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Adaptive synchronization of a class of continuous chaotic systems with uncertain parameters

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Abstract

This Letter investigates the problem of synchronization between two different chaotic systems. A generic adaptive synchronization scheme is proposed for a class of chaotic systems with uncertain parameters. Based on the Lyapunov stability theorem, an adaptive controller and the corresponding parameters update laws are designed to synchronize two chaotic systems. Numerical simulations are also given to show the effectiveness of the proposed method.

Keywords: Chaotic system; Synchronization; Adaptive control

1. Introduction

The study of synchronization phenomena in the dynamical systems started at Huggen's experiments in 17th century. Since then, synchronization has been widely explored in a variety of systems including physical, chemical and ecological systems. In the broadest sense, synchronization is often understood as the tendency to undergo resembling evolution in time. Synchronization is an important mechanism for creating order in complex systems.

Many nonlinear dynamical systems have been found to show a kind of behavior known as chaos, being characterized as chaotic systems by their extreme sensitivity to initial conditions and having noise-like behaviors. Because of these properties, chaotic systems are difficult to achieve synchronization. In 1990, Pecora and Carrol [1] introduced a method to synchronize two identical chaotic dynamical systems with different initial conditions and showed that for some chaotic systems synchronization is possible. Since then, the synchronization of chaotic dynamical systems attracted much attention due to its theoretical challenge and potential applications in secure communications, chemical reactions, biomedical science, social science, and many other fields. Many types of synchronization have been presented such as complete synchronization (CS) [1], phase synchronization (PS) [2,3], lag synchronization (LS) or anticipated synchronization (AS) [4,5], and generalized synchronization (GS) [6–8], etc. Up to now, a wide variety of methods have been proposed for the synchronization of chaotic systems, including PC method [1], active control method [9,10], feedback control method [11,12], impulse control method [13–15], fuzzy control method [16], etc.

The aforementioned methods and many other synchronization strategies are valid for chaotic systems only when the systems' parameters are exactly known, and the synchronization will be destroyed and broken with the effects of these uncertainties. However, in many practical situations, it is difficult to exactly determine the values of system parameters in advance. In order to overcome the upper obstacles, adaptive control method [17–23] is proposed to achieve synchronization between two

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chaotic systems with uncertain parameters. In Ref. [19], the authors have employed a combination of feedback control and adaptive control loop associated with the update law of estimation parameters to synchronize identically two coupled chaotic systems. In Ref. [22], several schemes for non-adaptive and adaptive synchronization of chaos in the colpitts oscillator have been presented and investigated. In Ref. [23], the authors have studied the adaptive synchronization and lag synchronization for uncertain dynamical system with time delay based on parameter identification. The main idea of the adaptive control is to utilize adaptive controllers to compensate the effects of parameters' uncertainty. The adaptive control method is significant to the problem of synchronization for chaotic systems.

Motivated by the above discussions, in this Letter, we study the adaptive synchronization of two different chaotic systems. A generic adaptive synchronization scheme is proposed for a class of chaotic systems which have the same structure but uncertain parameters. Based on the Lyapunov stability theorem, an adaptive controller and the corresponding parameters update laws are designed to synchronize two coupled chaotic systems where all parameters of the drive and response systems are uncertain. One illustrative example about the well-known Lorenz system is given to show the effectiveness of above-mentioned scheme.

2. Theoretical results

We consider the drive system in the form of

$$\dot{x} = f(x) + F(x)\theta \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ is the state variable, $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T : R^n \to R^n$ is a nonlinear vector function, $\theta \in R^m$ is the vector of the system parameters, $F \in C(R^n, R^{n\times m})$. The system considered here depends linearly on parameters, which is surely to be the case in many chaotic systems such as Lorenz system, Chen's system, Rössler system, Chua's circuits and so on. Let $\Omega \subset R^n$ be a chaotic bounded set of Eq. (1) which is globally attractive. For the vector function $f(x)$, we give a general assumption: For any $x, y \in \Omega$, there exists a constant $L > 0$ satisfying

