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A series of new explicit exact solutions for the coupled Klein–Gordon–Schrödinger equations

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Abstract

We devise an algebraic method to uniformly construct a series of explicit exact solutions for the coupled Klein–Gordon–Schrödinger equations.

1. Introduction

We consider the following coupled Klein–Gordon–Schrödinger (K–G–S) equations in three space dimension

$$\begin{aligned} & \mathrm{i}\frac{\partial}{\partial t}\Psi(X,t)+\Delta\Psi(X,t)+\rho\Psi(X,t)\Phi(X,t)=0, \\ & \frac{\partial^2}{\partial t^2}\Phi(X,t)-\Delta\Phi(X,t)+\mu^2\Phi(X,t)-\rho|\Psi(X,t)|^2=0, \quad t\in R,\ X\in R^3, \end{aligned} \tag{1}$$

which describes a system of conserved scalar nucleons interacting with neutral scalar mesons. Here, Ψ represents a complex scalar neucleon field and Φ a real scalar meson field. The real constant μ describes the mass of a meson, and ρ is a coupling constant. The first study for the coupled K–G–S equations was done by Fukuda and Tsutsumi. By using Galerkin method, they proved the existence of global strong solutions under the condition of initial boundary value [1]. Afterwards, much work has been done on the existence of global solutions, asymptotic behavior and stability for the K–G–S equations [2–7], but explicit exact solutions are still unknown to our knowledge. In this work, we are interested in solitary wave solutions and Jacobi doubly periodic wave solutions of K–G–S equations by using an algebraic method proposed recently [8,9]. Compared with the existing tanh methods and Jacobi function method [10–15], our method further exceeds their applicability in obtaining a series of exact wave solutions including the solitary wave, rational, triangular periodic, Jacobi, and Weierstrass doubly periodic solutions. More importantly, the method provides a guideline for the classification of the solutions based on the given parameters. The proposed method not only give an unified formulation to construct various travelling wave solutions, but also provides a rule to classify the types of solutions according to the given parameters. Furthermore, the proposed method is readily computerizable in solving equation by using symbolic software like *Mathematica* or *Maple*. In the following Section 2, we simply describe the proposed method, and then illustrate its application to the coupled K–G–S equations.

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2. Explicit exact solutions for coupled K–G–S equations

Let recall our proposed method whose main steps are outlined as follows [8,9]:

Step 1. For a given equation

$$H(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{2}$$

we first use wave transformation $u(x,t) = U(\xi) = U(x+ct)$, and change Eq. (2) into an ordinary differential equation (ODE) as:

$$H(U, U', U'', \ldots) = 0. \tag{3}$$

Step 2. We introduce a new variable $\varphi = \varphi(\xi)$ which is a solution of the following first-order ODE:

$$\varphi' = \varepsilon\sqrt{\sum_{j=0}^{m} c_j\varphi^j}, \tag{4}$$

where $\varepsilon = \pm 1$. The derivatives with respect to the variable ξ become the derivatives with respect to the variable φ as

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \to \varepsilon\sqrt{\sum_{j=0}^{m} c_j\varphi^j}\frac{\mathrm{d}}{\mathrm{d}\varphi}, \quad \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \to \varepsilon^2\left[\frac{1}{2}\sum_{j=1}^{m} jc_j\varphi^{j-1}\frac{\mathrm{d}}{\mathrm{d}\varphi} + \sum_{j=0}^{m} c_j\varphi^j\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\right], \ldots$$

By virtue of the variable φ, we expand the solution of Eq. (2) or (3) as the following series:

$$u(x,t) = U(\xi) = \sum_{i=0}^{n} a_i\varphi^i. \tag{5}$$

Balancing the highest derivative term with the nonlinear terms in Eq. (3) will give a relation for the positive integers n and m, from which the different possible values of n and m can be determined. These values lead to the series expansions of the exact solutions for Eq. (2). For example, in the case of KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{6}$$

we have

$$m = n + 2. \tag{7}$$

If we take $n = 1$ and $m = 3$ in (7), we obtain the following series expansion of an exact solution of the KdV equation (6) as

$$u = a_0 + a_1\varphi, \qquad \varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3}.$$

Similarly, if we take $n = 2$, $m = 4$ in (7), we have

$$u = a_0 + a_1\varphi + a_2\varphi^2, \qquad \varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}.$$

Step 3. Substituting the expansion (5) into Eq. (3) and setting the coefficients of all powers of φ^i and $\varphi^i\sqrt{\sum_{j=0}^{m} c_j\varphi^j}$ to zeros, we obtain a system of algebraic equations, from which the constants c, a_i, c_j ($i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$) can be found explicitly.

