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New Periodic Solutions for a Class of Singular Hamiltonian Systems

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Abstract In this paper, we apply a variant of the famous Mountain Pass Lemmas of Ambrosetti–Rabinowitz and Ambrosetti–Coti Zelati with $(\mathrm{PSC})_c$ type condition of Palais–Smale–Cerami to study the existence of new periodic solutions with a prescribed energy for symmetrical singular second order Hamiltonian conservative systems with weak force type potentials.

Keywords Singular Hamiltonian systems, periodic solutions, Mountain Pass Lemma, Palais–Smale–Cerami condition

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1 Introduction and Main Results

In 1975 and 1977, Gordon [1, 2] firstly used variational methods to study periodic solutions of 2-Body problems, later, many authors (see [3–26]) used variational methods to study singular Hamiltonian systems. Specially, in [7], Ambrosetti–Coti Zelati studied the periodic solutions of a fixed energy $h \in \mathbb{R}$ for Hamiltonian systems with singular potential $V \in C^2(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$:

$$\ddot{q} + V'(q) = 0, \tag{1.1}$$

$$\frac{1}{2}|\dot{q}|^2 + V(q) = h. \tag{1.2}$$

Using Ljusternik–Schnirelmann theory with classical $(\mathrm{PS})^+$ compact condition, they got the following theorems:

Theorem 1.1 ([7]) *Suppose* $V \in C^2(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$ *satisfies*

(A1) $3V'(x)\cdot x + V''(x)x\cdot x \neq 0, \forall\, x \in \Omega = \mathbb{R}^n\backslash\{0\}$;

(A2) $V'(x)\cdot x > 0, \forall\, x \in \Omega$;

(A3′) $\exists\, \alpha \in (0,2)$*, such that*

$$V'(x)\cdot x \geq -\alpha V(x), \quad \forall\, x \in \Omega;$$

(A4′) $\exists\, \delta \in (0,2)$ *and* $r > 0$*, such that*

$$V'(x)\cdot x \leq -\delta V(x), \quad \forall\, 0 < |x| \leq r;$$

(A5′) $\liminf_{|x|\to+\infty} \left[V(x) + \frac{1}{2}V'(x)\cdot x\right] \geq 0.$

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Then $\forall\, h < 0$, *the system* (1.1)–(1.2) *has at least a non-constant weak periodic solution which satisfies* (1.1)–(1.2) *pointwise except on a zero-measurable set.*

Theorem 1.2 ([7]) *Let* D_h *be the connected component of* $\Omega_h = \{x \in \Omega \,|\, V(x) \leq h\}$ *such that* $0 \in \overline{D}_h$ *and let* $\partial D_h = \{x \in D_h \,|\, V(x) = h\}$. *Let* $h < 0$ *be given. Suppose that* $\overline{D}_h$ *is compact and* $V : \Omega \to \mathbb{R}$ *satisfies* (A4′) *and*

(A1$_h$) $V \in C^2(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$ *and* $3V'(x)\cdot x + V''(x)x \cdot x > 0, \forall\, x \in D_h$;

(A2$_h$) $V'(x)\cdot x > 0, \forall\, x \in D_h$;

(A3$'_h$) $\exists\, 0 < \alpha' < 2$ *such that*

$$V'(x)\cdot x \geq -\alpha' V(x), \quad \forall\, x \in D_h;$$

(A6$_h$) $V \in C^4$ *in a neighborhood of* ∂D_h *and*

$$\max_{\xi\in\partial D_h} [V''(\xi)\xi\cdot\xi] < 0.$$

Then the system (1.1)–(1.2) *has at least a weak periodic solution for any* $h < 0$.

In [8], Ambrosetti–Coti Zelati used Mountain Pass Lemma with the $(PS)^+$ condition to study the existence of weak solutions for symmetrical N-Body problems with a fixed energy $h < 0$:

$$\begin{cases} m_i x_i'' + \nabla_{x_i} V(x_1, x_2, \ldots, x_N) = 0, \quad 1 \leq i \leq N, \\ \dfrac{1}{2}\sum m_i |\dot{x}_i(t)|^2 + V\left(x_1(t), x_2(t), \ldots, x_N(t)\right) = h. \end{cases} \tag{Ph}$$

They got

Theorem 1.3 ([7]) *Suppose* $V(x) = \frac{1}{2}\sum_{1\leq i\neq j\leq N} V_{ij}(x_i - x_j)$ *and* $V_{ij} \in C^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$ *satisfies*

(V1) $V_{ij}(\xi) = V_{ji}(\xi), \forall\, \xi \in \Omega = \mathbb{R}^n\backslash\{0\}$;

(V2) $\exists\, \alpha \in [1, 2)$, *such that*

$$\nabla V_{ij}(\xi)\cdot\xi \geq -\alpha V_{ij}(\xi) > 0, \quad \forall\, \xi \in \Omega;$$

(V3) $\exists\, \delta \in (0, 2)$ *and* $r > 0$, *such that*

$$\nabla V_{ij}(\xi)\cdot\xi \leq -\delta V_{ij}(\xi), \quad \forall\, 0 < |\xi| \leq r;$$

(V4) $V_{ij}(\xi) \to 0$, *as* $|\xi| \to +\infty$.

