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An efficient return mapping algorithm for general isotropic elastoplasticity in principal space

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ABSTRACT

This paper develops an efficient return mapping algorithm for implicit integration of general isotropic elastoplastic constitutive equations. A set of three mutually orthogonal unit base tensors in conjunction with a new set of three invariants are employed for the representation of arbitrary isotropic tensor valued and scalar valued functions of the stress tensor involved. The base tensors are constructed from the stress tensor by using the representation theorem and the three invariants are defined by the projection of the stress onto the base tensors. Geometrically, the base tensors are characterized by three mutually orthogonal unit vector and the three invariants are regarded as the components of a vector in principal space. With them, both the elastic constitutive equations and the flow rule of plasticity are represented as simple relationship among vectors in principal space. The return mapping algorithm associated with the representation is formulated in principal space and dimensions of the problem are reduced from six down to three. The explicit computation of the principal directions and the coordinate transformation from the principal reference frame to the global reference frame are omitted. The expressions for the consistent tangent operator for the proposed algorithm are derived in an efficient and closed-form manner. It consists of two parts: one is consistent with the return mapping in the fixed principal stress directions and another reflects the changes in the principal stress directions. Finally, a numerical example demonstrates the performances of the proposed implementation.

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1. Introduction

It is of crucial importance to integrate the constitutive equations for stress update in numerical analysis of elastoplastic boundary value problem. Many engineering materials, such as concrete, soils and rocks, exhibit the complex mechanical behaviors which prompt the use of the elastoplastic model whose yield function and the plastic potential function depend on all the three stress invariants. In this case, the constitutive equations can not be integrated analytically to obtain a closed form. The solution to the problem is to use the numerical integration technique, see e.g. Refs. [1–4]. The integration algorithm predominantly affects the overall numerical accuracy, stability and efficiency of an elastoplastic analysis since the numerical integration needs to be carried out at all yielded integration points for each equilibrium iteration. For this reason, the integration algorithm has been the subject of numerous papers in computational elastoplasticity for the last decades.

The return mapping method, originally proposed by Wilkins [5], seems to have been the most popular in recent years. This integration scheme has been investigated by Krieg and Krieg [6]. An overview of this integration algorithm can be found in the book by Crisfield [7]. Among the return mapping methods the fully implicit backward Euler difference scheme is the predominant, due to its properties of stability and accuracy for large strain increment, see e.g. Refs. [8,9]. The solution of the constitutive problems by this procedure will require an iterative process. For each iteration, at all yielded integration points, one needs to evaluate the inversion of a positive definite fourth-order tensor given by the sum of the elasticity tensor and the second derivatives of the plastic potential function. Clearly, the inversion of the fourth order tensor constitutes a main task to improve the efficiency of the solution procedure.

For isotropic elastoplastic material, the fourth-order tensor is the isotropic tensor valued function of the stress and has the same index symmetry as the elasticity tensor. Palazzo and his colleagues [1] exploit an approach based on those important features. By employing the representation theorem, they establish the general expression of the fourth-order tensor and its inversion with respect to such base tensors as the second-order identity tensor, the stress deviator and its square. The unknown coefficients in the expression are computed based on the tensor operation. It is shown that only a linear system of order three needs to be solved. This enables the dimensions of the problem to be reduced from six down to three. However, the tensor operations involving the inversion are

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complicated and lengthy because of the non-orthogonality of the base tensors. Rosati and Valoroso [10] improved previous work by introducing the spectral decomposition of the stress tensor and carrying out the return mapping in principal stress space. In contrast with other approaches in principal stress space, see, e.g. Refs. [11–13], they presented a fully tensorial description starting from the derivatives of the eigenvalues and eigenprojectors of a symmetric order-two tensor with respect to the tensor itself. The principal space representation and inversion of the fourth order tensor obtained as the derivative of scalar isotropic functions of a symmetric tensor argument are established by using the dyadic and square tensor products between the eigenprojectors of this symmetric tensor. Since the eigenprojectors are mutually orthogonal, the derivation and results involved in it are compact. All the tensors involved in the return mapping and the material tangent operators are directly expressed in the global co-ordinate system. The usual procedure of expressing the updated stress and the constitutive matrix in the principal reference frame and transforming back to the global reference frame can be omitted. But the numerical determination of the principal directions to obtain the eigenprojectors is required as usual. Tamagnini et al. [14] and Borja et al. [3,4] proposed an algorithm based on the spectral decomposition of stress and strain for isotropic three invariants plasticity model. The return mapping is performed in principal elastic strain space. The consistent tangent operators are derived for the proposed algorithm in an efficient, closed form manner. Only the inversion of a 3×3 matrix is required.

In this paper, similar to the mentioned above approaches, we develop a return mapping algorithm in three dimensional space. This work is based on the general representation of the constitutive equations for isotropic elastoplastic material given by our earlier work [15,16]. The remarking feature is that we employ a set of three mutually orthogonal unit base tensors and a new set of the invariants of the stress. The base tensors are constructed from the stress by making use of the representation theorem in conjunction with the simple tensor operations. They are expressed in the global coordinate system and can be characterized geometrically by three mutually orthogonal unit vectors in three dimensional principal space. The three invariants are defined by the projection components of the stress onto the base tensors. Then the return mapping algorithm associated with the proposed representation of the constitutive equations is formulated in principal space. But in contrast with usual approaches in principal space, all tensorial quantities entering the return mapping and the expression of the consistent tangent tensor can be directly expressed in terms of the proposed base tensors and the invariants. The explicit computation of the principal axes and the coordinate transformation from the principal reference frame to the global reference frame can be avoided. In addition, the matrix which needs to be inversed often takes the simple form. Therefore, the algorithm is made more efficient.

This paper is organized as follows. Section 2 addresses the establishment of three mutually orthogonal unit base tensors and the definition of a new set of the invariants of the stress. Their geometrical interpretations are given in principal space. In Section 3, the general invariant representation of the constitutive equations is discussed for isotropic elastoplastic material, based on the proposed base tensors and three invariants. Section 4 describes the fully implicit backward Euler algorithm associated with the representation. In Section 5, the relevant expressions for the consistent tangent operators for the proposed algorithm are provided in closed form. Finally, in Section 6, a numerical example demonstrates the performances of the proposed implementation.

We shall use boldface letters to indicate vectors and the second order tensors and symbols such as $\mathbb{A}$, $\mathbb{C}$, $\mathbb{I}$ to denote fourth-order tensors, where $\mathbb{I}$ is the fourth-order identity tensor. Other notations are list below: $\boldsymbol{I}$ denotes the second-order identity. tr implies the trace operator. For two tensors $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{ST}$ represents the dot product of tensors defined as $(\boldsymbol{ST})_{ij} = S_{ik}T_{kj}$ The symbol "$\otimes$" denotes tensor product of tensors, for example, $(\boldsymbol{S} \otimes \boldsymbol{T})_{ijkl} = S_{ij}T_{kl}$. Similarly, the symbol ":" denotes the contraction of the innermost two indices of two tensors, for example, $(\boldsymbol{S} : \boldsymbol{T})_{ij} = S_{ij}T_{ij}$, or $(\mathbb{C} : \boldsymbol{S})_{ij} = \mathbb{C}_{ijkl}\boldsymbol{S}_{kl}$. A square product $\boldsymbol{D} \boxtimes \boldsymbol{E}$ between tensors is defined by

$$(\boldsymbol{D} \boxtimes \boldsymbol{E}) : \boldsymbol{A} = \boldsymbol{DAE}, \quad \text{or} \quad (\boldsymbol{D} \boxtimes \boldsymbol{E})_{ijkl} = D_{ik}E_{jl} \tag{1}$$

Three invariants of a second order tensor such as the stress $\boldsymbol{\sigma}$ are denoted by

$$p = \frac{1}{\sqrt{3}}\mathrm{tr}\boldsymbol{\sigma}, \quad q = \sqrt{\mathrm{tr}\boldsymbol{S}^2}, \quad \theta = \frac{1}{3}\sin^{-1}\left[-\frac{\sqrt{6}\mathrm{tr}\boldsymbol{S}^3}{(\mathrm{tr}\boldsymbol{S}^2)^{3/2}}\right] \tag{2}$$

where $\boldsymbol{S}$ is the deviatoric stress and θ is called the Lode angle.

2. A set of mutually orthogonal unit base tensors

In the description of the constitutive equations for isotropic elastoplastic material, we have to deal with the isotropic second order tensor valued functions of the stress (or strain) tensor. According to the representation theorem [17], it can be expressed as linear combination of the second identity tensor $\boldsymbol{I}$, the stress $\boldsymbol{\sigma}$ and its square $\boldsymbol{\sigma}^2$. The three tensors are not mutually orthogonal. Chen [15] obtained a set of mutually orthogonal base tensors by constructing a tensor $\boldsymbol{t}$ which is an isotropic tensor-valued function of the stress and orthogonal to the stress $\boldsymbol{\sigma}$ and the identity tensor $\boldsymbol{I}$, namely, $\mathrm{tr}(\boldsymbol{\sigma t}) = 0$, $\mathrm{tr}(\boldsymbol{t}) = 0$. For convenience, let $\boldsymbol{t}$ to be a unit tensor, $\mathrm{tr}(\boldsymbol{t}^2) = 1$. To satisfy all the requirements, tensor $\boldsymbol{t}$ is shown to take the form

$$\boldsymbol{t} = \frac{1}{\cos 3\theta}\left(\sqrt{2}\boldsymbol{i} - \sin 3\theta \boldsymbol{s} - \sqrt{6}\boldsymbol{s}^2\right) \tag{3}$$

where $\boldsymbol{i}$ and $\boldsymbol{s}$ are the normalization of $\boldsymbol{I}$ and the deviatoric stress $\boldsymbol{S}$ which are given respectively by

$$\boldsymbol{i} = \frac{\boldsymbol{I}}{\sqrt{\mathrm{tr}\boldsymbol{I}^2}} = \frac{\boldsymbol{I}}{\sqrt{3}}, \quad \boldsymbol{s} = \frac{\boldsymbol{S}}{\sqrt{\mathrm{tr}\boldsymbol{S}^2}} = \frac{\boldsymbol{S}}{q} \tag{4}$$

By definition, we have that $\mathrm{tr}(\boldsymbol{i}^2) = \mathrm{tr}(\boldsymbol{s}^2) = \mathrm{tr}(\boldsymbol{t}^2) = 1$, $\mathrm{tr}(\boldsymbol{is}) = \mathrm{tr}(\boldsymbol{st}) = \mathrm{tr}(\boldsymbol{ti}) = 0$. Therefore, $\boldsymbol{i}$, $\boldsymbol{s}$, $\boldsymbol{t}$ form a set of mutually orthogonal unit base tensors.

