

Nonextensivity and the q -distribution of a relativistic gas under an external electromagnetic field

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We investigate the nonextensivity and the q -distribution of a relativistic gas under an external electromagnetic field. We derive an expression for the nonextensive parameter q based on the relativistic generalized Boltzmann equation, the relativistic q - H theorem and the relativistic version of the q -power-law distribution function in the nonextensive q -kinetic theory. We thus provide the connection between the parameter $q \neq 1$ and the spatial-temporal derivatives of the temperature field of the gas as well as the four-potential, and thereby present a clearly physical meaning for the nonextensivity for the relativistic gas.

nonextensivity, q -distribution, relativistic gas

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In recent years, nonextensive statistical mechanics (NSM) has been considered as one of the generalizations of Boltzmann-Gibbs (BG) statistics. NSM has been studied on the basis of Tsallis entropy [1], with the nonextensive parameter q different from unity, in the form of

$$S_q = \frac{k}{1-q} \left(\sum_{i=1}^W p_i^q - 1 \right), \quad (1)$$

where p_i is the probability that the system under consideration is in its i th configuration and k is Boltzmann constant. This entropy is nonextensive for the parameter $q \neq 1$ and the deviation of q from unity is said to represent the degree of nonextensivity. Taking the limit $q \rightarrow 1$, S_q reduces to the BG entropy, $S = k \sum_i p_i \ln p_i$, and extensivity is recovered.

NSM has attracted a great deal of attention both from theoretical study and experimental observation [2]. When many traditional theories in BG statistics have been generalized in the framework of NSM and have been applied to various interesting fields, we need to know under what

circumstances, e.g. which class of nonextensive systems and under what physical situations, can NSM be used for the statistical description. Thus, understanding the physical meaning of q and determining this parameter from the underlying microscopic dynamics of the systems under consideration has become very important problems in NSM and for its applications. In this aspect, some theoretical research has been done (e.g. see [3–11] and the references therein), in which self-gravitating systems and plasma systems have offered the best framework to research the nonextensive effects [4–8, 12–24].

In particular, the expression for the nonextensive parameter q has been determined rigorously by the generalized Maxwell-Boltzmann (MB) distribution, the generalized Boltzmann equation and the q - H theorem in the framework of NSM, thus the nonextensivity is related to some nonequilibrium properties of the systems with long-range interactions (e.g. gravitational forces and Coulombian forces) [4–11]. In other words, we now know that, for NSM, the parameter q different from unit is related to the temperature gradient and the potentials of the systems such as self-gravitating systems and plasma systems. Thus NSM can be

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reasonably applied to describing the thermodynamic properties of the systems under an external field when they are in the nonequilibrium stationary-state. Recently, the above theory has led to an experimental test of NSM using the solar sound speeds in the helioseismological measurements [21].

Recently, the stationarity, ergodicity and entropy in relativistic systems were analyzed [25], and a close connection between time parameters and entropy in special relativity was revealed. The generalized Boltzmann equation and the q - H theorem for a relativistic case are studied in the q -kinetic theory, where an expression of the nonextensive four-entropy flux is given [26] by

$$S_q^\mu = -k_B c^2 \int \frac{d^3 p}{E} p^\mu f^q \ln_q f. \quad (2)$$

Following the lines from the kinetic theory presented in [4,5,8], in this work, we derive the expression of the nonextensive parameter $q \neq 1$ for the relativistic gas and then present it a physical interpretation for a relativistic gas under an external electromagnetic field.

1 The physical interpretation for q -parameter

We consider a relativistic gas of N -point particles of mass m enclosed in a volume V and under the action of an external Lorentz four-force field,

$$F^\mu(x, p) = -\frac{Q}{mc} F^{\mu\nu}(x) p_\nu, \quad (3)$$

where the particles have four-momentum $p \equiv p^\mu = (E/c, \mathbf{p})$ at the point $x \equiv x^\mu = (ct, \mathbf{r})$ of the space-time, with the energy $E/c = \sqrt{\mathbf{p}^2 + m^2 c^2}$, Q is the charge of the particle, and $F^{\mu\nu}$ is the Maxwell electromagnetic tensor. The index μ takes its usual values 0, 1, 2, 3, which identify one time and three space coordinates. The states of this gas can be characterized by a Lorentz invariant one-particle distribution function, $f_q(x, p)$. Thus at each time t , $f_q(x, p) d^3 x d^3 p$ gives the number of particles in the volume element $d^3 x d^3 p$ around the space position \mathbf{x} and momentum \mathbf{p} . The evolution equation of the relativistic distribution function is assumed to be the relativistic generalized Boltzmann equation in the q -kinetic theory [26]:

$$p^\mu \partial_\mu f_q + m F^\mu \frac{\partial f_q}{\partial p^\mu} = C_q(f_q), \quad (4)$$

where $\partial_\mu = (c^{-1} \partial_t, \nabla)$ are the respective derivatives with respect to the time-space coordinates, and $C_q(f)$ is the relativistic q -collisional term. The relativistic version of the power-law distribution in the framework of NSM can be obtained as a natural consequence of the relativistic q - H

theorem:

$$f_q(x, p) = \left\{ 1 - (1-q) \left[\alpha(x) + \beta_\mu p^\mu \right] \right\}^{1/(1-q)}, \quad (5)$$

where $\alpha(x)$ and $\beta_\mu(x)$ are arbitrary space-time-dependent parameters. For the above relativistic gas in the presence of an external electromagnetic field, it can be written [26,27] as

$$f_q(x, p) = n B_q \left[1 - (1-q) \left(\frac{u - [p^\mu + c^{-1} Q A^\mu(x)] U_\mu}{k_B T} \right) \right]^{1/(1-q)}, \quad (6)$$

where n is the particle number density, B_q a normalization constant, u the Gibbs function per particles, U_μ the mean four-velocity of the gas and $A^\mu(x)$ is the four-potential. It is clear that, in the limit $q \rightarrow 1$, eq. (6) recovers to the well-known relativistic Jüttner distribution [28,29]:

$$f(x, p) = n B \exp \left(\frac{u - [p^\mu + c^{-1} Q A^\mu(x)] U_\mu}{k_B T} \right). \quad (7)$$

To determine the expression for the nonextensive parameter, according to the relativistic q - H theorem, the solution of eq. (4) will evolve towards the power-law distribution function, eq. (6), the q -collision term vanishes, $C_q(f_q) = 0$, and eq. (4) reduces to

$$p^\mu \partial_\mu f_q + m F^\mu \frac{\partial f_q}{\partial p^\mu} = 0. \quad (8)$$

In other words [30], the distribution function (6) must satisfy eq. (8). For the sake of convenience, we can write the equation for $f_q^{1-q}(x, p)$ as

$$p^\mu \partial_\mu f_q^{1-q} + m F^\mu \frac{\partial f_q^{1-q}}{\partial p^\mu} = 0. \quad (9)$$

From eq. (6), we have

$$f_q^{1-q}(x, p) = n^{1-q} B_q^{1-q} \left[1 - (1-q) \left(\frac{\mu - [p^\mu + c^{-1} Q A^\mu(x)] U_\mu}{k_B T} \right) \right]. \quad (10)$$

Then,

$$\begin{aligned} \partial_\mu f_q^{1-q}(x, p) = & \left[1 - (1-q) \left(\frac{\mu - [p^\mu + c^{-1} Q A^\mu(x)] U_\mu}{k_B T} \right) \right] \\ & \times \partial_\mu \left(n B_q \right)^{1-q} - \left[\left(n B_q \right)^{1-q} \right] \left[\frac{-u}{k_B T^2} \partial_\mu T \right. \\ & + \frac{p^\mu U_\mu}{k_B T^2} \partial_\mu T + \frac{c^{-1} Q A^\mu(x) U_\mu}{k_B T^2} \partial_\mu T \\ & \left. - \frac{c^{-1} Q \partial_\mu A^\mu(x)}{k_B T} U_\mu \right], \end{aligned} \quad (11)$$

and

$$\frac{\partial f^{1-q}}{\partial p^\mu} = (nB_q)^{1-q} \frac{U_\mu}{k_B T}. \quad (12)$$