$$\|f(y) - f(x)\| \leqslant L\|y - x\|,$$

i.e., $f(x)$ satisfies Lipschitz condition in Ω (Lipschitz constant is L). It is worth noting that this condition is very loose, for example, the condition holds as long as $\frac{\partial f_i}{\partial x_j}$ $(i, j = 1, 2, \ldots, n)$ are bounded. Therefore the assumption above is valid for almost all well-known chaotic and hyperchaotic systems. The response system is given by

$$\dot{y} = f(y) + F(y)\mu \tag{2}$$

which has the same structure as the drive system, $\mu \in R^m$ is the parameter vector. The goal of control is to find out a appropriate controller U that is independent of all uncertain parameters such that the controlled response system

$$\dot{y} = f(y) + F(y)\mu + U(x, y) \tag{3}$$

is synchronous with the drive system (1). At first, we recall the definition of complete synchronization.

Definition. The unidirectionally coupled response system (3) and the drive system (1) are said to be completely synchronizing if, for all $x(t_0), y(t_0) \in R^n$, $\lim_{t\to\infty} \|x(t) - y(t)\| = 0$.

Let $e = y - x$ be the error between the states of systems (1) and (3). Then the error dynamical system of systems (1) and (3) can be obtained:

$$\dot{e} = \dot{y} - \dot{x} = f(y) + F(y)\mu + U - f(x) - F(x)\theta. \tag{4}$$

From the definition of complete synchronization, we know it is sufficient for synchronizing the systems (1) and (3) that the trivial solution of error system (4) is asymptotically stable.

Let $\hat{\theta}$, $\hat{\mu}$ be the estimated values of the parameters θ, μ of the drive and response systems, we can get the following theorem.

Theorem. *Let the adaptive controller be*

$$U(x, y) = F(x)\hat{\theta} - F(y)\hat{\mu} - (E + \epsilon)e, \tag{5}$$

where adaptation $\dot{E} = \|e\|^2$, $\varepsilon > 0$ *and the parameters estimation update law as follows*:

$$\dot{\hat{\theta}} = -F^T(x)e,$$
$$\dot{\hat{\mu}} = F^T(y)e.$$

Then, the synchronization of systems (1) *and* (3) *can be achieved.*

Proof. Let $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{\mu} = \mu - \hat{\mu}$. Substituting Eq. (5) into Eq. (4), the error dynamical system of systems (1) and (3) can be rewritten as

$$\begin{aligned}\dot{e} &= f(y) + F(y)\mu + F(x)\hat{\theta} - F(y)\hat{\mu} - (E + \epsilon)e \\ &\quad - f(x) - F(x)\theta \\ &= f(y) - f(x) + F(y)\tilde{\mu} - F(x)\tilde{\theta} - (E + \epsilon)e.\end{aligned} \tag{6}$$

Construct a Lyapunov function candidate in the form of

$$V = \frac{1}{2}e^T e + \frac{1}{2}(E - \hat{L})^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} + \frac{1}{2}\tilde{\mu}^T\tilde{\mu}$$

where $\hat{L}$ is a large positive constant satisfying $\hat{L} \geqslant L$. Without losing generality as Ω is globally attractive, we assume $x, y \in \Omega$ and use the Lipschitz condition of $f(x)$. By differentiating the function V along the trajectories of the error system (4), we obtain

$$\begin{aligned}\dot{V} &= e^T\dot{e} + (E - \hat{L})\dot{E} + \dot{\tilde{\theta}}^T\tilde{\theta} + \dot{\tilde{\mu}}^T\tilde{\mu} \\ &= e^T\left[f(y) - f(x) + F(y)\tilde{\mu} - F(x)\tilde{\theta} - (\epsilon + E)e\right] \\ &\quad + (E - \hat{L})\|e\|^2 + \left[\left(F^T(x)\right)e\right]^T\tilde{\theta} - \left[F^T(y)e\right]^T\tilde{\mu} \\ &\leqslant L\|e\|^2 - (\epsilon + E)\|e\|^2 + (E - \hat{L})\|e\|^2 \\ &\leqslant -\epsilon\|e\|^2.\end{aligned}$$

From the Lyapunov stability theorem, the trivial solution of the error system (4) is asymptotically stable which implies the synchronization of systems (1) and (3) is achieved. □

Remark 1. The synchronization of systems (1) and (3) with the adaptive controller (5) can be ensured because of the free choice of the constant $\hat{L}$ in the Lyapunov function V. Even if the Lipschitz constant L is unknown, one can choose a constant $\hat{L}$ such that $\hat{L} \gg L$. This means the synchronization of systems (1) and (3) is independent of the constant L.