The partial derivative of the three invariants of the stress with respect to the stress itself can be obtained by some tensor operations in the form

$$\frac{\partial p}{\partial \boldsymbol{\sigma}} = \frac{\boldsymbol{I}}{\sqrt{3}} = \boldsymbol{i}, \quad \frac{\partial q}{\partial \boldsymbol{\sigma}} = \frac{1}{q}\boldsymbol{S} = \boldsymbol{s}, \quad q\frac{\partial \theta}{\partial \boldsymbol{\sigma}} = \boldsymbol{t} \tag{5}$$

It is easily shown that three base tensors $\boldsymbol{i}$, $\boldsymbol{s}$, $\boldsymbol{t}$ are coaxial. Clearly, arbitrary isotropic tensor valued functions expressed as their linear combinations are also coaxial with them. Tensor algebra operations among coaxial tensors of order two, such as the addition, subtraction and scalar product, can be performed as they are vectors whose three components are the representation coefficients of the tensors with respective to the base tensors. If two tensors are orthogonal, the corresponding vectors are also orthogonal.

Denote the three principal axes of the stress by $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$. Three eigenvalue bases are defined by

$$\boldsymbol{a}_i = \boldsymbol{n}_i \otimes \boldsymbol{n}_i \quad (i = 1, 2, 3, \text{no sum over} \quad i). \tag{6}$$

Three coordinate axes are associated with $\boldsymbol{a}_i$ ($i = 1,2,3$) respectively to establish the so called principal space. Then, coaxial tensors can conveniently be characterized by vectors in this principal space. Geometrically, $\boldsymbol{i}$ is along the axis which subtends equal angles with the three coordinate axes, $\boldsymbol{s}$ and $\boldsymbol{t}$ are in the deviatoric plane or π plane and mutually orthogonal, as shown in Fig. 1(a).

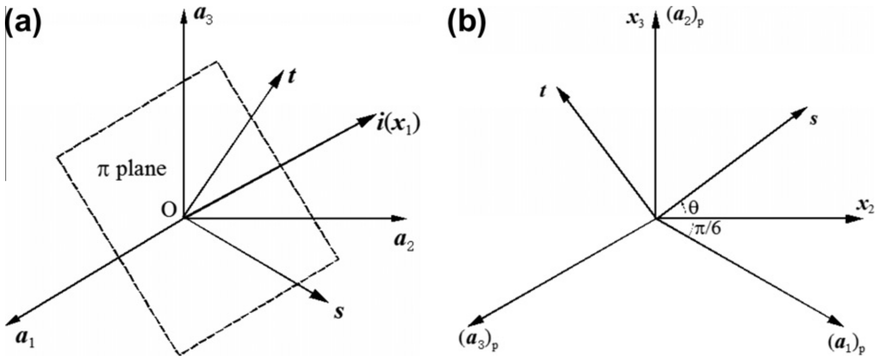

Fig. 1. (a) Principal space and a set of the base tensors $\boldsymbol{i}$, $\boldsymbol{s}$, $\boldsymbol{t}$. (b) Geometrical relations between two sets of the bases $\boldsymbol{s}$, $\boldsymbol{t}$ and $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ in π plane.

Solving the characteristic equation of the normalized stress derivator $\boldsymbol{s}$, ones obtain its three principal values in terms of the Lode angle. Then we can express $\boldsymbol{s}$ in the spectral form

$$\boldsymbol{s} = \sqrt{\frac{2}{3}}\sin\left(\theta + \frac{2\pi}{3}\right)\boldsymbol{a}_1 + \sqrt{\frac{2}{3}}\sin\theta\boldsymbol{a}_2 + \sqrt{\frac{2}{3}}\sin\left(\theta - \frac{2\pi}{3}\right)\boldsymbol{a}_3 \tag{7a}$$

Inserting (7a) into (3), $\boldsymbol{t}$ is also expressed in the spectral form

$$\boldsymbol{t} = \sqrt{\frac{2}{3}}\cos\left(\theta + \frac{2\pi}{3}\right)\boldsymbol{a}_1 + \sqrt{\frac{2}{3}}\cos\theta\boldsymbol{a}_2 + \sqrt{\frac{2}{3}}\cos\left(\theta - \frac{2\pi}{3}\right)\boldsymbol{a}_3 \tag{7b}$$

Comparing (7b) with (7a), one easily obtains (7b) by the substitution of $\theta + \pi/2$ for θ in (7a). This shows that the Lode angle of $\boldsymbol{t}$ is $\theta + \pi/2$. In view of (7), the base tensors $\boldsymbol{s}$ and $\boldsymbol{t}$ depend on both the principal values (the Lode angle) and principal axes of the stress. When the base tensors are employed in the return mapping algorithm, it is disadvantageous to some degree since the Lode angle changes with stress update.

Consider two deviatoric tensors $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ which are coaxial with $\boldsymbol{s}$ and $\boldsymbol{t}$, and have the fixed Lode angle of 0 and $\pi/2$ respectively. Upon the substitution of 0 and $\pi/2$ for θ in (7a) respectively, the two tensors can be written as

$$x_2 = \frac{\sqrt{2}}{2}(\boldsymbol{a}_1 - \boldsymbol{a}_3), \quad x_3 = \frac{1}{\sqrt{6}}(-\boldsymbol{a}_1 + 2\boldsymbol{a}_2 - \boldsymbol{a}_3) \tag{8}$$

Obviously, the two defined tensors depend only on the principal axes. Denote $\boldsymbol{x}_1 = \boldsymbol{i}$, as shown in Fig. 1(a). After a simple operation, one has

$$\mathrm{tr}\boldsymbol{x}_1^2 = \mathrm{tr}\boldsymbol{x}_2^2 = \mathrm{tr}\boldsymbol{x}_3^2 = 1, \quad \mathrm{tr}\boldsymbol{x}_1\boldsymbol{x}_2 = \mathrm{tr}\boldsymbol{x}_2\boldsymbol{x}_3 = \mathrm{tr}\boldsymbol{x}_3\boldsymbol{x}_1 = 0 \tag{9}$$

It follows that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are mutually orthogonal unit tensors. They serve as a new set of the base tensors, with which we work in the sequel.

Since $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are deviatoric tensors, they can be expressed as linear combination of $\boldsymbol{s}$ and $\boldsymbol{t}$. Using (7) and (8), one obtains $\mathrm{tr}(\boldsymbol{s}\boldsymbol{x}_2) = \mathrm{tr}(\boldsymbol{t}\boldsymbol{x}_3) = \cos\theta$, $\mathrm{tr}(s\boldsymbol{x}_3) = -\mathrm{tr}(\boldsymbol{t}\boldsymbol{x}_2) = \sin\theta$. Therefore, the relations between two sets of the base tensors are

$$\boldsymbol{x}_2 = \boldsymbol{s}\cos\theta - \boldsymbol{t}\sin\theta, \quad \boldsymbol{x}_3 = \boldsymbol{s}\sin\theta + \boldsymbol{t}\cos\theta \tag{10}$$

Denote the projection axes of the coordinate axes $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ onto π plane by $(\boldsymbol{a}_2)_\mathrm{p}$ and $(\boldsymbol{a}_3)_\mathrm{p}$ respectively. Using (8), (10) and the definition of π plane, it can be shown that the vector axis associated with $\boldsymbol{x}_3$ coincides with $(\boldsymbol{a}_3)_\mathrm{p}$ and the vector axis associated with $\boldsymbol{x}_2$ is perpendicular to it in π plane. Moreover, the angle between $\boldsymbol{s}$ and $\boldsymbol{x}_2$ is the Lode angle, which is measured anti-clockwise from the positive $\boldsymbol{x}_2$-axis, as depicted in Fig. 1(b).

Eq. (10) clearly shows that the transformation between $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{s}$, $\boldsymbol{t}$ follows the transformation rule of vectors, in other words, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are obtained from $\boldsymbol{s}$ and $\boldsymbol{t}$ by clockwise rotation by an angle θ about the axis $\boldsymbol{i}$, as depicted in Fig. 1(b). With this expression, we can evaluate $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ directly from the normalized stress deviator $\boldsymbol{s}$, instead of the principal axes. This is advantageous in the return mapping algorithm since the explicit computation of the principal axes will be avoided.

Using (10) to express $\boldsymbol{s}$ in terms of $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, and recalling that the decomposition of the stress $\boldsymbol{\sigma} = p\boldsymbol{i} + q\boldsymbol{s}$, one can express the stress referred to this new set of the base tensors

$$\boldsymbol{\sigma} = \sum_{I=1}^{3}\hat{\sigma}_I\boldsymbol{x}_I \tag{11a}$$

where

$$\hat{\sigma}_1 = p, \quad \hat{\sigma}_2 = q\cos\theta, \quad \hat{\sigma}_3 = q\sin\theta \tag{11b}$$

Those three coefficients constitute a new set of the invariants of the stress, which geometrically indicate the projection of the stress onto the axes $\boldsymbol{x}_I$ $(I = 1,2,3)$ respectively.

Using the chain rule and (5), the derivatives of the last two invariants in (11b) with respect to stress are obtained as

$$\frac{\partial\hat{\sigma}_2}{\partial\boldsymbol{\sigma}} = \cos\theta\frac{\partial q}{\partial\boldsymbol{\sigma}} - \sin\theta\left(q\frac{\partial\theta}{\partial\boldsymbol{\sigma}}\right) = \boldsymbol{x}_2 \tag{12a}$$

$$\frac{\partial\hat{\sigma}_3}{\partial\boldsymbol{\sigma}} = \sin\theta\frac{\partial q}{\partial\boldsymbol{\sigma}} + \cos\theta\left(q\frac{\partial\theta}{\partial\boldsymbol{\sigma}}\right) = \boldsymbol{x}_3 \tag{12b}$$

Combing the above equation and the first equation in (5), one has the simple expression

$$\frac{\partial\hat{\sigma}_I}{\partial\boldsymbol{\sigma}} = \boldsymbol{x}_I \quad (I = 1,2,3) \tag{13}$$

In the sequel, we need to deal with the derivatives of the three base tensors x_I $(I = 1,2,3)$ with respect to the stress tensor for deriving the fourth order tangent operator. Using (10) and (5), one readily obtains

$$\frac{\partial x_2}{\partial\boldsymbol{\sigma}} = \cos\theta\frac{\partial\boldsymbol{s}}{\partial\boldsymbol{\sigma}} - \frac{\sin\theta}{q}\boldsymbol{s}\otimes\boldsymbol{t} - \sin\theta\frac{\partial t}{\partial\boldsymbol{\sigma}} - \frac{\cos\theta}{q}\boldsymbol{t}\otimes\boldsymbol{t} \tag{14a}$$

$$\frac{\partial x_3}{\partial\boldsymbol{\sigma}} = \sin\theta\frac{\partial\boldsymbol{s}}{\partial\boldsymbol{\sigma}} + \frac{\cos\theta}{q}\boldsymbol{s}\otimes\boldsymbol{t} + \cos\theta\frac{\partial\boldsymbol{t}}{\partial\boldsymbol{\sigma}} - \frac{\sin\theta}{q}\boldsymbol{t}\otimes\boldsymbol{t} \tag{14b}$$