Then $\forall\, h < 0$, *the problem* (Ph) *has a periodic solution.*

Theorem 1.4 ([8]) *Suppose* V *satisfies* (V1), (V3), (V4) *and*

(V2′) $\exists\, \alpha \in (0, 2)$, *such that*

$$\nabla V_{ij}(\xi)\cdot\xi \geq -\alpha V_{ij}(\xi) > 0, \quad \forall\, \xi \in \Omega;$$

(V5) $V_{ij} \in C^2(\Omega, \mathbb{R})$ *and*

$$3\nabla V_{ij}(\xi)\cdot\xi + V''_{ij}(\xi)\xi\cdot\xi > 0.$$

Then $\forall\, h < 0$, (Ph) *has a weak periodic solution.*

Motivated by these two papers, we have the following theorem:

Theorem 1.5 *Suppose $V \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ satisfies*

(V_1) $V(-q) = V(q)$;

(V_2) *There exists a constant $0 < \alpha < 2$ such that*

$$\langle V'(q), q\rangle \geq -\alpha V(q) > 0, \quad \forall\, q \in \mathbb{R}^n \backslash \{0\};$$

(V_3) $\exists\, \delta \in (0, 2), r > 0$, *such that*

$$\langle V'(q), q\rangle \leq -\delta V(q), \quad \forall\, 0 < |q| \leq r;$$

(V_4) $V(q) \to 0$, *as $|q| \to +\infty$.*

Then for any given $h < 0$, the system (1.1)–(1.2) *has at least a non-constant weak periodic solution which can be obtained by Mountain Pass Lemma.*

Corollary 1.6 *Suppose $0 < \alpha = \delta < 2$ and*

$$V(x) = -|x|^{-\alpha}.$$

Then for any $h < 0$, (1.1)–(1.2) *has at least one non-constant weak periodic solution with the given energy h.*

Remark 1.7 We guess (V5) in Theorem 1.4 can be deleted by combining the arguments of this paper and [7, 8].

2 Some Lemmas

Lemma 2.1 ([7]) *Let $f(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 dt \int_0^1 (h - V(u))dt$ and $\tilde{u} \in H^1$ be such that $f'(\tilde{u}) = 0$ and $f(\tilde{u}) > 0$. Set*

$$\frac{1}{T^2} = \frac{\int_0^1 (h - V(\tilde{u}))\, dt}{\frac{1}{2}\int_0^1 |\dot{\tilde{u}}|^2 dt}. \tag{2.1}$$

Then $\tilde{q}(t) = \tilde{u}(t/T)$ is a non-constant T-periodic solution for (1.1)–(1.2).

Lemma 2.2 ([27]) *Let σ be an orthogonal representation of a finite or compact group G in the real Hilbert space H such that for $\forall\, \sigma \in G$,*

$$f(\sigma \cdot x) = f(x),$$

where $f \in C^1(H, \mathbb{R})$.

Let $S = \{x \in H \,|\, \sigma x = x, \forall\, \sigma \in G\}$. Then the critical point of f in S is also a critical point of f in H.

Let

$$\Lambda_0 = \{u \in H^1 = W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), u(t + 1/2) = -u(t), u(t) \neq 0\}.$$

By Lemmas 2.1–2.2 and (V_1), we have

Lemma 2.3 ([7]) *If $\bar{u} \in \Lambda_0$ is a critical point of $f(u)$ and $f(\bar{u}) > 0$, then $\bar{q}(t) = \bar{u}(t/T)$ is a non-constant T-periodic solution of* (1.1)–(1.2).

Cerami [13] introduced the following $(\mathrm{CPS})_c$ condition:

Definition 2.4 ([13]) *Let X be a Banach space, $\{q_n\} \subset X$ satisfy*

$$f(q_n) \to c, \quad (1 + \|q_n\|)f'(q_n) \to 0. \tag{2.2}$$

Then $\{q_n\}$ *has a strongly subsequence, then we call that* $\{q_n\}$ *satisfies Cerami–Palais–Smale condition at level c, we denote it by* $(\mathrm{CPS})_c$. *If* $(\mathrm{CPS})_c$ *holds for all c, we call* $f(q)$ *satisfies the* (CPS) *condition.*

Combining the different forms of the Mountain Pass Lemmas in [8, 10, 28, 29], it is not difficult to get

Lemma 2.5 *Suppose* $f \in C^1(\Lambda_0, \mathbb{R})$ *and*

(AR_1) $\exists\, \rho, \beta > 0$, *s.t.* $f(u) \geq \beta, \forall\, u \in \Lambda_0$, $\|u\|_{H^1} = \rho$,

(AR_2) $\exists\, u_0, u_1 \in \Lambda_0$ *with* $\|u_0\|_{H^1} < \rho < \|u_1\|_{H^1}$ *s.t.* $\max\{f(u_0), f(u_1)\} < \beta$.

