semi-parametric estimators, based on 2000 random samples of size 100 from Clayton family with various parameter values. Here, $\tilde{\theta}_n$ and $\tilde{\theta}_n^*$ stand for the estimator based on Kendall's τ in Genest and Rivest (1993) and maximum pseudo-likelihood in Genest et al. (1995) respectively.

From Table 1, we see that the biases of $\hat{\theta}_n$ are often smaller than those of $\tilde{\theta}_n^*$. Especially, when γ are 1.9 and 2.0, the biases of $\hat{\theta}_n$ are much smaller than those of the other two estimators. When $\gamma \leq 1$, the variances of $\hat{\theta}_n$ are a little larger than those of $\tilde{\theta}_n$ and $\tilde{\theta}_n^*$; otherwise, the variances of $\hat{\theta}_n$ are smaller than those of $\tilde{\theta}_n$.

After the estimation, we use the RIC to do the model selection following the four steps proposed in Section 4. In order to find the Archimedean copula that fits the sample data best, the Eq. (7) is used to measure the distance between the empirical and estimated radial distributions. The results using Monte Carlo method to simulate 1000 runs for $\tau = 0.2$, 0.5 and 0.7 respectively, are listed in Table 2.

It can be seen from Table 2 that the RIC gives a good model selection from Clayton, Gumbel, Frank, AMH and Joe copulas, and leads to more accuracy as the sample size increases. So the RIC method converges to the right copula, i.e., the probability of a successful model selection approaches one as n increases. However, when τ approaches 0, larger samples are needed for a successful identification.

The average biases and variances of the parameter estimators for $\tau = 0.2$ ($n = 600$), 0.5 and 0.7 ($n = 300$) are given in Table 3. We can see that our method gives a quite accurate estimation of the parameters.

Table 2
Percentage of successful model selection over 1000 runs for two-dimensional data.

Copula	$\tau = 0.2$			$\tau = 0.5$			$\tau = 0.7$		
	n			n			n		
	100	300	600	30	100	300	30	100	300
Clayton	56.9	75.6	85.6	66.9	91.5	99.4	73.7	95.6	99.9
Gumbel	35.7	57.9	73.1	41.2	67.1	89.8	47.9	79.5	97.7
Frank	35.9	50.7	65.8	51.6	78.7	95.2	62.0	88.2	98.8
AMH	29.7	51.4	65.8						
Joe	60.4	73.6	84.7	51.6	75.7	92.3	55.5	82.2	97.4

Table 3
Average biases ($\times 100$) and variances ($\times 100$) of the parameter estimators over 1000 runs for two-dimensional data.

Copula	$\tau = 0.2$		$\tau = 0.5$		$\tau = 0.7$	
	Bias	Var	Bias	Var	Bias	Var
Clayton	1.18	0.89	1.28	6.40	2.50	21.20
Gumbel	0.03	0.22	1.49	1.79	0.75	5.55
Frank	−1.04	9.89	1.73	30.48	−1.44	70.04
AMH	−1.00	0.72		.	.	.
Joe	0.58	0.67	0.96	5.84	−0.70	20.75

Table 4
Percentage of successful model selection over 1000 runs for three-dimensional data.

Copula	$\tau = 0.2$			$\tau = 0.5$			$\tau = 0.7$		
	n			n			n		
	100	300	600	30	100	300	30	100	300
Clayton	88.3	99.2	100.0	86.2	97.4	100.0	80.8	98.2	100.0
Gumbel	80.1	89.5	96.7	72.4	92.0	100.0	69.4	93.5	100.0
Frank	66.3	84.5	95.1	71.9	88.6	98.4	79.6	93.5	99.8

Table 5
Average biases ($\times 100$) and variances ($\times 100$) of the parameter estimators over 1000 runs for three-dimensional data.

Copula	$\tau = 0.2$		$\tau = 0.5$		$\tau = 0.7$	
	Bias	Var	Bias	Var	Bias	Var
Clayton	−0.21	0.41	2.98	4.13	1.59	16.64
Gumbel	−0.18	0.13	2.11	0.82	−0.56	3.48
Frank	0.28	4.49	3.63	7.97	−5.00	43.80

4.2. Three-dimensional simulation

Now, we consider three-dimensional situations for three key Archimedean families: Clayton, Gumbel and Frank copulas. The results of percentage of successful selections using Monte Carlo method to simulate 1000 runs for the same sample sizes as two-dimensional situations are listed in Table 4. The average biases and variances of the parameter estimator for $\tau = 0.2$ ($n = 600$), 0.5 and 0.7 ($n = 300$) are given in Table 5, where the τ represents the dependence between the two out of three-dimensional data.

It can be seen from Table 4 that the RIC also gives a good model identification from Clayton, Gumbel and Frank copulas, and leads to more accuracy as the sample size increases. It is very useful to model the data by selecting the Archimedean copula among lower tail dependence, upper tail dependence and symmetric dependence. Also, as we expected, it gives a considerably correct estimation of the parameters.