After the lengthy tensor operation, we obtain the following derivative [16]

$$q\frac{\partial\boldsymbol{s}}{\partial\boldsymbol{\sigma}} = -\boldsymbol{s}\otimes\boldsymbol{s} + (\mathbb{I} - \boldsymbol{i}\otimes\boldsymbol{i}) \tag{15a}$$

$$\begin{aligned} q\frac{\partial\boldsymbol{t}}{\partial\boldsymbol{\sigma}} = {} & \tan 3\theta\boldsymbol{i}\otimes\boldsymbol{i} + \frac{2\sqrt{2}}{\cos\theta 3\theta}(\boldsymbol{i}\otimes\boldsymbol{s} + \boldsymbol{s}\otimes\boldsymbol{i} - \tan 3\theta\boldsymbol{s}\otimes\boldsymbol{s} \\ & - (3\boldsymbol{s}\otimes\boldsymbol{t} + 2\boldsymbol{t}\otimes\boldsymbol{s}) + 3\tan 3\theta\boldsymbol{t}\otimes\boldsymbol{t} - \tan 3\theta\mathbb{I} \\ & - \frac{3\sqrt{2}}{\cos 3\theta}(\boldsymbol{i}\boxtimes\boldsymbol{s} + \boldsymbol{s}\boxtimes\boldsymbol{i}) \end{aligned} \tag{15b}$$

Inserting the above results into (14), after rearrangement, we have

$$
\begin{aligned}
q\cos 3\theta \frac{\partial \boldsymbol{x}_2}{\partial \boldsymbol{\sigma}} = & -\cos 2\theta \boldsymbol{i} \otimes \boldsymbol{i} - 2\sqrt{2}\sin\theta(\boldsymbol{i}\otimes\boldsymbol{s} + \boldsymbol{s}\otimes\boldsymbol{i}) \\
& - \cos 4\theta \boldsymbol{s}\otimes\boldsymbol{s} + 2\sin\theta\cos 3\theta(\boldsymbol{s}\otimes\boldsymbol{t} + \boldsymbol{t}\otimes\boldsymbol{s}) \\
& - (2\sin\theta\sin 3\theta + \cos 2\theta)\boldsymbol{t}\otimes\boldsymbol{t} + \cos 2\theta\mathbb{I} \\
& + 3\sqrt{2}\sin\theta(\boldsymbol{i}\boxtimes\boldsymbol{s} + \boldsymbol{s}\boxtimes\boldsymbol{i})
\end{aligned} \tag{16a}
$$

$$
\begin{aligned}
q\cos 3\theta \frac{\partial \boldsymbol{x}_2}{\partial \boldsymbol{\sigma}} = & \sin 2\theta \boldsymbol{i} \otimes \boldsymbol{i} + 2\sqrt{2}\cos\theta(\boldsymbol{i}\otimes\boldsymbol{s} + \boldsymbol{s}\otimes\boldsymbol{i}) \\
& - \sin 4\theta \boldsymbol{s}\otimes\boldsymbol{s} - 2\cos\theta\cos 3\theta(\boldsymbol{s}\otimes\boldsymbol{t} + \boldsymbol{t}\otimes\boldsymbol{s}) \\
& + (2\cos\theta\sin 3\theta + \sin 2\theta)\boldsymbol{t}\otimes\boldsymbol{t} - \sin 2\theta\mathbb{I} \\
& - 3\sqrt{2}\cos\theta(\boldsymbol{i}\boxtimes\boldsymbol{s} + \boldsymbol{s}\boxtimes\boldsymbol{i})
\end{aligned} \tag{16b}
$$

By using (10), the above derivatives are expressed in terms of the base tensors $\boldsymbol{x}_I$ ($I = 1,2,3$)

$$\frac{\partial \mathbf{x}_2}{\partial \boldsymbol{\sigma}} = \sum_{I=1}^{3}\sum_{J=1}^{3} B_{IJ}^{1}(\boldsymbol{\sigma})\mathbf{x}_I \otimes \mathbf{x}_J + \mathbb{F}^1(\boldsymbol{\sigma}) \tag{17a}$$

$$\frac{\partial \mathbf{x}_3}{\partial \boldsymbol{\sigma}} = \sum_{I=1}^{3}\sum_{J=1}^{3} B_{IJ}^{2}(\boldsymbol{\sigma})\mathbf{x}_I \otimes \mathbf{x}_J + \mathbb{F}^2(\boldsymbol{\sigma}) \tag{17b}$$

where

$$B_{IJ}^{1}(\boldsymbol{\sigma}) = \frac{1}{q\cos 3\theta}\begin{bmatrix} -\cos 2\theta & -\sqrt{2}\sin 2\theta & -2\sqrt{2}\sin^2\theta \\ & 1-2\cos 2\theta & \sin 2\theta \\ \text{sym} & & -1 \end{bmatrix} \tag{18a}$$

$$B_{IJ}^{2}(\boldsymbol{\sigma}) = \frac{1}{q\cos 3\theta}\begin{bmatrix} \sin 2\theta & 2\sqrt{2}\cos^2\theta & \sqrt{2}\sin 2\theta \\ & 0 & -2\cos^2\theta \\ \text{sym} & & 2\sin 2\theta \end{bmatrix} \tag{18b}$$

are symmetrical matrix. Two fourth order tensors in (17) are respectively

$$
\begin{aligned}
\mathbb{F}^1(\boldsymbol{\sigma}) = \frac{1}{q\cos 3\theta}\Big[& 3\sqrt{2}\sin\theta\cos\theta(\boldsymbol{x}_1\boxtimes\boldsymbol{x}_2 + \boldsymbol{x}_2\boxtimes\boldsymbol{x}_1) \\
& + 3\sqrt{2}\sin^2\theta(\boldsymbol{x}_1\boxtimes\boldsymbol{x}_3 + \boldsymbol{x}_3\boxtimes\boldsymbol{x}_1) + \cos 2\theta\mathbb{I}\Big]
\end{aligned} \tag{18c}
$$

$$
\begin{aligned}
\mathbb{F}^2(\boldsymbol{\sigma}) = \frac{1}{q\cos 3\theta}\Big[& -3\sqrt{2}\cos^2\theta(\boldsymbol{x}_1\boxtimes\boldsymbol{x}_2 + \boldsymbol{x}_2\boxtimes\boldsymbol{x}_1) \\
& - 3\sqrt{2}\sin\theta\cos\theta(\boldsymbol{x}_1\boxtimes\boldsymbol{x}_3 + \boldsymbol{x}_3\boxtimes\boldsymbol{x}_1) - \sin 2\theta\mathbb{I}\Big]
\end{aligned} \tag{18d}
$$

The description given above is referred to the stress tensor. When referred to other tensors such as the elastic strain tensor, it is straightforward to write the corresponding expressions with the substitution of the elastic strain tensor for the stress tensor.

In using the above analytical expression (3) and (18), it is assumed that the eigenvalues of the stress are all distinct and nonzero. If the stress admits a double eigenvalue (including two zero eigenvalues), then θ is equal to $-\pi/6$ or $\pi/6$. It appears that (3) and (18) might become singular because the denominator $\cos 3\theta$ goes to zero as $\theta \to \pm\pi/6$. This case does not represent a real problem from the computational standpoint. The singularity can be eliminated by resorting to a simple perturbation technique similar to that proposed in Refs. [18–20]. That is, a small perturbation is given to arbitrary one of the stress components so that θ deviates slightly from $-\pi/6$ or $\pi/6$. If the stress admits a triple eigenvalues (including three zero eigenvalues), (18) might become also singular because the invariant q vanishes. As shown at the end of Section 5, the derivative of the base tensors will be unnecessary for deriving the fourth order tangent operator.

3. General representation of the isotropic constitutive equations

The constitutive equations are formulated within the framework of small deformation elastoplasticity. The material behavior is assumed to be isotropic. The strain tensor is additively decomposed into the elastic and plastic parts

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}} \tag{19}$$

Due to isotropy, the stress response of the material is characterized by an isotropic tensor valued function of the elastic strain tensor. By the representation theorem, the principal directions of the stress response tensor and the elastic strain tensor coincide. Here is introduced the order preserving hypothesis as described by Laine et al. [21]: "the eigenvalues of stress and strain tensors are classified in the same order: the eigenvector associated with the highest eigenvalue of the stress tensor is also associated with the highest eigenvalue of the strain tensor, etc". Then the base tensors associated with the stress are identical to that associated with the elastic strain in principal space, that is

$$\boldsymbol{x}_I(\boldsymbol{\sigma}) = \boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\mathrm{e}}) \quad (\boldsymbol{I} = 1,2,3) \tag{20}$$

Inserting (20) into (11a), we obtain the general expressions of the elastic constitutive equations

$$\boldsymbol{\sigma} = \sum_{I=1}^{3} \hat{\sigma}_I \boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\mathrm{e}}) \tag{21}$$

where $\hat{\sigma}_I$ $(I = 1,2,3)$ is the scalar valued function of the elastic strain tensor.