Let

$$C = \inf_{P \in \Gamma_\rho} \max_{0 \leq \xi \leq 1} f(P(\xi)),$$

where

$$\Gamma_\rho = \{P \in C([0,1], \Sigma_\rho) \,|\, \|P(0)\|_{H^1} = \rho, P(1) = u_1\},$$
$$\Sigma_\rho = \{u \in \Lambda_0 \,|\, \|u\|_{H^1} \geq \rho\}.$$

Then there exists $\{u_n\} \subset \Lambda_0$ *such that*

$$f(u_n) \to C, \quad (1 + \|u_n\|)f'(u_n) \to 0.$$

Furthermore, if f satisfies $(\mathrm{CPS})_C$ *condition, that is* $\{u_n\}$ *has a convergent subsequence. If*

$$f(u_n) \to +\infty, \quad \forall\, u_n \rightharpoonup u \in \partial\Lambda_0,$$

then C is a critical value of f, so there exists $u \in \Lambda_0$ *such that* $f'(u) = 0$, *and* $f(u) = C \geq \beta > 0$.

Lemma 2.6 ([1]) *Let V satisfy the so called Gordon's strong force condition*: *There exists a neighborhood* $\mathcal{N}$ *of* 0 *and a function* $U \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ *such that*

(i) $\lim_{x \to 0} U(x) = -\infty$;

(ii) $-V(x) \geq |U'(x)|^2$ *for every* $x \in \mathcal{N} \setminus \{0\}$.

Let

$$\partial\Lambda_0 = \{u \in H^1 = W^{1.2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), u(t+1/2) = -u(t),\, \exists\, t_0,\, u(t_0) = 0\}.$$

Then we have

$$\int_0^1 V(u_n)dt \to -\infty, \quad \forall\, u_n \rightharpoonup u \in \partial\Lambda_0.$$

Lemma 2.7 (Sobolev–Rellich–Kondrachov, Compact Imbedding Theorem [30, 31])

$$W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n) \subset C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$$

and the embedding is compact.

Lemma 2.8 (Eberlein–Shmulyan [32]) *A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.*

Lemma 2.9 ([30]) *Let* $q \in W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ *and* $\int_0^T q(t)dt = 0$. *Then we have*

(i) *Poincare–Wirtinger's inequality*:

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt.$$

(ii) *Sobolev's inequality*:

$$\max_{0\le t\le T}|q(t)|=\|q\|_\infty\le\sqrt{\frac{T}{12}}\left(\int_0^T|\dot q(t)|^2dt\right)^{1/2}.$$

It is not difficult to prove

Lemma 2.10 $\forall\, u\in\Lambda_0$, *we have*

$$\int_0^1 u(t)dt=0.$$

By Lemmas 2.9 and 2.10, $\forall\, u\in\Lambda_0$, $\|u\|=(\int_0^1|\dot u|^2dt)^{1/2}$ is equivalent to the $H^1=W^{1,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ norm:

$$\|u\|_{H^1}=\left(\int_0^1|\dot u|^2dt\right)^{1/2}+\left(\left|\int_0^1 udt\right|\right)^{1/2}.$$

3 Proof of Theorem 1.5

In singular Hamiltonian systems, in order to apply Mountain Pass Lemma for the variational functional defined on Λ_0 (an open set of Banach space), we need a complete condition:

$$f(u_n)\to+\infty,\quad u_n\rightharpoonup\partial\Lambda_0,\tag{3.1}$$

which can guarantee that the critical point is in Λ_0, not on its boundary. But the assumptions of Theorem 1.5, we do not have the strong force condition, so we need to revise the potential V as V_ε,

$$V_\varepsilon(u)=V(u)+W_\varepsilon(u),\ W_\varepsilon(u)=-\frac{\varepsilon}{|u|^\gamma},\quad\gamma>2.\tag{3.2}$$

We also need to revise the functional $f(u)$ as

$$f_\varepsilon(u)=\frac12\int_0^1|\dot u|^2dt\int_0^1(h-V_\varepsilon(u))\,dt.\tag{3.3}$$

Remark 3.1 Different from earlier papers, here we use $W_\varepsilon(u)$ with $\gamma>2$ not $\gamma=2$ to perturb V in order that f_ε satisfies (3.1) and we can verify all conditions of Mountain Pass Lemma.