3. Numerical simulations

One illustrative examples about the well-known Lorenz system is given to show the effectiveness of above-mentioned scheme. The Lorenz system can be described by following nonlinear ODE:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha(x_2 - x_1) \\ \gamma x_1 - x_1 x_3 - x_2 \\ x_1 x_2 - \beta x_3 \end{pmatrix}, \tag{7}$$

which has a chaotic attractor when $\alpha = 10$, $\beta = \frac{8}{3}$, $\gamma = 28$. The Lorenz chaotic attractor is shown in Fig. 1.

There are many results about the Lorenz system. One of them is that there exists a bounded region $\Omega \subset R^3$ containing the whole Lorenz attractor such that each orbit starting in Ω never leave it [24].

$$\Omega = \left\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + (x_3 - \alpha - \beta)^2 \leqslant M\right\}$$

where $M = \frac{b^2(\alpha+\beta)^2}{4(\gamma-1)}$.

Let $x^T = (x_1, x_2, x_3)$, $\theta^T = (\theta_1, \theta_2, \theta_3)$ and rewrite the Lorenz system as

$$\dot{x} = f(x) + F(x)\theta \tag{8}$$

where

$$f(x) = \begin{pmatrix} 0 \\ -x_2 - x_1 x_3 \\ x_1 x_2 \end{pmatrix}$$

and

$$F(x) = \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{pmatrix}.$$

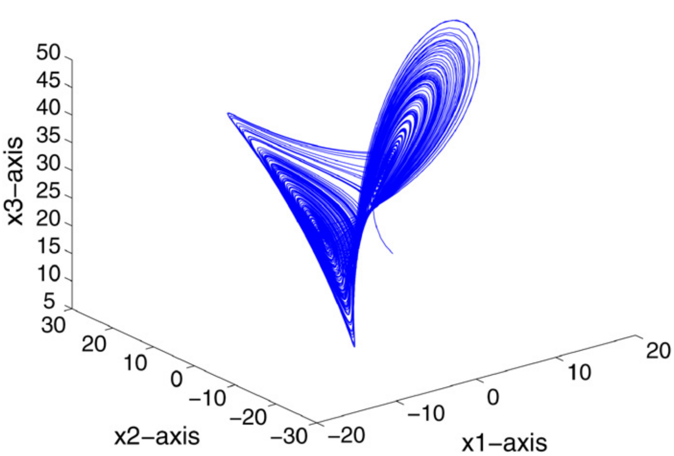

Fig. 1. The chaotic attractor of Lorenz system.

Assume that Eq. (8) is the drive system. The response system is given by the following controlled system:

$$\dot{y} = f(y) + F(y)\mu + U(x, y) \tag{9}$$

where $\mu^T = (\mu_1, \mu_2, \mu_3)$ and $U(x, y)$ is an adaptive controller.

Let us choose the adaptive controller U, adaptation E and the parameters estimation update law as follow:

$$U(x, y) = F(x)\hat{\theta} - F(y)\hat{\mu} - (E + \epsilon)e, \quad \varepsilon > 0, \tag{10}$$

i.e.,

$$\begin{cases} U_1 = (x_2 - x_1)\hat{\theta}_1 - (y_2 - y_1)\hat{\mu}_1 - (E + \varepsilon)(y_1 - x_1), \\ U_2 = x_1\hat{\theta}_2 - y_1\hat{\mu}_2 - (E + \varepsilon)(y_2 - x_2), \\ U_3 = -x_3\hat{\theta}_3 + y_3\hat{\mu}_3 - (E + \varepsilon)(y_3 - x_3), \end{cases}$$