For the isotropic hyperelasticity, we assume the existence of a stored strain energy function such that

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} \tag{22}$$

where the function W depends on the three invariants of the elastic strain tensor. By analogy with (11b), those three invariants are given as

$$\hat{\varepsilon}_1^e = p^{\varepsilon^e}, \quad \hat{\varepsilon}_2^e = q^{\varepsilon^e}\cos\theta^{\varepsilon^e}, \quad \hat{\varepsilon}_3^e = q^{\varepsilon^e}\sin\theta^{\varepsilon^e} \tag{23a}$$

where p^{ε^e}, q^{ε^e} and θ^{ε^e} are defined by (2) with the substitution of $\boldsymbol{\varepsilon}^{\mathrm{e}}$ for $\boldsymbol{\sigma}$. Then the function W can be expressed as

$$W = W(\hat{\varepsilon}_1^{\mathrm{e}}, \hat{\varepsilon}_2^{\mathrm{e}}, \hat{\varepsilon}_3^{\mathrm{e}}) \tag{23b}$$

Using (23b) and the chain rule, Eq. (22) is rewritten as

$$\boldsymbol{\sigma} = \sum_{I=1}^{3} \frac{\partial W}{\partial \hat{\varepsilon}_I^{\mathrm{e}}} \frac{\partial \hat{\varepsilon}_I^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} \tag{24}$$

By analogy with (13), one obtain the following expression

$$\frac{\partial \hat{\varepsilon}_I^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = \boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\mathrm{e}}) \quad (I = 1,2,3) \tag{25}$$

Inserting (25) into (24) and comparing (24) with (21), we have the elastic constitutive equation in the vector form

$$\hat{\sigma}_I = \frac{\partial W}{\partial \hat{\varepsilon}_I^{\mathrm{e}}} \quad (I = 1,2,3) \quad \text{or} \quad \hat{\boldsymbol{\sigma}} = \frac{\partial W}{\partial \hat{\boldsymbol{\varepsilon}}^{\mathrm{e}}} \tag{26}$$

where the following vector notations are introduced

$$\hat{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \begin{Bmatrix} \hat{\varepsilon}_1^e \\ \hat{\varepsilon}_2^e \\ \hat{\varepsilon}_3^e \end{Bmatrix}, \quad \hat{\boldsymbol{\sigma}} = \begin{Bmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{Bmatrix} \tag{27}$$

The above equation indicates that the vector corresponding to the stress tensor is expressed as the gradient of the strain energy function W with respect to the vector corresponding to the strain tensor in principal space. Laine et al. [21] gave a general expression of the elastic constitutive equations similar to (26) and (21), where the base tensors are expressed only in terms of the principal axes.

The strain energy function is required to be convex for the stability. Then, the order preserving hypothesis is automatically satisfied [22].

To derive the elastic constitutive equations in the increment form, we differentiate both sides of (24), with the help of (25) and (26)

$$d\boldsymbol{\sigma} = \mathbb{C}^e : d\boldsymbol{\varepsilon}^e \tag{28}$$

where

$$\mathbb{C} = \sum_{I=1}^{3} \boldsymbol{x}(\boldsymbol{\varepsilon}^e) \otimes \frac{\partial}{\partial \boldsymbol{\varepsilon}^e}\left(\frac{\partial W}{\partial \hat{\varepsilon}_I^e}\right) + \sum_{I=1}^{3} \hat{\sigma}_I \frac{\partial \boldsymbol{x}_I(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} \tag{29}$$

is the fourth order tangent elasticity tensor. With the help of (25), the partial derivative in the first term is written as

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}^e}\left(\frac{\partial W}{\partial \hat{\varepsilon}_I^e}\right) = \frac{\partial^2 W}{\partial \hat{\varepsilon}_I^e \partial \hat{\varepsilon}_J^e}\frac{\partial \hat{\varepsilon}_J^e}{\partial \boldsymbol{\varepsilon}^e} = \sum_{J=1}^{3} C_{IJ}^e \boldsymbol{x}_J(\boldsymbol{\varepsilon}^e) \quad (I = 1,2,3) \tag{30}$$

where

$$C_{IJ}^e = \frac{\partial^2 W}{\partial \hat{\varepsilon}_I^e \partial \hat{\varepsilon}_J^e} = \frac{\partial \hat{\sigma}_I}{\partial \hat{\varepsilon}_J^e} \quad (I = 1,2,3\ J = 1,2,3) \tag{31}$$

is the elastic tangent moduli in principal space. It reflects the change of the stress with respect to the elastic strain when the principal axes are held fixed. The derivative of the base tensors with respect to the elastic strain in the second term of the right side of (29) can be obtained by substituting the elastic strain tensor for the stress tensor in (17) and (18).

In view of (29) and (30) as well as (17), we obtain the elastic tangent operator in the form

$$\begin{aligned}\mathbb{C}^e =& \sum_{I=1}^{3}\sum_{J=1}^{3}\left[C_{IJ}^e + \hat{\sigma}_2 \boldsymbol{B}_{IJ}^1(\boldsymbol{\varepsilon}^e) + \hat{\sigma}_3 \boldsymbol{B}_{IJ}^2(\boldsymbol{\varepsilon}^e)\right]\boldsymbol{x}_I(\boldsymbol{\varepsilon}^e) \otimes \boldsymbol{x}_J(\boldsymbol{\varepsilon}^e) \\ &+ \hat{\sigma}_2 \mathbb{F}^1(\boldsymbol{\varepsilon}^e) + \hat{\boldsymbol{\sigma}}_3 \mathbb{F}^2(\boldsymbol{\varepsilon}^e)\end{aligned} \tag{32}$$

For the plastic response, it is assumed that there exists a potential function

$$G = G(\boldsymbol{\sigma}, \xi) \tag{33}$$

where $\xi = \xi_\alpha$ ($\alpha = 1,2,\ldots,n$) is an array of internal variables to characterize the past history of the plastic deformation. The plastic strain increment $d\boldsymbol{\varepsilon}^p$ is assumed to be orthogonal to the potential, that is, parallel to the gradient direction of the potential,

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial G}{\partial \boldsymbol{\sigma}} \tag{34}$$

where $d\lambda$ is plastic multiplier.

For the isotropic hardening material, assuming the internal variables are a set of scalars, the potential is the function of the invariant of the stress and the internal variables,

$$G = G(\hat{\boldsymbol{\sigma}}, \xi) \tag{35}$$

Inserting (35) into (34), using the chain rule of the partial derivatives and considering (13) and (20), we obtain the result

$$d\boldsymbol{\varepsilon}^p = d\lambda \sum_{I=1}^{3} \frac{\partial G}{\partial \hat{\sigma}_I} \boldsymbol{x}_I(\boldsymbol{\varepsilon}^e) \tag{36}$$

The above equation shows that the plastic strain increment and the stress are also coaxial. In accordance with the properties of coaxial tensors as described above, the plastic strain increment takes the general form

$$d\boldsymbol{\varepsilon}^p = \sum_{I=1}^{3} d\hat{\varepsilon}_I^p \boldsymbol{x}_I(\boldsymbol{\varepsilon}^e) \tag{37}$$

A direct comparison of (37) and (36) yields

$$d\hat{\varepsilon}_I^p = d\lambda \frac{\partial G}{\partial \hat{\sigma}_I} (I = 1,2,3) \quad \text{or } d\hat{\boldsymbol{\varepsilon}}^p = d\lambda \frac{\partial G}{\partial \hat{\boldsymbol{\sigma}}} \tag{38}$$

Obviously, the constitutive equations are revealed to be a simple relationship between two vectors $d\hat{\boldsymbol{\varepsilon}}^p$ and $\hat{\boldsymbol{\sigma}}$ in principal space.

In the phenomenological isotropic hardening model, the evolution of the scalar internal variables is related to three invariants of the plastic strain increment and/or plastic work increment. Due to coaxiality of the plastic strain increment and stress, plastic work increment can be expressed in terms of the invariants of both. Therefore as an approximation, we can assume that the evolution of the internal variables is linear combination of the invariants of the plastic strain increment with coefficients given by the invariants of the stress. In view of (38) and (35), the internal variable increment can be expressed as

$$d\xi = d\lambda \boldsymbol{h}(\hat{\sigma}, \xi) \tag{39}$$

For isotropic case, the yield function takes the general form

$$F(\boldsymbol{\sigma}, \xi) = F(\hat{\boldsymbol{\sigma}}, \xi) = 0 \tag{40}$$

The plastic multiplier $d\lambda$ is determined with the aid of the loading/unloading criterion. This can be expressed in Kuhn–Tucker form as follows

$$F(\hat{\boldsymbol{\sigma}}, \xi) \le 0, \quad d\lambda \ge 0, \quad d\boldsymbol{F}(\hat{\boldsymbol{\sigma}}, \xi) d\lambda = 0 \tag{41}$$

4. Return mapping algorithm for implicit integration

Given a strain increment $\Delta\varepsilon$ from the global current finite element solution and the value of the stress and the internal variables at time t_n, the goal of the constitutive integration is to solve for the values of these variables at time t_{n+1}. For the elastoplastic material the constitutive equations are non-linear and the solution to the problem is to use an approximate numerical technique. The fully implicit backward Euler scheme is found to be numerically stable for larger strain increment and also is considerably simple to implement. Therefore, this scheme is used to integrate the constitutive equations.

We start with the definition of the trial elastic strain by assuming that the increment step is elastic

$$\boldsymbol{\varepsilon}_{n+1}^{tr} = \boldsymbol{\varepsilon}_n^e + \Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n + \boldsymbol{\varepsilon}_n^e \tag{42}$$

The subscript $n + 1$ denotes the evaluation of quantities at time t_{n+1}. The trial elastic stress associated with the trial elastic strain is obtained by the constitutive equation

$$\boldsymbol{\sigma}_{n+1}^{tr} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}^e}(\boldsymbol{\varepsilon}_{n+1}^{tr}) \tag{43}$$

If $F(\boldsymbol{\sigma}_{n+1}^{tr}, \xi_n) \leqslant 0$, the stress lies inside the yield locus, the trial state represents the actual final state of the material.

If $F(\boldsymbol{\sigma}_{n+1}^{tr}, \xi_n) > 0$, the trial state lies outside the yield locus and the consistency condition is violated. The process is then declared plastic. The objective of returning mapping is to map the trial state back to the yield surface so that the consistency condition is restored. Such a task is performed by the plastic corrector. Using the fully implicit integration scheme, we have

$$\boldsymbol{\varepsilon}_{n+1}^e = \boldsymbol{\varepsilon}_{n+1}^{tr} - \Delta\boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}_{n+1}^{tr} - \Delta\lambda_{n+1}(G_{,\boldsymbol{\sigma}})_{n+1} \tag{44a}$$

$$\xi_{n+1} = \xi_n + \Delta\xi = \xi_n + \Delta\lambda_{n+1}\boldsymbol{h}_{n+1} \tag{44b}$$

The computed value of the stress and the internal variables at the end of time t_{n+1} must satisfy the consistency condition

$$F(\boldsymbol{\sigma}_{n+1}, \xi_{n+1}) = 0 \tag{45}$$

Eq. (44) together with (45) constitutes a system of nonlinear equations, which is iteratively solved by a Newton–Raphson method. To improve the efficiency of the solution procedures and save computation time, we make use of the representation of the constitutive relationship with respect to three mutually orthogonal unit base tensors $\boldsymbol{x}_I$ $(I = 1,2,3)$ as described in Section 2. For isotropic hardening models, the trial elastic strain $\boldsymbol{\varepsilon}^{\rm tr}_{n+1}$, the plastic correction $\Delta\boldsymbol{\varepsilon}^{\rm p}_{n+1}$, the converged elastic strain $\boldsymbol{\varepsilon}^{\rm e}_{n+1}$ and the stress $\boldsymbol{\sigma}_{n+1}$ at time t_{n+1} have the same principal directions. Recall that those base tensors depend only on the principal directions. Therefore, they remain unchanged during iteration. If we compute the base tensors $x_I(\boldsymbol{\varepsilon}^{\rm tr}_{n+1})(I = 1, 2, 3)$ associated with the trial elastic strain by using (3) and (10) with the substitution of the elastic trial strain for the stress, then all the second order tensors involved in the algorithm, such as the elastic strain, the plastic strain increment and the stress, can be considered as vectors of three components which are the projection of the corresponding tensors on the three base tensors respectively. Therefore, there are only three unknowns, the projected components or the representation coefficients, needed to be iterated upon. This number is half of the six unknowns needed to determine the stress tensor using traditional algorithms. By reducing the number of equations by three, this algorithm is made more efficient.