Step 4. Substituting the constants c_j ($j = 0, 1, \ldots, m$) obtained in Step 3 into Eq. (4), we can then obtain all the possible solutions. We remark here that the travelling wave solutions of Eq. (3) depend on the explicit solvability of Eq. (4). The solution of the system of algebraic equations will be getting tedious with the increase of the values of n and m. When $m = 4$, Eq. (4) gives a series of interesting solutions such as solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic wave solutions. We then consider only the case $m = 4$ in this paper and hence

$$\varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}. \tag{8}$$

We have the following results:

Case A. If $c_3 = c_4 = 0$, Eq. (8) possesses:
two polynomial type solutions

$$\varphi = \varepsilon\sqrt{c_0}\xi, \quad c_1 = c_2 = 0,\ c_0 > 0 \tag{9}$$

and

$$\varphi = -\frac{c_0}{c_1} + \frac{1}{4}c_1\xi^2, \quad c_2 = 0,\ c_1 \neq 0, \tag{10}$$

a exponential type solution

$$\varphi = -\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}\xi), \quad c_0 = \frac{c_1^2}{4c_2},\ c_2 > 0, \tag{11}$$

a triangular type solution

$$\varphi = -\frac{c_1}{2c_2} + \frac{\varepsilon c_1}{2c_2}\sin(\sqrt{-c_2}\xi),\ c_0 = 0,\ c_2 < 0 \tag{12}$$

and a hyperbolic type solution

$$\varphi = -\frac{c_1}{2c_2} + \frac{\varepsilon c_1}{2c_2}\sinh(2\sqrt{c_2}\xi), \quad c_0 = 0,\ c_2 > 0. \tag{13}$$

Case B. If $c_3 = c_1 = 0$, Eq. (8) admits:
a bell shaped solitary wave solution

$$\varphi = \sqrt{-\frac{c_2}{c_4}}\operatorname{sech}(\sqrt{c_2}\xi), \quad c_0 = 0,\ c_2 > 0,\ c_4 < 0, \tag{14}$$

a kink shaped solitary wave solution

$$\varphi = \varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right), \quad c_0 = \frac{c_2^2}{4c_4},\ c_2 < 0,\ c_4 > 0, \tag{15}$$

two triangular type solutions

$$\varphi = \sqrt{-\frac{c_2}{c_4}}\sec(\sqrt{-c_2}\xi), \quad c_0 = 0,\ c_2 < 0,\ c_4 > 0 \tag{16}$$

and

$$\varphi = \varepsilon\sqrt{\frac{c_2}{c_4}}\tan\left(\sqrt{\frac{c_2}{2}}\xi\right), \quad c_0 = \frac{c_2^2}{4c_4},\ c_2 > 0,\ c_4 > 0, \tag{17}$$

a rational type solution

$$\varphi = -\frac{\varepsilon}{\sqrt{c_4}\xi}, \quad c_0 = c_2 = 0,\ c_4 > 0, \tag{18}$$

three Jacobi elliptic doubly periodic type solutions

$$\varphi = \sqrt{\frac{-c_2k^2}{c_4(2k^2-1)}}\operatorname{cn}\left(\sqrt{\frac{c_2}{2k^2-1}}\xi\right), \quad c_4 < 0,\ c_2 > 0,\ c_0 = \frac{c_2^2k^2(1-k^2)}{c_4(2k^2-1)^2}, \tag{19}$$

$$\varphi = \sqrt{\frac{-k^2}{c_4(2-k^2)}}\operatorname{dn}\left(\sqrt{\frac{c_2}{2-k^2}}\xi\right), \quad c_4 < 0,\ c_2 > 0,\ c_0 = \frac{c_2^2(1-k^2)}{c_4(2-k^2)^2} \tag{20}$$

and

$$\varphi = \varepsilon\sqrt{\frac{-c_2k^2}{c_4(k^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{c_2}{k^2+1}}\xi\right), \quad c_4 > 0,\ c_2 < 0,\ c_0 = \frac{c_2^2k^2}{c_4(k^2+1)^2}, \tag{21}$$

where k is a modulus. The Jacobi elliptic functions are doubly periodical and possess properties of triangular functions:

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \quad \mathrm{dn}^2\xi = 1 - m^2\mathrm{sn}^2\xi,$$

$$(\mathrm{sn}\xi)' = \mathrm{cn}\xi\mathrm{dn}\xi, \quad (\mathrm{cn}\xi)' = -\mathrm{sn}\xi\mathrm{dn}\xi, \quad (\mathrm{dn}\xi)' = -m^2\mathrm{sn}\xi\mathrm{cn}\xi.$$

When $k \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}\xi \to \tanh\xi, \quad \mathrm{cn}\xi \to \mathrm{sech}\xi.$$

When $k \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\mathrm{sn}\xi \to \sin\xi, \quad \mathrm{cn}\xi \to \cos\xi.$$

As $k \to 1$, the Jacobi doubly periodic solutions (19) and (20) degenerate to the solitary wave solutions (14), and the solution (21) degenerates to (15).

Case C. If $c_4 = 0$, Eq. (8) admits:
a bell shaped solitary wave solution

$$\varphi = -\frac{c_2}{c_3}\mathrm{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right), \quad c_0 = c_1 = 0,\ c_2 > 0, \tag{22}$$

a triangular type solution

$$\varphi = -\frac{c_2}{c_3}\sec^2\left(\frac{\sqrt{-c_2}}{2}\xi\right), \quad c_0 = c_1 = 0,\ c_2 < 0, \tag{23}$$

a rational type solution

$$\varphi = \frac{1}{c_3\xi^2}, \quad c_0 = c_1 = c_2 = 0 \tag{24}$$

and a Weierstrass elliptic doubly periodic type solution

$$\varphi = \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right), \quad c_2 = 0,\ c_3 > 0, \tag{25}$$

where $g_2 = -4c_1/c_3$, and $g_3 = -4c_0/c_3$ are called invariants of Weierstrass elliptic function.

Case D. If $c_0 = c_1 = 0$, Eq. (8) admits:
a solitary wave solution

$$\varphi = \frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\varepsilon\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}, \quad c_2 > 0 \tag{26}$$

and a triangular type solution

$$\varphi = -\frac{c_2\sec^2\left(\frac{1}{2}\sqrt{-c_2}\xi\right)}{2\varepsilon\sqrt{-c_2c_4}\tan\left(\frac{1}{2}\sqrt{-c_2}\xi\right) + c_3}, \quad c_2 < 0. \tag{27}$$

In the case when $c_4 = 0$, the solution (26) and (27) degenerate to the solution (22) and (23) respectively. As $c_3 = 2\varepsilon\sqrt{c_2c_4}$, the solution (26) degenerates to the following solution

$$\varphi = \frac{1}{2}\varepsilon\sqrt{\frac{c_2}{c_4}}\left[1 + \tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)\right],$$

which is the same kind of solution with (13).

Remark. The other types of travelling wave solutions such as $\csc\xi$, $\cot\xi$, $\operatorname{csch}\xi$, and $\coth\xi$ can also be obtained by considering the different values of c_j $(j = 0, 1, \ldots, 4)$ in Eq. (8). These solutions appear in pairs of the functions $\sec\xi$, $\tan\xi$, $\operatorname{sech}\xi$ and $\tanh\xi$. In this paper we also omit all singular solutions which consist of $\csc\xi$, $\cot\xi$, $\operatorname{csch}\xi$, and $\coth\xi$ and rational solution (24).