$$\dot{E} = \|e\|^2, \tag{11}$$

$$\dot{\hat{\theta}} = -F^T(x)e, \tag{12}$$

i.e.,

$$\begin{cases} \dot{\hat{\theta}}_1 = -(x_2 - x_1)(y_1 - x_1), \\ \dot{\hat{\theta}}_2 = -x_1(y_2 - x_2), \\ \dot{\hat{\theta}}_3 = x_3(y_3 - x_3), \end{cases}$$

$$\dot{\hat{\mu}} = F^T(y)e, \tag{13}$$

i.e.,

$$\begin{cases} \dot{\hat{\mu}}_1 = (y_2 - y_1)(y_1 - x_1), \\ \dot{\hat{\mu}}_2 = y_1(y_2 - x_2), \\ \dot{\hat{\mu}}_3 = -y_3(y_3 - x_3). \end{cases}$$

In the simulations, the program OED45 in Matlab is used to solve ordinary differential equations (8), (9), (11), (12) and (13). Suppose that the "unknown" parameters of the drive and response systems are chosen as $\theta^T = (\theta_1, \theta_2, \theta_3) = (10, 28, 8/3)$, $\mu^T = (\mu_1, \mu_2, \mu_3) = (10, 30, 3)$. The initial values of the drive and response systems are taken as $(x_1(0), x_2(0),$

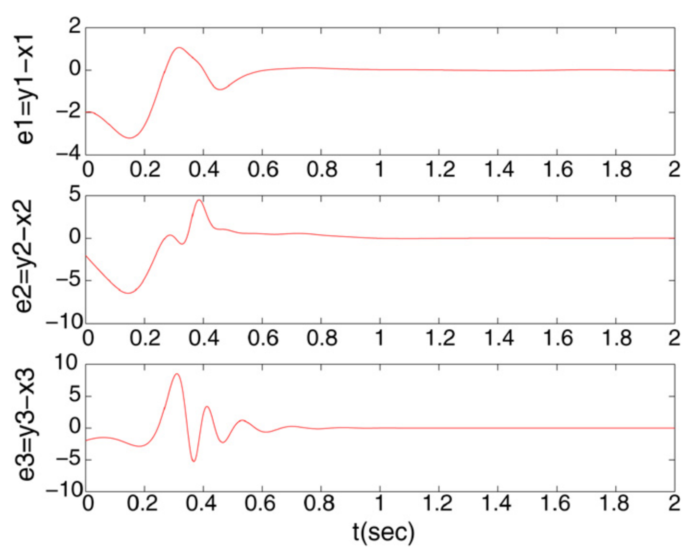

Fig. 2. State errors of coupled Lorenz system.

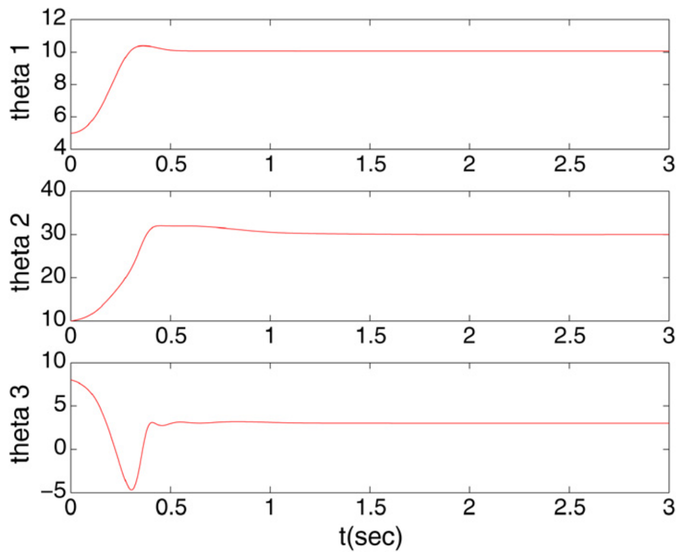

Fig. 3. Estimated parameters of the drive system.