Introduce the following partial derivative notation, for convenience

$$G_{,\hat{\boldsymbol{\sigma}}} = \frac{\partial G}{\partial \hat{\boldsymbol{\sigma}}} \tag{46}$$

Once the base tensors $x_I(\boldsymbol{\varepsilon}^{\rm tr}_{n+1})$ $(I = 1, 2, 3)$ are known, we readily obtain the three projected components $\hat{\boldsymbol{\varepsilon}}^{\rm tr}_{n+1}$ of the trial elastic strain onto them. Then, Eq. (44a) can conveniently be expressed in vector form as

$$\hat{\boldsymbol{\varepsilon}}^{\rm e}_{n+1} = \hat{\boldsymbol{\varepsilon}}^{\rm tr}_{n+1} - \Delta\lambda_{n+1}(G_{,\hat{\boldsymbol{\sigma}}})_{n+1} \tag{47}$$

The elastic constitutive equations between the vectors corresponding to the stress and elastic strain is given by (26), or in the increment form

$$d\hat{\boldsymbol{\sigma}}_{n+1} = \boldsymbol{C}^{\rm e}_{n+1} d\hat{\boldsymbol{\varepsilon}}^{\rm e}_{n+1} \tag{48}$$

where $\boldsymbol{C}^{\rm e}_{n+1}$ is a 3×3 elastic tangent matrix given by (31).

The above nonlinear system is rewritten in the residual form as

$$\boldsymbol{R}_{\hat{\varepsilon}} = \hat{\boldsymbol{\varepsilon}}^{\rm e}_{n+1} - \hat{\boldsymbol{\varepsilon}}^{\rm tr}_{n+1} + \Delta\lambda_{n+1}(G_{,\hat{\boldsymbol{\sigma}}})_{n+1} \tag{49a}$$

$$\boldsymbol{R}_{\xi} = -\xi_{n+1} + \xi_n + \Delta\lambda_{n+1}\boldsymbol{h}_{n+1} \tag{49b}$$

$$F = F(\hat{\boldsymbol{\sigma}}_{n+1}, \xi_{n+1}) \tag{49c}$$

Linearization of (49) around the ith estimate of the solution yields

$$\boldsymbol{R}^i_{\hat{\varepsilon}} + (\boldsymbol{C}^e)^{-1}\delta\hat{\boldsymbol{\sigma}}^i + \Delta\lambda^i \delta G^i_{,\hat{\boldsymbol{\sigma}}} + \delta\lambda^i G^i_{,\hat{\boldsymbol{\sigma}}} = 0 \tag{50a}$$

$$\boldsymbol{R}^i_{\xi} - \delta\xi^i + \Delta\lambda^i \delta\boldsymbol{h}^i + \delta\lambda^i \boldsymbol{h}^i = 0 \tag{50b}$$

$$F^i + F^i_{,\hat{\boldsymbol{\sigma}}}\delta\hat{\boldsymbol{\sigma}}^i + F^i_{,\xi}\delta\xi^i = 0 \tag{50c}$$

where the superscript i is the iteration number and (48) is used. For simplicity, the subscript $n+1$ is omitted. It is noted that the trial elastic strain is constant during iteration and does not contribute. With the help of (35) and (39), we have

$$\delta G^i_{,\hat{\boldsymbol{\sigma}}} = G^i_{,\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\sigma}}}\delta\hat{\boldsymbol{\sigma}}^i + G^i_{,\hat{\boldsymbol{\sigma}}\xi}\delta\xi^i \tag{51a}$$

$$\delta\boldsymbol{h}^i = \boldsymbol{h}^i_{,\hat{\boldsymbol{\sigma}}}\delta\hat{\boldsymbol{\sigma}}^i + \boldsymbol{h}^i_{,\xi}\delta\xi^i \tag{51b}$$

Inserting (51) into (50), we have an equation in matrix form as

$$\begin{Bmatrix} \delta\hat{\boldsymbol{\sigma}}^i \\ \delta\xi^i \end{Bmatrix} = -(\boldsymbol{A}^i)^{-1}\boldsymbol{R}^i - \delta\lambda^i(\boldsymbol{A}^i)^{-1}\nabla G^i \tag{52}$$

where

$$\boldsymbol{A}^i = \begin{bmatrix} (\boldsymbol{C}^{\rm e})^{-1} + \Delta\lambda^i G^i_{,\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\sigma}}} & \Delta\lambda^i G^i_{,\hat{\boldsymbol{\sigma}}\xi} \\ \Delta\lambda^i \boldsymbol{h}^i_{,\hat{\boldsymbol{\sigma}}} & -\boldsymbol{I} + \Delta\lambda^i \boldsymbol{h}^i_{,\xi} \end{bmatrix} \tag{53}$$

$$\nabla G^i = \begin{Bmatrix} G^i_{,\hat{\boldsymbol{\sigma}}} \\ \boldsymbol{h}^i \end{Bmatrix}, \quad \boldsymbol{R}^i = \begin{Bmatrix} \boldsymbol{R}^i_{\hat{\varepsilon}} \\ \boldsymbol{R}^i_{\xi} \end{Bmatrix} \tag{54}$$

Combining (52) and (50c), we obtain

$$\delta\lambda^i = \frac{F^i - (\nabla F^i)^T(\boldsymbol{A}^i)^{-1}\boldsymbol{R}^i}{(\nabla F^i)^T(\boldsymbol{A}^i)^{-1}\nabla G^i} \tag{55}$$

where $\nabla F^i = \{F^i_{,\hat{\boldsymbol{\sigma}}} \quad F^i_{,\xi}\}^{\rm T}$ and "T" denotes the transpose of the matrix.

The equation system (52) is solved iteratively for the subincrements $\delta\hat{\boldsymbol{\sigma}}^i, \delta\xi^i$ and the corresponding variables are updated until the residuals vanish to be within the prescribed tolerances.

$$\hat{\boldsymbol{\varepsilon}}^{{\rm e},i+1} = \hat{\boldsymbol{\varepsilon}}^{{\rm e},i} + \delta\hat{\boldsymbol{\varepsilon}}^{{\rm e},i} = \hat{\boldsymbol{\varepsilon}}^{{\rm e},i} + (\boldsymbol{C}^{\rm e})^{-1}\delta\hat{\boldsymbol{\sigma}}^i \tag{56a}$$

$$\xi^{i+1} = \xi^i + \delta\xi^i \quad \Delta\lambda^{i+1} = \Delta\lambda^i + \delta\lambda^i \tag{56b}$$

$$\hat{\boldsymbol{\sigma}}^{i+1} = \hat{\boldsymbol{\sigma}}^i + \delta\hat{\boldsymbol{\sigma}}^i \tag{56c}$$

Table 1
Return mapping algorithm based on the base tensors $x_I(\boldsymbol{\varepsilon}^{\rm tr}_{n+1})$ $(I = 1, 2, 3)$.

1. Calculate the base tensors and the components of the vector corresponding to the elastic trial strain and stress

$$x_I(\boldsymbol{\varepsilon}^{\rm tr}_{n+1}), \quad \hat{\boldsymbol{\varepsilon}}^{\rm tr}_{n+1}, \quad \hat{\boldsymbol{\sigma}}^{\rm tr}_{n+1}$$

2. Initialize

$$i = 0, \quad \hat{\boldsymbol{\varepsilon}}^{{\rm e}\,0} = \hat{\boldsymbol{\varepsilon}}^{\rm tr}_{n+1}, \quad \xi^0 = \xi_n, \quad \Delta\lambda^0 = 0, \hat{\boldsymbol{\sigma}}^0 = \hat{\boldsymbol{\sigma}}^{\rm tr}_{n+1}$$

3. Check convergence

if $F^i < TOL_1$, $\|\boldsymbol{R}^i_{\hat{\varepsilon}}\| < TOL_2$ and $\|\boldsymbol{R}^i_{\xi}\| < TOL_3$, then
$\boldsymbol{\varepsilon}^{\rm e}_{n+1} = \sum_{I=1}^3 \hat{\varepsilon}^{e,i+1}_I \boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\rm tr}_{n+1})$, $\boldsymbol{\sigma}_{n+1} = \sum_{I=1}^3 \hat{\sigma}^{i+1}_I \boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\rm tr}_{n+1})$ and exit

4. Obtain the consistency parameter

$$\delta\lambda^i = \frac{F^i - (\nabla F^i)^T(\boldsymbol{A}^i)^{-1}\boldsymbol{R}^i}{(\nabla F^i)^T(\boldsymbol{A}^i)^{-1}\nabla G^i}$$

5. Update $\delta\hat{\boldsymbol{\sigma}}$ and $\delta\xi$

$$\begin{Bmatrix} \delta\hat{\boldsymbol{\sigma}}^i \\ \delta\xi^i \end{Bmatrix} = -(\boldsymbol{A}^i)^{-1}\boldsymbol{R}^i - \delta\lambda^i(\boldsymbol{A}^i)^{-1}\nabla G^i$$

6. Update solution

$$\hat{\boldsymbol{\varepsilon}}^{{\rm e},i+1} = \hat{\boldsymbol{\varepsilon}}^{{\rm e},i} + \delta\hat{\boldsymbol{\varepsilon}}^{{\rm e},i} = \hat{\boldsymbol{\varepsilon}}^{{\rm e},i} + (\boldsymbol{C}^{\rm e})^{-1}\delta\hat{\boldsymbol{\sigma}}^i$$

$$\xi^{i+1} = \xi^i + \delta\xi^i, \quad \Delta\lambda^{i+1} = \Delta\lambda^i + \delta\lambda^i$$

$$\hat{\boldsymbol{\sigma}}^{i+1} = \hat{\boldsymbol{\sigma}}^i + \delta\hat{\boldsymbol{\sigma}}^i$$

7. Increase iteration counter: $i = i + 1$ and go to 3

Then, the elastic strain and stress tensor can be easily obtained from

$$\boldsymbol{\varepsilon}^e_{n+1} = \sum_{I=1}^{3} \hat{\varepsilon}_I^{e,i+1} \boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \tag{57a}$$

$$\boldsymbol{\sigma}_{n+1} = \sum_{I=1}^{3} \hat{\sigma}_I^{i+1} x_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \tag{57b}$$