Now we consider the K–G–S equations (1). Let $X = (x, y, z)$ and rewrite the Eq. (1) as

$$\begin{aligned}&\mathrm{i}\Psi_t + \Psi_{xx} + \Psi_{yy} + \Psi_{zz} + \rho\Psi\Phi = 0,\\ &\Phi_{tt} - (\Phi_{xx} + \Phi_{yy} + \Phi_{zz}) + \mu^2\Phi + \rho|\Psi|^2 = 0.\end{aligned} \tag{28}$$

In order to obtain the travelling wave solutions of system (28), we make the transformation

$$\Psi = U(\xi)\exp(\mathrm{i}\eta), \quad \Phi = V(\xi),$$

where $\xi = x + cy + dz + et$, $\eta = px + qy + rz + st$ and reduce system (16) to the following system of ODEs:

$$\begin{aligned}&(1 + c^2 + d^2)U'' - (s + p^2 + q^2 + r^2)U + \rho UV = 0,\\ &(e^2 - 1 - c^2 - d^2)V'' + \mu^2 V - \rho U^2 = 0,\\ &e + p + qc + dr = 0.\end{aligned} \tag{29}$$

Suppose that

$$U = \sum_{i=0}^{n_1} a_i\varphi^i, \quad V = \sum_{i=0}^{n_2} b_i\varphi^i$$

and φ satisfies (4). Balancing the highest linear term with the nonlinear terms in (29) gives

$$n_2 - 2 + m = 2n_1, \quad n_1 - 2 + m = n_1 + n_2.$$

By choosing $m = 4$, $n_1 = n_2 = 2$ we have

$$U = a_0 + a_1\varphi + a_2\varphi^2, \quad V = b_0 + b_1\varphi + b_2\varphi^2. \tag{30}$$

Substituting (30) into (29) leads to the following system of algebraic equations

$$\begin{aligned}&-2\rho a_0^2 + 2\mu^2 b_0 + 4\varepsilon^2(e^2 - 1 - c^2 - d^2)b_2c_0 + \varepsilon^2(e^2 - 1 - c^2 - d^2)b_1c_1 = 0,\\ &-4\rho a_0a_1 + 2\mu^2 b_1 + 6\varepsilon^2(e^2 - 1 - c^2 - d^2)b_2c_1 + 8\varepsilon^2(e^2 - 1 - c^2 - d^2)b_2c_2 = 0,\\ &-2\rho a_1^2 - 4\rho a_0a_2 + 2\mu^2 b_2 + 8\varepsilon^2(e^2 - 1 - c^2 - d^2)b_2c_2 + 3\varepsilon^2(e^2 - 1 - c^2 - d^2)b_1c_3 = 0,\\ &-4\rho a_1a_2 + 10\varepsilon^2(e^2 - 1 - c^2 - d^2)b_2c_3 + 4\varepsilon^2(e^2 - 1 - c^2 - d^2)b_1c_4 = 0,\\ &-2\rho a_2^2 + 12\varepsilon^2(e^2 - 1 - c^2 - d^2)b_2c_4 = 0,\\ &-2(s + p^2 + q^2 + r^2)a_0 + 2\rho a_0b_0 + 4\varepsilon^2(1 - c^2 - d^2)(a_2c_0 + a_1c_1) = 0,\\ &-2(s + p^2 + q^2 + r^2)a_1 + 2\rho(a_1b_0 + a_0b_1) + 6\varepsilon^2(1 - c^2 - d^2)(a_2c_1 + a_1c_2) = 0,\\ &-2(s + p^2 + q^2 + r^2)a_2 + 2\rho(a_2b_0 + a_1b_1 + a_0b_2) + 8\varepsilon^2(1 - c^2 - d^2)(a_2c_2 + a_1c_3) = 0,\\ &2\rho(a_2b_1 + a_1b_2) + \varepsilon^2(1 - c^2 - d^2)(10a_2c_3 + 4a_1c_4) = 0,\\ &2\rho a_2b_2 + 12\varepsilon^2(1 - c^2 - d^2)a_2c_4 = 0.\end{aligned}$$