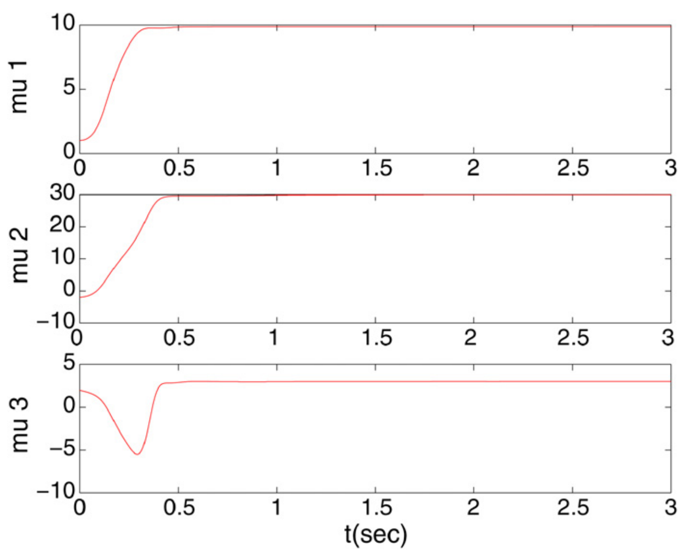

Fig. 4. Estimated parameters of the response system.

$x_3(0)) = (1,1,1)$, $(y_1(0), y_2(0), y_3(0)) = (3,3,3)$, respectively. The initial values estimated for "unknown" parameters are taken as $\hat{\theta}_1 = 5$, $\hat{\theta}_2 = 10$, and $\hat{\theta}_3 = 8$, $\hat{\mu}_1 = 1$, $\hat{\mu}_2 = -2$, and $\hat{\mu}_3 = 2$. Let $E_0 = 2$, $\varepsilon = 0.1$, the simulated results are shown in Figs. 2, 3, 4, and 5. In Fig. 2, three state errors versus time are shown and the state errors tend to zero asymptotically as time evolves. Fig. 3 shows that arbitrary initial estimated values of parameters $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ finally evolve to unknown parameters $(10, 28, 8/3)$. Fig. 4 shows that arbitrary initial estimated values of parameters $(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3)$ finally evolve to $(10, 30, 3)$. Fig. 5 shows that estimated value of E is finally convergent to a constant.

4. Conclusion

This Letter has investigated the problem of synchronization between two different chaotic systems. A generic adaptive synchronization scheme has been proposed for a class of

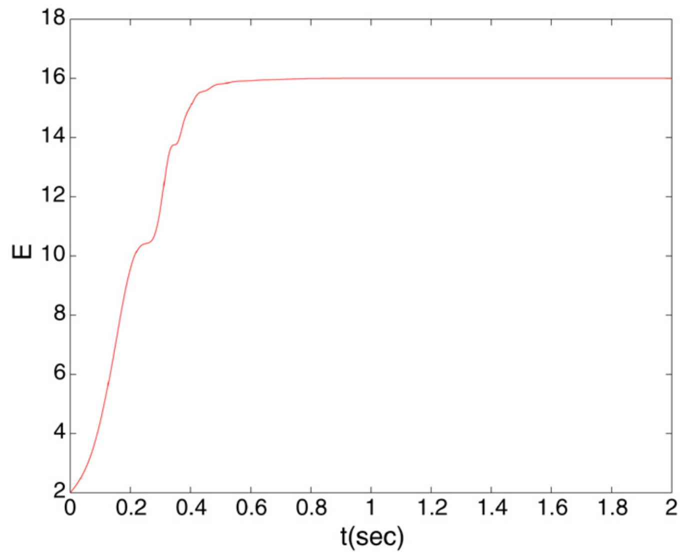

Fig. 5. Estimated of E.

chaotic systems which have the same structure but uncertain parameters. Based on the Lyapunov stability theorem, an adaptive controller and the corresponding parameter update laws are designed to synchronize two coupled chaotic systems. One example about the well-known Lorenz system has been simulated, demonstrating the effectiveness of above-mentioned scheme.

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