At the beginning of the iteration, we should make

$$\hat{\boldsymbol{\varepsilon}}^{e,0} = \hat{\boldsymbol{\varepsilon}}^{\text{tr}}_{n+1}, \quad \xi^0 = \xi_n, \quad \Delta\lambda^0 = 0, \quad \hat{\boldsymbol{\sigma}}^0 = \hat{\boldsymbol{\sigma}}^{\text{tr}}_{n+1} \tag{58}$$

It is often assumed that the potential function G and the functions $\boldsymbol{h}$ are independent of the internal variables ξ. Therefore, from (50) and (51), we have

$$\delta\hat{\boldsymbol{\sigma}}^{\text{i}} = -(\boldsymbol{A}^i)^{-1}\boldsymbol{R}^i_{\hat{\varepsilon}} - \delta\lambda^{\text{i}}(\boldsymbol{A}^{\text{i}})^{-1}\text{G}^{\text{i}}_{,\hat{\sigma}} \tag{59a}$$

$$\delta\xi^i = \boldsymbol{R}^i_{\xi} + \Delta\lambda^i \boldsymbol{h}^i_{,\hat{\sigma}}\delta\hat{\sigma}^i + \delta\lambda^i\boldsymbol{h}^i \tag{59b}$$

where

$$\boldsymbol{A}^i = (\boldsymbol{C}^e)^{-1} + \Delta\lambda^i G^i_{,\hat{\sigma}\hat{\sigma}} \tag{60}$$

is a 3×3 matrix. Combining (59) and (50c), we obtain

$$\delta\lambda^i = \frac{F^i - (F^i_{,\hat{\sigma}} + \Delta\lambda^i F^i_{,\xi}\boldsymbol{h}^i_{,\hat{\sigma}})(\boldsymbol{A}^i)^{-1}\boldsymbol{R}^i_{\hat{\varepsilon}} + \boldsymbol{F}^i_{,\xi}\boldsymbol{R}^i_{\xi}}{(\boldsymbol{F}^i_{,\hat{\sigma}} + \Delta\lambda^i \boldsymbol{F}^i_{,\xi}\boldsymbol{h}^i_{,\hat{\sigma}})(\boldsymbol{A}^i)^{-1}\boldsymbol{G}^i_{,\hat{\sigma}} - \boldsymbol{F}^i_{,\xi}\boldsymbol{h}^i} \tag{61}$$

The above describes a complete iterative solution process of the constitutive integration referred to the proposed base tensors. For the sake of clarity, a step-by-step description of the algorithm has been summarized in Table 1.

5. Consistent tangent operator

The continuum tangent operator which relates infinitesimal strain and stress increments can be easily derived from the constitutive equations described in Section 3. Its use is compatible with an exact integration of the constitutive equations, which are continuum in nature. However, the proposed constitutive integration algorithm does not represent an exact integration; it is finite difference based and in essence a secant approach. If the continuum tangent operator is used in the global Newton iterations the convergence will be slow, as the stress and strain increments are finite rather than infinitesimal. In order to preserve the quadratic rate of asymptotic convergence of the global Newton iterations, it is necessary to adopt the consistent tangent operator relevant to the proposed constitutive integration algorithms [23]. The consistent tangent operator is defined by

$$\mathbb{C}^{\text{ep}} = \frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} = \frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}} \tag{62}$$

where the last equality follows directly from the definition of trial elastic strain (42).

Consider that the stress and the elastic strain tensor are coaxial, see (20). The stress at time t_{n+1} is expressed as

$$\boldsymbol{\sigma}_{n+1} = \sum_{I=1}^{3}(\hat{\sigma}_I)_{n+1}\boldsymbol{x}_I(\boldsymbol{\sigma}_{n+1}) = \sum_{I=1}^{3}(\hat{\sigma}_I)_{n+1}\boldsymbol{x}_I(\boldsymbol{\sigma}^{\text{tr}}_{n+1}) = \sum_{I=1}^{3}(\hat{\sigma}_I)_{n+1}\boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \tag{63}$$

Differentiating the above equation with respect to the trial elastic strain, one obtains

$$\mathbb{C}^{\text{ep}} = \frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}} = \sum_{I=1}^{3}\boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \otimes \frac{\partial(\hat{\sigma}_I)_{n+1}}{\partial\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}} + \sum_{I+1}^{3}(\hat{\sigma}_I)_{n+1}\frac{\partial\boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1})}{\partial\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}} \tag{64}$$

The above expression is similar to (29) in form. Using the chain rule, one has

$$\frac{\partial(\hat{\sigma}_I)_{n+1}}{\partial\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}} = \sum_{J=1}^{3}\frac{\partial(\hat{\sigma}_I)_{n+1}}{\partial(\hat{\varepsilon}^{\text{tr}}_J)_{n+1}}\frac{\partial(\hat{\varepsilon}^{\text{tr}}_J)_{n+1}}{\partial\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}} = \sum_{J=1}^{3}C^{\text{ep}}_{IJ}\boldsymbol{x}_J(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \tag{65}$$

where the matrix

$$C^{\text{ep}}_{IJ} = \frac{\partial(\hat{\sigma}_I)_{n+1}}{\partial(\hat{\varepsilon}^{\text{tr}}_J)_{n+1}} \tag{66}$$

denotes the variation of the stress with respect to the trial elastic strain with their principal directions held fixed. In the derivation of (65), Eq. (25) is used with the substitution of the elastic trial strain for the elastic strain.

In view of (66), the consistent tangent operator (64) can be expressed as

$$\mathbb{C}^{\text{ep}} = \sum_{I=1}^{3}\sum_{J=1}^{3}C^{\text{ep}}_{IJ}\boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \otimes \boldsymbol{x}_J(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) + \sum_{I=1}^{3}(\hat{\sigma}_I)4_{n+1}\frac{\partial\boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1})}{\partial\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}} \tag{67}$$

The second term of the above equations results only from the rotation of the principal directions alone. Notice that the principal directions remain unchanged during the return mapping. Therefore, it depends not on the return mapping algorithm. In other words, it depends not on the specific plasticity model used. On the contrary, the first term of the above equation is a function of the constitutive response and the algorithm used to track this response. A little detailed derivation of the constitutive matrix $\boldsymbol{C}^{\text{ep}}$ in this term is given in the sequel.

At convergence in return mapping algorithm we have computed a stress state $\hat{\boldsymbol{\sigma}}^{i+1}$ which satisfied the yield conditions exactly. Then we insert a new increment $\text{d}\hat{\boldsymbol{\varepsilon}}^{\text{tr}}$ into (52). It will result in a non-zero value for $\boldsymbol{R}_{\hat{\varepsilon}}$, that is, $\boldsymbol{R}_{\hat{\varepsilon}} = \text{d}\hat{\boldsymbol{\varepsilon}}^{\text{tr}}$, but that $\boldsymbol{R}_{\xi}$ and F remain zero until subsequent iterations. As proposed by Zienkiewicz and Taylor [24], the required matrix $\boldsymbol{C}^{\text{ep}}$ is derived directly now by using (52) and (55) from

$$\begin{Bmatrix} \text{d}\hat{\boldsymbol{\sigma}} \\ \text{d}\xi \end{Bmatrix} = \begin{bmatrix} \boldsymbol{C}^{\text{ep}} & * \\ * & * \end{bmatrix}\begin{Bmatrix} \text{d}\hat{\boldsymbol{\varepsilon}}^{\text{tr}} \\ 0 \end{Bmatrix} \tag{68a}$$

where

$$\begin{bmatrix} \boldsymbol{C}^{\text{ep}} & * \\ * & * \end{bmatrix} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\nabla G(\nabla F)^T\boldsymbol{A}^{-1}}{(\nabla F)^T\boldsymbol{A}^{-1}\nabla G} \tag{68b}$$

If the potential function G and the function $\boldsymbol{h}$ are independent of the internal variable ξ, we have, from (59)–(61)

$$C^{\text{ep}} = (\boldsymbol{A}^i)^{-1} - \frac{(\boldsymbol{A}^i)^{-1}G^i_{,\hat{\sigma}}(F^i_{,\hat{\sigma}} + \Delta\lambda^i F^i_{,\xi}\boldsymbol{h}^i_{,\hat{\sigma}})(\boldsymbol{A}^i)^{-1}}{(\boldsymbol{F}^i_{,\hat{\sigma}} + \Delta\lambda^i\boldsymbol{F}^i_{,\xi}\boldsymbol{h}^i_{,\hat{\sigma}})(\boldsymbol{A}^i)^{-1}\boldsymbol{G}^i_{,\hat{\sigma}} - \boldsymbol{F}^i_{,\xi}\boldsymbol{h}^i} \tag{69}$$

where the superscript i is the convergent iteration number.

A direct comparison between (29) and (64) shows that the substitution of the matrix $\boldsymbol{C}^{\text{ep}}$ and the trail elastic strain for the matrix $\boldsymbol{C}^{\text{e}}$ and the elastic strain respectively in (32) will lead to the detailed expression of (67). Therefore, Eq. (67) is rewritten as

$$\mathbb{C}^{\text{ep}} = \sum_{I=1}^{3}\sum_{J=1}^{3}\left[C^{\text{ep}}_{IJ} + (\hat{\sigma}_2)_{n+1}B^1_{IJ}(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) + (\hat{\sigma}_3)_{n+1}B^2_{IJ}(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1})\right]\boldsymbol{x}_I(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \otimes \boldsymbol{x}_J(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) + (\hat{\sigma}_2)_{n+1}\mathbb{F}^1(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) + (\hat{\sigma}_3)_{n+1}\mathbb{F}^2(\boldsymbol{\varepsilon}^{\text{tr}}_{n+1}) \tag{70}$$

The singularity in (70) to which a double eigenvalue of the argument tensor might lead has been discussed at the end of Section 2. If the argument tensor, namely, the elastic trial strain, admits a triple eigenvalue, the symmetry requires that the stress must be a pressure, that is, a spherical tensor. The second term of the tangent operator must vanish and no singularity needs to be treated.

6. Numerical example

To demonstrate the validity and numerical efficiency of the proposed method, a numerical example is presented. We implement the proposed method in a FORTRAN subroutine UMAT provided by ABAQUS. For comparison purposes, we also implement the method formulated in the general stress space (six dimensional space) and method formulated in principal stresses (principal directions are needed) in the same manner. Each method is both implemented with a consistent tangent operator and a continuum tangent operator. The results and computation time of the numerical example using these methods are compared. Although the proposed method is discussed in tensor form, it is implemented in matrix form in the program as other methods.