Since $\varepsilon = \pm 1$ implies that $\varepsilon^2 = \varepsilon$, we may eliminate the variable ε from the above system. With the aid of *Mathematica*, we obtain two kinds of solutions, namely,

$$\begin{aligned}&c_1 = c_3 = a_1 = b_1 = 0, \quad a_0 = \frac{\mu^2 b_2}{\rho a_2},\\ &b_0 = \frac{\mu^2 b_2^2}{\rho a_2^2}, \quad c_2 = \frac{\mu^2(b_2^2 - a_2^2)}{4a_2^2e^2}, \quad c_4 = \frac{\rho(a_2^2 - b_2^2)}{6e^2 b_2},\end{aligned} \tag{31}$$

where a_2, b_2, c, d, e, p, q, r and s being constants satisfying

$$c^2 + d^2 = \frac{a_2^2 + b_2^2(e^2 - 1)}{b_2^2 - a_2^2}, \quad s + p^2 + q^2 + r^2 = \frac{\mu^2 b_2^2}{a_2^2}, \quad e + p + qc + rd = 0 \tag{32}$$

and

$$\begin{aligned} &c_2 = c_4 = a_2 = b_2 = 0, \quad a_0 = \frac{\mu^2 b_1}{2\rho a_1}, \\ &b_0 = \frac{\mu^2 b_1^2}{2\rho a_1^2}, \quad c_1 = \frac{\mu^4 b_1 (b_1^2 - a_1^2)}{4\rho e^2 a_1^4}, \quad c_3 = \frac{2\rho(a_1^2 - b_1^2)}{3e^2 b_1}, \end{aligned} \tag{33}$$