From the above simulation experiments, we can see that our method cannot only estimate the unknown parameter θ of Archimedean generator to fit the data but also select the right copula model. It is an efficient solution to the open question about Archimedean copula selection shown in Section 1.

4.3. Real data example

In this subsection, a real data set which contains the time series of three stock indices (Dow Jones, Nasdaq and Standard & Poor's)

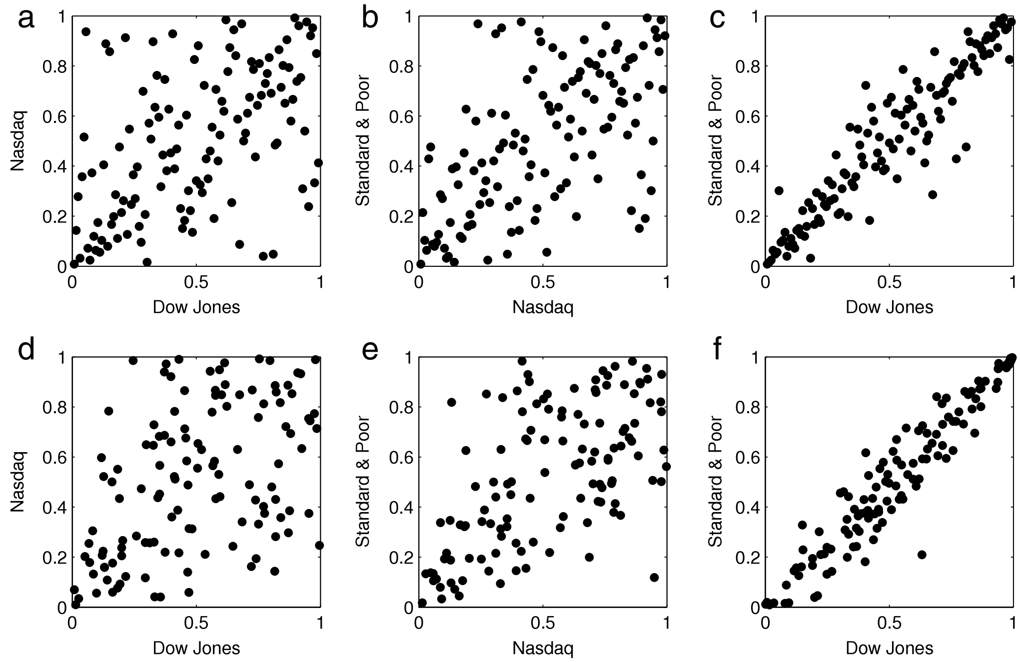

Fig. 1. The scatter plots of samples of each pair of three stock indices (a)–(c) and the simulation of the selected models corresponding to them (d)–(f).

was analyzed for illustrative purpose. Let us consider the monthly log returns from January 1999 to May 2009, for a total of $n = 125$ observations, which are obtained from http://www.economy.com/freelunch/default.asp. We use the empirical margins to map the observations into the uniform space and the scatter plots of the resulted data are illustrated in Fig. 1(a)–(c), from which we see that the three stock indices have different dependence structures between each other, so we should apply model selection for every pair in order to capture the true dependence structures among them.

Using the RIC method to do the model selection from Archimedean copulas by the four steps proposed in Section 3, we have the following results:

(1) For Dow Jones and Nasdaq indices, the RICs of the Clayton, Gumbel, Frank, AMH and Joe copulas are 0.0640, 0.0447, 0.0575, 0.0280 and 0.0852 respectively, which implies that the data are more likely coming from the AMH copula with parameter $\theta = 0.9986$;
(2) For Nasdaq and Standard & Poor's indices, the corresponding RICs are 0.0384, 0.0588, 0.0434, 0.0516 and 0.1163 respectively, which implies that the data are more likely coming from the Clayton copula with parameter $\theta = 1.8181$;
(3) For Dow Jones and Standard & Poor's indices, the corresponding RICs are 0.0317, 0.0221, 0.0298, 0.2997 and 0.0426 respectively, which implies that the data are more likely coming from the Gumbel copula with parameter $\theta = 5.2003$.

The scatter plots of stochastic simulation of the selected models with the same sample size of the observations are shown in Fig. 1(d)–(f), which illustrate that our model selection method could capture accurate dependence structure of the three stock indices. Therefore, our method provides an effective and convenient way to compare and justify which Archimedean copula model fits the data best, which is different from other model selection method that requires additional prior information such as the Bayesian method (e.g. Huard et al. (2006)).

5. Conclusion

In this paper, using the relationship between Archimedean copulas and l_1-norm symmetric distributions, we realize a decomposition of the random vector into two independent parts which are utilized respectively to estimate the model parameter and select an appropriate Archimedean copula based on radial information criteria for a given data. Furthermore, it is extended to multivariate cases conveniently.

The numerical simulations illustrate that the presented method cannot only precisely estimate the parameter but also select the right copula family to fit the simulated data. The application in modelling the real stock indices data shows that the proposed approach can capture accurate dependence structures among them. Although only some most important families are considered in the competing model set, the RIC method can be extended to other Archimedean copula families without any difficulty.

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