The simulation refers to an isotropic model which is extensions of the original Drucker–Prager model [25]. The extensions developed by ABAQUS include the use of noncircular yield surfaces in the deviatoric stress plane. The model is defined through the following yield function

$$F = \sqrt{\frac{3}{2}}\frac{q}{2}\left[1+\frac{1}{\kappa}-\left(1-\frac{1}{\kappa}\right)\sin 3\theta\right]+\frac{1}{\sqrt{3}}p\tan\phi - d \tag{71}$$

and potential function

$$G = \sqrt{\frac{3}{2}}\frac{q}{2}\left[1+\frac{1}{\kappa}-\left(1-\frac{1}{\kappa}\right)\sin 3\theta\right]+\frac{1}{\sqrt{3}}p\tan\psi \tag{72}$$

The parameters included in (71) and (72) are defined as follows: κ is the ratio of the yield stress in triaxial tension to the yield stress in triaxial compression and, thus, controls the dependence of the yield surface on the value of the intermediate principal stress, see Fig. 2; ϕ and ψ are the friction angle and the dilation angle respectively; d is the cohesion defined by $d = (1-\frac{1}{3}\tan\phi)\sigma_c$, if hardening is defined by the uniaxial compression yield stress σ_c, $d = (\frac{1}{\kappa}+\frac{1}{3}\tan\phi)\sigma_t$, if hardening is defined by the uniaxial tension yield stress σ_t. The international variable that describes the hardening is chosen to be the equivalent plastic strain

With the help of (11b), one can rewrite the yield function and potential function in terms of the defined invariants as

$$F = \alpha\sqrt{\hat{\sigma}_2^2+\hat{\sigma}_3^2}+3\beta\hat{\sigma}_3-\frac{4\beta\hat{\sigma}_3^3}{\hat{\sigma}_2^2+\hat{\sigma}_3^2}+\gamma_F\hat{\sigma}_1-d \tag{73}$$

$$G = \alpha\sqrt{\hat{\sigma}_2^2+\hat{\sigma}_3^2}+3\beta\hat{\sigma}_3-\frac{4\beta\hat{\sigma}_3^3}{\hat{\sigma}_2^2+\hat{\sigma}_3^2}+\gamma_G\hat{\sigma}_1 \tag{74}$$

where

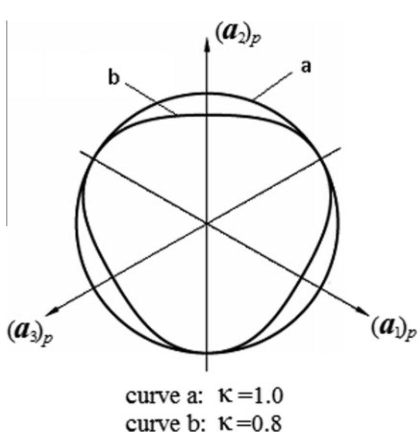

Fig. 2. Typical yield/flow surfaces in the deviatoric plane.

$$\alpha = \frac{\sqrt{6}}{4}\left(1+\frac{1}{\kappa}\right),\quad \beta = \frac{\sqrt{6}}{4}\left(1-\frac{1}{\kappa}\right),\quad \gamma_F = \frac{1}{\sqrt{3}}\tan\phi,\quad \gamma_G = \frac{1}{\sqrt{3}}\tan\psi \tag{75}$$

Using (74), we obtain the matrix needed in the constitutive iterations, see (60)

$$G_{,\hat{\sigma}\hat{\sigma}} = \frac{\alpha-8\beta\sin 3\theta}{q}\begin{bmatrix}0 & 0 & 0\\ 0 & \sin^2 & -\sin\theta\cos\theta\\ 0 & -\sin\theta\cos\theta & \cos^2\theta\end{bmatrix} \tag{76}$$

where all non-zero elements are included in a 2 × 2 submatrix. This case always occurs when there exists no coupled terms of the first invariant and other two invariants in the potential function.

Elastic response is assumed to be linear. With the help of (11b), the elastic strain energy function is

$$W = \frac{3}{2}K\hat{\varepsilon}_1^2+\mu(\hat{\varepsilon}_2^2+\hat{\varepsilon}_3^2) \tag{77}$$

where K is the bulk modulus and μ is the shear modulus. Using (31), the elastic tangent matrix takes the diagonal form

$$\boldsymbol{C}^e = \begin{bmatrix}3K & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & 2\mu\end{bmatrix} \tag{78}$$

Inserting (78) and (76) into (60), the matrix $\boldsymbol{A}$ takes the simple form. Obviously, only the inversion of a 2 × 2 submatrix is required during the constitutive iteration.

The example refers to a full three dimensional problem of a square strip with a circular hole, subjected to increasing extension in a direction perpendicular to the axis of the strip and a uniformly distributed load over one surface of the strip in the thickness direction, see Fig. 3. The analysis is performed using the three-dimensional finite element procedure as described above. Along the thickness direction, the strip is divided into six equal layers of elements. A boundary condition is applied by restraining the degree of freedom of the other surface of the strip in the thickness direction. Due to symmetry of geometry and load only one-quarter of the strip is modeled and the finite element mesh employed is shown in Fig. 4.

The elastic properties of the material are taken as E = 30 GPa and υ = 0.2. The hardening is defined by the tangent modulus in uniaxial compression $\frac{d\sigma_c}{d\varepsilon^p} = 2.5$ GPa, where the uniaxial compression initial yield stress is taken as σ_c = 20 MPa. We assume the associated flow laws, $\phi = \psi = 30°$. Other parameters are assumed to be κ = 0.78. The uniform displacement is applied in increments on two lateral sides. The first two time step size is 0.28%. The ratio of next time step size to last one is chosen to be 1.3. The applied total displacement δ is 0.4 m and the uniformly distributed load q is 30 MPa.

In order to demonstrate the validity, the main computational results corresponding to the ultimate load are compared between the proposed method and other existing methods at both global and local level. At global level, the contours of maximum principal stress and equivalent plastic strain are compared, as shown in Figs. 5 and 6. "GS" denotes the method formulated in the general stress space, "PS" denotes the method formulated in principal stresses and "BT" denotes the proposed method. At local level, we choose eight elements (see Fig. 4) and compare the results of their first integration point, as shown in Table 2.

The contours of maximum principal stress and equivalent plastic strain are almost the same between the proposed method and other existing methods. The illustrations on the integration point level show identical results using the three different methods. These show evidently the validity of the proposed method.

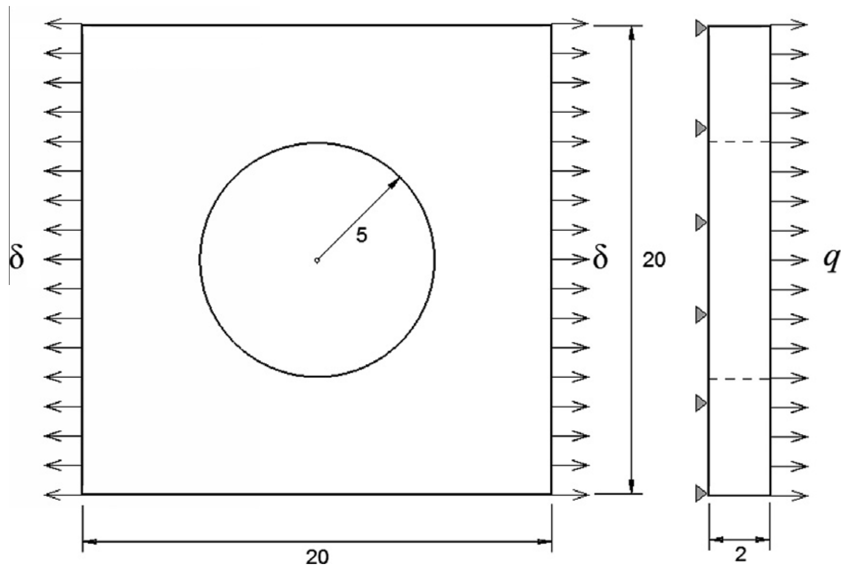

Fig. 3. Perforated strip subjected to increasing extension.

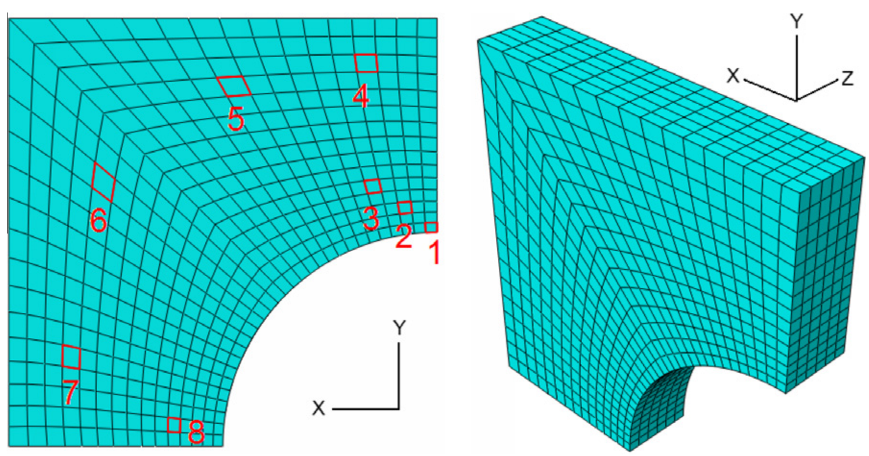

Fig. 4. Finite element mesh.

In order to demonstrate the validity of the consistent tangent operator using the proposed method, we compare the iteration numbers required to attain convergence between the proposed method and other existing methods. The iteration numbers with consistent tangent operators and that with continuum tangent operators are also compared. These are summarized in Table 3.

Table 3 shows that the iteration numbers required to attain convergence using the three different methods are exactly the same, either with consistent tangent operators or continuum tangent operators. It also shows that the iteration numbers with consistent tangent operators are considerably less than the ones with continuum tangent operators. Since the rate of asymptotic convergence of the GS or PS method is quadratic, the quadratic rate of asymptotic convergence of the proposed method is thus preserved.

In order to demonstrate the numerical efficiency, we compare the computation time for the proposed method and other existing methods. The results are shown in Table 4. T_T denotes the total computation time which includes time spent on equilibrium iterations and other system operations, in addition to the time spent on constitutive integration. T_C denotes the computation time spent only on constitutive integration. It is noted that the calculations are carried out with sufficient physical memory and only one processor.

As shown in Table 4, in this numerical example, the proposed method saves up to 68.7% of the time for the constitutive integration compared to GS method and 39.8% of the time compared to PS method. This shows that the proposed method is more efficient than the existing methods. The main reasons for this is mentioned above, namely that the explicit computation of the principal axes and the coordinate transformation from the principal reference frame to the global reference frame is avoided. In addition, the expressions for the matrix which needs to be inversed become simple using the proposed method, as shown in (76) and (78), and the evaluation of inversion becomes easier.