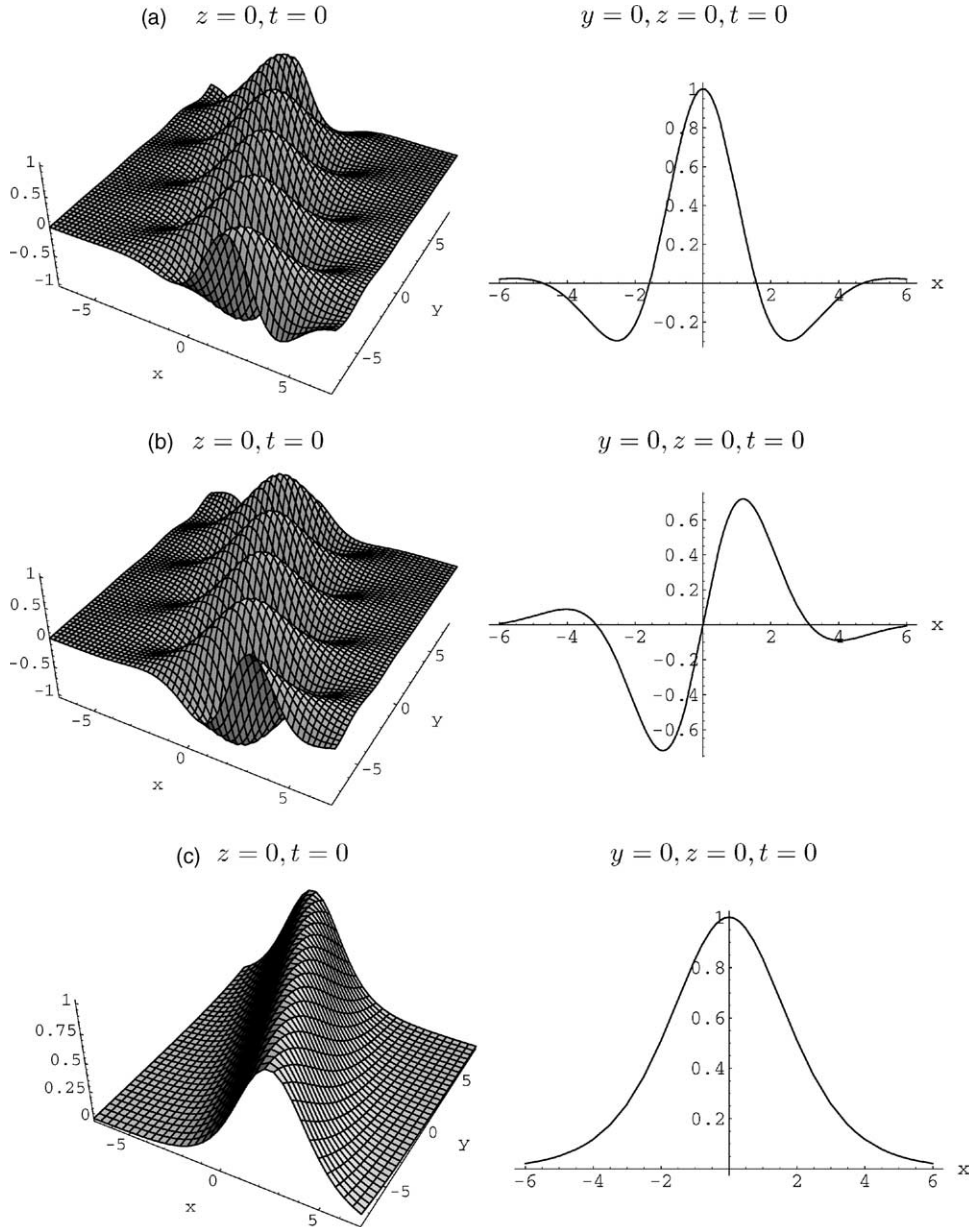

Fig. 1. The solitary wave solution $\Psi_1(x, y, z, t)$, where parameters satisfy (32). (a) The real part; (b) the imaginary part; (c) the modulus.