Substituting eqs. (11) and (12) into eq. (9), we have

$$\begin{aligned} & p^\mu \partial_\mu (nB_q)^{1-q} - p^\mu (1-q) \frac{u}{k_B T} \partial_\mu (nB_q)^{1-q} \\ & + \frac{[p^\mu + c^{-1}QA^\mu(x)]U_\mu}{k_B T} \partial_\mu (nB_q)^{1-q} \\ & - p^\mu (nB_q)^{1-q} \left[\frac{-u + p^\mu U_\mu + c^{-1}QA^\mu(x)U_\mu}{k_B T^2} \partial_\mu T \right. \\ & \left. - \frac{c^{-1}Q\partial_\mu A^\mu(x)}{k_B T} U_\mu \right] \\ & - \frac{Q}{c} F^{\mu\nu} p_\nu (nB_q)^{1-q} (1-q) \frac{U_\mu}{k_B T} = 0. \end{aligned} \quad (13)$$

Eq. (13) is considered identically null for any arbitrary variable p , and hence the sum of the coefficients of each power for p must be zero. We therefore derive the sum of the coefficients in eq. (13), for the first-order term in p , as

$$\begin{aligned} & p^\mu \partial_\mu (nB_q)^{1-q} - p^\mu (1-q) \frac{u}{k_B T} \partial_\mu (nB_q)^{1-q} \\ & + p^\mu (1-q) \frac{c^{-1}QA^\mu(x)U_\mu}{k_B T} \partial_\mu (nB_q)^{1-q} \\ & - p^\mu (nB_q)^{1-q} (1-q) \left[\frac{1}{k_B T^2} (-u + c^{-1}QA^\mu(x)U_\mu) \partial_\mu T \right. \\ & \left. - \frac{c^{-1}QA^\mu(x)}{k_B T} U_\mu \right] - \frac{Q}{c} F^{\mu\nu} p_\nu (nB_q)^{1-q} (1-q) \frac{U_\mu}{k_B T} = 0, \end{aligned} \quad (14)$$

and, for the second-order term in p , as

$$\partial_\mu (nB_q)^{1-q} - (nB_q)^{1-q} \frac{\partial_\mu T}{T} = 0. \quad (15)$$

Substituting eq. (15) into eq. (14), we find the relation

$$\partial_\mu T = (1-q) \frac{Q}{ck_B} (-\partial_\mu A^\nu + F^{\mu\nu}) U_\mu. \quad (16)$$

Furthermore, let $F^{\mu\nu}$ be a dissymmetry tensor and make use of the relation [30]

$$F^{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu. \quad (17)$$

Eq. (16) becomes

$$\partial_\mu T = (1-q) \frac{Q}{ck_B} (\partial_\nu A^\mu) U_\mu. \quad (18)$$

Thus, we obtained an expression for the nonextensive parameter $q \neq 1$ for the relativistic gas under an external

electromagnetic field. Eq. (18) provides a connection between the parameter $q \neq 1$ and the space-time derivatives of the temperature field of the gas as well as the four-potential, and therefore provides a clearly physical interpretation for the nonextensivity for the relativistic gas. We find that q is different from unity if and only if the quantity $\partial_\mu T$ is different from zero. If the temperature field satisfies $\partial_\mu T = 0$, then we have $q=1$, corresponding to the standard case in BG statistics. If the temperature field is $\partial_\mu T \neq 0$, then we have $q \neq 1$, corresponding to the case of NSM. Thus, the nonextensive parameter $q \neq 1$ is related closely to the spatial-temporal inhomogeneity of the temperature field of a nonequilibrium relativistic gas in an external electromagnetic field.

2 Conclusion

In summary, we investigated the nonextensivity and the power-law distribution of a relativistic gas under the influence of an external electromagnetic field. We derived an expression for the nonextensive parameter q based on the generalized relativistic Boltzmann equation, the relativistic q -H theorem, and the relativistic version of the q -power-law distribution function in the nonextensive q -kinetic theory. We thus provided the connection between the parameter $q \neq 1$ and the space-time derivatives of the temperature field of the gas as well as the four-potential, and thereby we can present a clearly physical meaning for the nonextensive parameter $q \neq 1$ for the relativistic gas. Nonextensivity is related closely to the spatial-temporal inhomogeneity of the temperature field of a nonequilibrium relativistic gas under an external electromagnetic field.

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