By the way, the simulation is also carried out by using the ABAQUS built-in extended Drucker–Prager model. The total time for this computation is 121.6s which is longer than the total time for all the three (GS, PS and BT) methods. The reason for this is perhaps that as commercial software, ABAQUS adopts a more general method which is applicable to the general case of more than one internal variables and the potential function including the internal variables. That is, the equation system (52)–(55) rather than

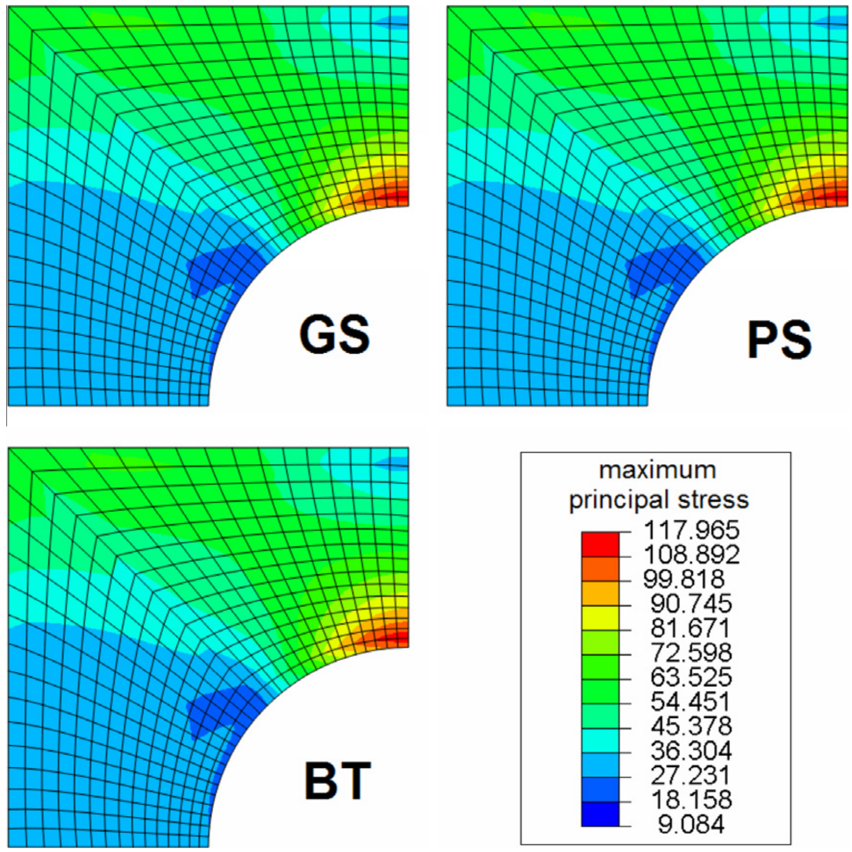

Fig. 5. Contours of maximum principal stress (MPa).

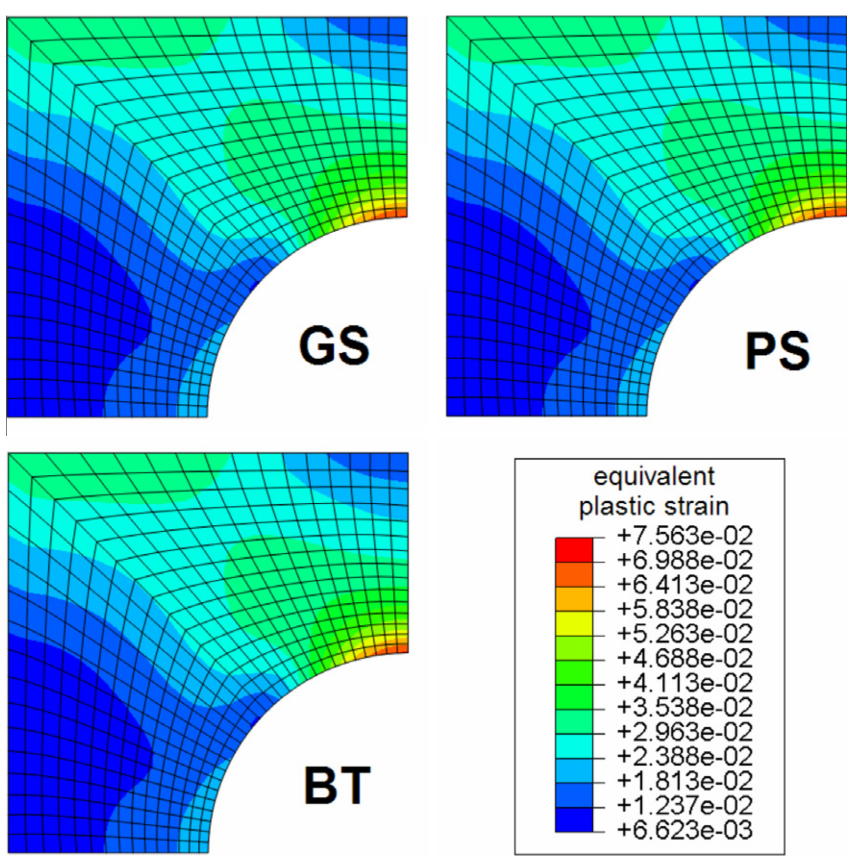

Fig. 6. Contours of equivalent plastic strain.

Table 2
Comparison of the results on the integration point level.

Element (integration point)	Maximum principal stress			Equivalent plastic strain		
	Method	Value (MPa)	Error	Method	Value	Error
1(1)	GS	123.654709	0.000E+00	GS	6.68126717E−02	0.000E+00
	PS	123.654709	0.000E+00	PS	6.68126717E−02	0.000E+00
	BT	123.654709		BT	6.68126717E−02	
2(1)	GS	98.5085831	0.000E+00	GS	4.96679768E−02	0.000E+00
	PS	98.5085831	0.000E+00	PS	4.96679768E−02	0.000E+00
	BT	98.5085831		BT	4.96679768E−02	
3(1)	GS	79.9415207	9.507E−08	GS	3.78767103E−02	0.000E+00
	PS	79.9415131	0.000E+00	PS	3.78767103E−02	0.000E+00
	BT	79.9415131		BT	3.78767103E−02	
4(1)	GS	52.4177361	1.469E−07	GS	2.18483862E−02	0.000E+00
	PS	52.4177361	1.469E−07	PS	2.18483862E−02	0.000E+00
	BT	52.4177284		BT	2.18483862E−02	
5(1)	GS	56.0904198	6.775E−08	GS	2.91363318E−02	−6.178E−08
	PS	56.0904198	6.775E−08	PS	2.91363336E−02	0.000E+00
	BT	56.090416		BT	2.91363336E−02	
6(1)	GS	36.6049347	0.000E+00	GS	1.97020415E−02	−9.644E−08
	PS	36.6049347	0.000E+00	PS	1.97020415E−02	−9.644E−08
	BT	36.6049347		BT	1.97020434E−02	
7(1)	GS	31.3021069	0.000E+00	GS	8.69037304E−03	0.000E+00
	PS	31.3021049	−6.389E−08	PS	8.69037304E−03	0.000E+00
	BT	31.3021069		BT	8.69037304E−03	
8(1)	GS	28.5417137	0.000E+00	GS	1.61733646E−02	0.000E+00
	PS	28.5417156	6.657E−08	PS	1.61733646E−02	0.000E+00
	BT	28.5417137		BT	1.61733646E−02	

Error = (GS − BT)/GS or (PS − BT)/PS.

Table 3
Comparison of iteration numbers.

Time increment (%)	Consistent tangent operator			Continuum tangent operator		
	GS	PS	BT	GS	PS	BT
0.280	1	1	1	1	1	1
0.280	5	5	5	5	5	5
0.364	5	5	5	10	10	10
0.473	4	4	4	10	10	10
0.615	4	4	4	9	9	9
0.800	4	4	4	10	10	10
1.040	3	3	3	10	10	10
1.352	3	3	3	10	10	10
1.757	3	3	3	10	10	10
2.284	3	3	3	10	10	10
2.969	3	3	3	10	10	10
3.860	3	3	3	10	10	10
5.018	3	3	3	11	11	11
6.523	3	3	3	15	15	15
8.481	4	4	4	20	20	20
11.02	4	4	4	26	26	26
14.33	4	4	4	37	37	37
18.63	3	3	3	44	44	44
19.92	2	2	2	36	36	36

Table 4
Comparison of computation time.

Method	T_T (s)	1.0 − BT/(GS, PS, BT)	T_C (s)	1.0 − BT/(GS, PS, BT)
GS consistent	103.2	13.3	22.7	68.7
PS consistent	93.7	4.5	11.8	39.8
BT consistent	89.5	0.00	7.1	0.00

(59)–(61) is used in the solution and therefore more computation time is spent.

7. Conclusions

An efficient return mapping algorithm is developed for implicit integration of general isotropic elastoplastic constitutive equations. A remarking feature is that we employ a set of three mutually orthogonal unit base tensors in conjunction with a new set of three invariants for the representation of arbitrary isotropic tensor valued and scalar valued functions of the stress tensor involved. The base tensors are constructed from the stress tensor by using the representation theorem in conjunction with the simple tensor operations. They are expressed in the global coordinate system and characterized geometrically by mutually orthogonal vectors in principal space. The three invariants are defined by the projection of the stress onto the base tensors. By making use of the representation theorem, all the second-order, the fourth-order tensor-valued functions of the stress involved in the integration algorithms can be represented in terms of those three base tensors with the coefficients given by the invariants of the stress and scalar internal variables. Due to the fact that the principal axes of the stress remain unchanged during the stress return and the defined base tensors depend only on the principal directions, therefore, those base tensors serve as the fixed base. Only the three coefficients need to be iterated upon. This number is half of the six unknowns needed to determine the stress tensor using traditional algorithms. The consistent tangent operator is derived for the proposed algorithm in an efficient, closed form manner. It consists of two parts: one is the tangent operator consistent with the return mapping in the principal stress direction and another reflects the changes in principal stress directions with total applied strain. Compared with the usual approaches in principal space, the main advantage of the proposed approach is that only two base tensors need to be derived from the stress tensor with simple tensor operations. All tensorial quantities entering the return mapping and the expression of the consistent tangent tensor are directly expressed in terms of the proposed base tensors in the global co-ordinate system. The usual procedure of expressing the updated stress and the constitutive matrix in the principal reference frame and transforming back to the global reference frame can be omitted. The explicit computation of the principal axes can be avoided. In addition, the matrix which needs to be inversed often takes the simple form. The algorithm is made more efficient.

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