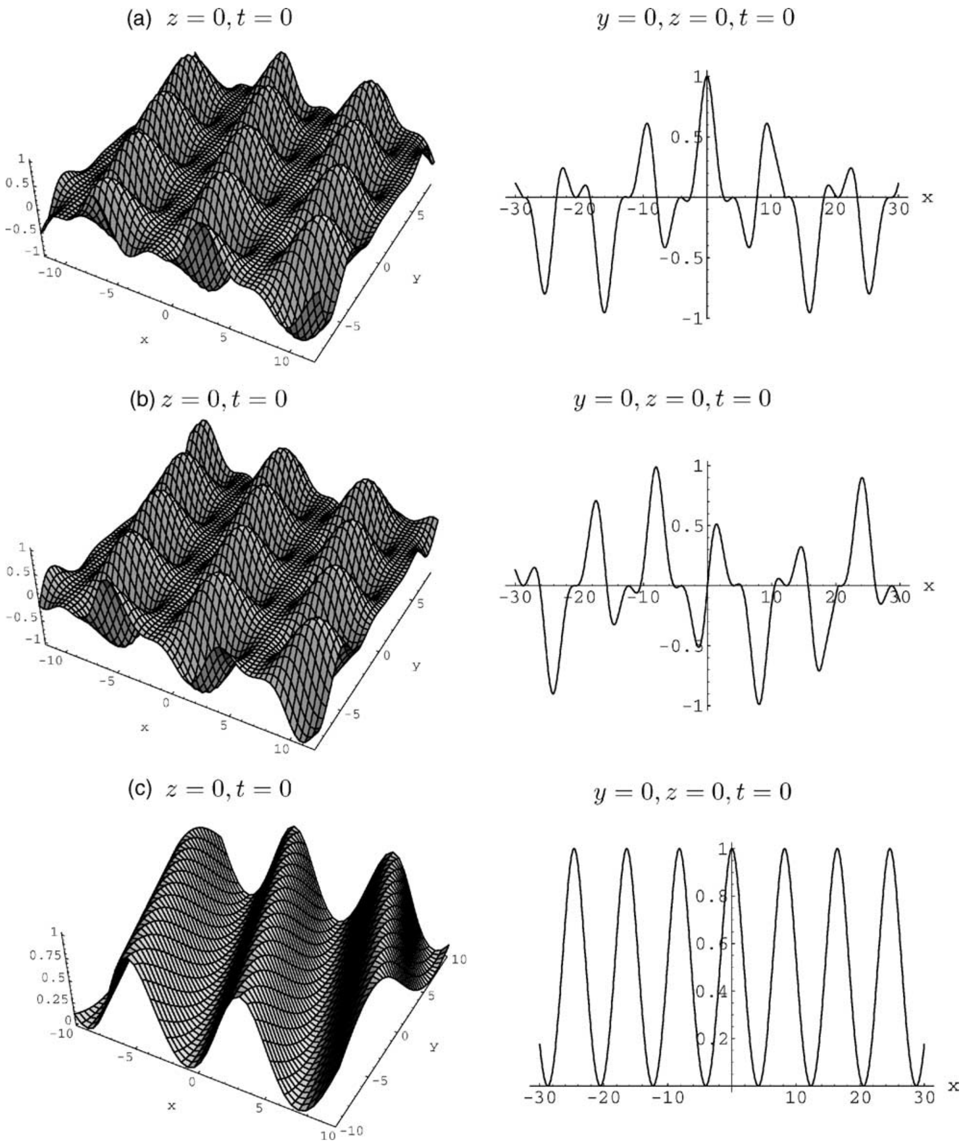

Fig. 2. The Jacobi doubly periodic solution $\Psi_2(x, y, z, t)$, where parameters satisfy (32). (a) The real part; (b) the imaginary part; (c) the modulus.

where a_1, b_1, c_0, c, d, e, p, q, r and s being constants satisfying

$$c^2 + d^2 = \frac{a_1^2 + b_1^2(e^2 - 1)}{b_1^2 - a_1^2}, \quad s + p^2 + q^2 + r^2 = \frac{\mu^2 b_1^2}{a_1^2}, \quad e + p + qc + rd = 0. \tag{34}$$

By using (14), (19) and (31), we obtain two kinds of travelling wave solutions, namely, a bell-shaped solitary wave solution

$$\begin{aligned} \Psi_1 &= \frac{\mu^2 b_2}{\rho a_2} \exp(\mathrm{i}\eta) \left\{ 1 - \frac{3}{2} \mathrm{sech}^2 \left(\frac{\mu}{2ea_2} \sqrt{b_2^2 - a_2^2} \xi \right) \right\}, \\ \Phi_1 &= \frac{\mu^2 b_2^2}{\rho a_2^2} \left\{ 1 - \frac{3}{2} \mathrm{sech}^2 \left(\frac{\mu}{2ea_2} \sqrt{b_2^2 - a_2^2} \xi \right) \right\}, \end{aligned} \tag{35}$$

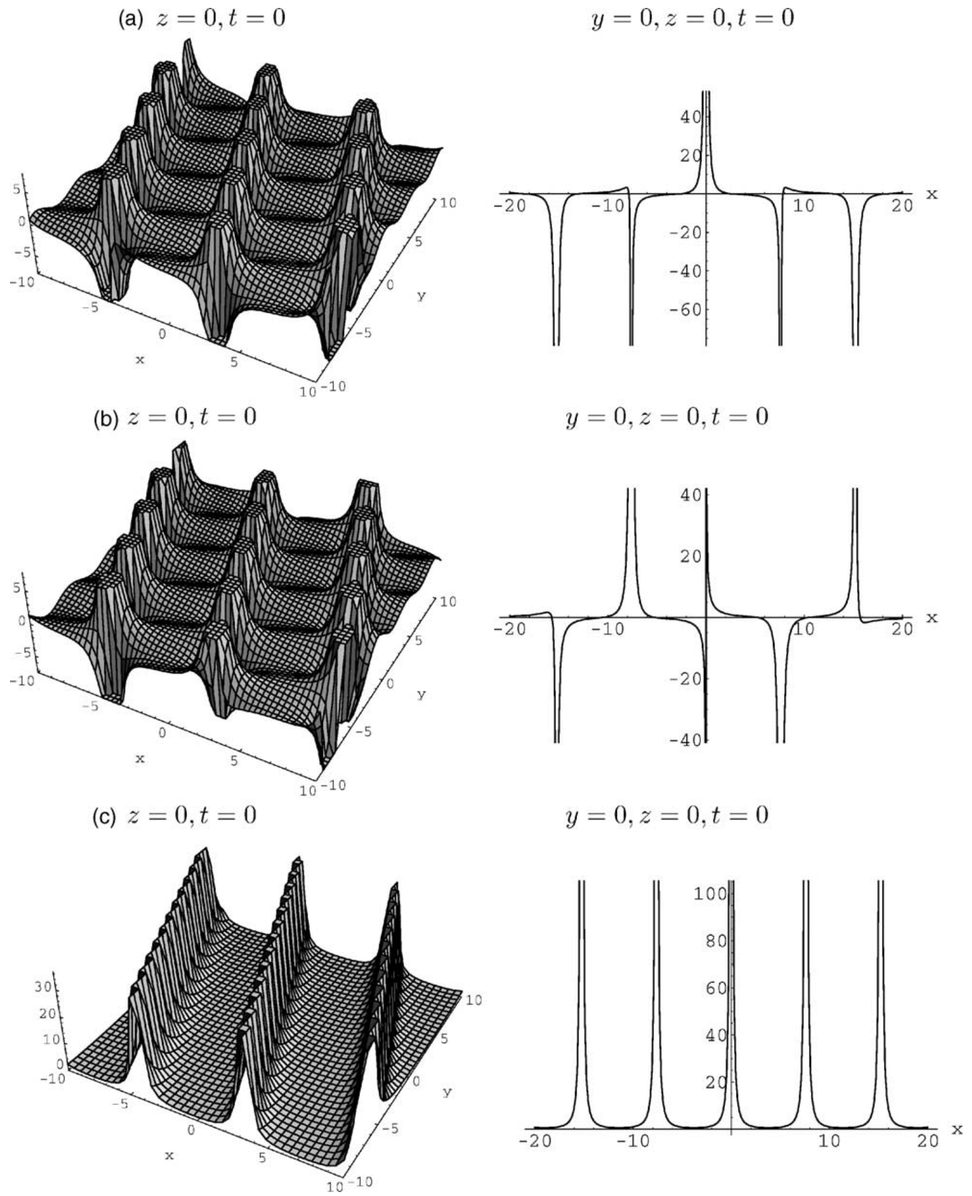

Fig. 3. The Weierstrass doubly periodic solution $\Psi_3(x, y, z, t)$, where parameters satisfy (34). (a) The real part; (b) the imaginary part; (c) the modulus.

two Jacobi doubly periodic wave solutions

$$\begin{aligned}
\Psi_2 &= \frac{\mu^2 b_2}{\rho a_2}\exp(i\eta)\left\{1 - \frac{3k^2}{2(2k^2-1)}\mathrm{cn}^2\left(\frac{\mu}{2ea_2}\sqrt{\frac{b_2^2-a_2^2}{2k^2-1}}\xi\right)\right\},\\
\Phi_2 &= \frac{\mu^2 b_2^2}{\rho a_2^2}\left\{1 - \frac{3m^2}{2(2k^2-1)}\mathrm{cn}^2\left(\frac{\mu}{2ea_2}\sqrt{\frac{b_2^2-a_2^2}{2k^2-1}}\xi\right)\right\},
\end{aligned} \tag{36}$$

where $\xi = x + cy + dz + et$, $\eta = px + qy + rz + st$, among them all parameters satisfy (31) and (32). As $k \to 1$, the Jacobi doubly periodic wave solutions (36) degenerates to the solitary wave solution (35).

In the case of (33), by using (25), we obtain a Weierstrass elliptic doubly periodic wave solution

$$\Psi_3 = \exp(\mathrm{i}\eta)\left\{\frac{\mu^2 b_1}{2\rho a_1} + a_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)\right\},$$
$$\Phi_3 = \frac{\mu^2 b_1^2}{2\rho a_1^2} + b_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right),$$

where $\xi = x + cy + dz + et$, $\eta = px + qy + rz + st$, $g_2 = -4c_1/c_3$ and $g_3 = -4c_0/c_3$, among them all parameters satisfy (33) and (34). Since Φ_1, Φ_2 and Φ_3 have similar properties with the modulus of Ψ_1, Ψ_2 and Ψ_3, we only draw some figures for the solutions Ψ_1, Ψ_2 and Ψ_3 (see Figs. 1–3).

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