After we apply Mountain Pass Lemma to the variational functional f_ε to get the critical point u_ε, we let $\varepsilon\to0$ to get the limit point, which is a weak solution satisfying (1.1)–(1.2) except on a Lebegue's zero-measurable set.

In order to find the critical point of f_ε in Λ_0, we need to verify all conditions of Mountain Pass Lemma. Let us begin to prove

Lemma 3.2 *Let* (V_1)–(V_2) *hold. For all* $C>0$, *if* $\{u_n\}\subset\Lambda_0$ *such that*

$$f_\varepsilon(u_n)\to C>0,\quad(1+\|u_n\|)f'_\varepsilon(u_n)\to0,\tag{3.4}$$

then $\{u_n\}\subset\Lambda_0$ *has a strongly convergent subsequence, the limit must be in* Λ_0, *that is,* f_ε *satisfies the* $(\mathrm{CPS})_C$ *condition in* Λ_0.

Proof The proof will be divided into three steps:

Step 1 We show that $\{u_n\}$ is bounded.

In fact, by $f_\varepsilon(u_n)\to C$, we have

$$-\frac12\|u_n\|^2\int_0^1 V_\varepsilon(u_n)dt\to C-\frac h2\|u_n\|^2.\tag{3.5}$$

So when n is large enough, it follows that

$$-\frac{1}{2}\|u_n\|^2\int_0^1 V_\varepsilon(u_n)dt \le C+1-\frac{h}{2}\|u_n\|^2. \tag{3.6}$$

By calculations, we get

$$\langle V_\varepsilon'(u_n),u_n\rangle = \langle V'(u_n),u_n\rangle - \gamma W_\varepsilon(u_n). \tag{3.7}$$

Note that

$$-\gamma W_\varepsilon \ge -\alpha W_\varepsilon. \tag{3.8}$$

From (V_2), (3.7) and (3.8), we have

$$\langle V_\varepsilon'(u_n),u_n\rangle \ge -\alpha V_\varepsilon(u_n) > 0. \tag{3.9}$$

So

$$\begin{aligned}\langle f_\varepsilon'(u_n),u_n\rangle &= \|u_n\|^2\int_0^1\left(h-V_\varepsilon(u_n)-\frac{1}{2}\langle V_\epsilon'(u_n),u_n\rangle\right)dt\\ &\le \|u_n\|^2\int_0^1\left(h-V_\varepsilon(u_n)+\frac{\alpha}{2}V_\varepsilon(u_n)\right)dt\\ &= \|u_n\|^2\int_0^1\left(h-\left(1-\frac{\alpha}{2}\right)V_\varepsilon(u_n)\right)dt.\end{aligned} \tag{3.10}$$

Since $0<\alpha<2$, using (3.6) and (3.10), we have

$$\langle f_\varepsilon'(u_n),u_n\rangle \le h\|u_n\|^2+\left(1-\frac{\alpha}{2}\right)\left[2(C+1)-h\|u_n\|^2\right] = \frac{\alpha}{2}h\|u_n\|^2+C_1, \tag{3.11}$$

where $C_1=2(1-\frac{\alpha}{2})(C+1)>0, 0<\alpha<2$.

By (3.4), we have

$$\langle f_\varepsilon'(u_n),u_n\rangle \le \|u_n\|\|f_\varepsilon'(u_n)\| \to 0. \tag{3.12}$$

(3.11), (3.12) and $h<0$ imply

$$\|u_n\| \le C_2. \tag{3.13}$$

Step 2 We prove $u_n \rightharpoonup u\in\Lambda_0$.

Since H^1 is a reflexive Banach space, by Lemma 2.8 and (3.13), $\{u_n\}$ has a weakly convergent subsequence still denoted by $\{u_n\}$ such that $u_n\rightharpoonup u$.

To prove $u\in\Lambda_0$, we need two lemmas.

Lemma 3.3 *Assume the potential V_ε satisfies Gordon's strong force condition. Then for any weakly convergent sequence $u_n\rightharpoonup u\in\partial\Lambda_0$, there holds*

$$f_\varepsilon(u_n)\to+\infty.$$

Proof First of all, recall that

$$f_\varepsilon(u_n)=\frac{1}{2}\int_0^1|\dot u_n|^2dt\int_0^1(h-V_\varepsilon(u_n))dt.$$

(1) If $u\equiv$ constant, from $u_\varepsilon\in\partial\Lambda_0$, we deduce $u\equiv 0$. By Sobolev's embedding theorem, we have

$$\|u_n\|_\infty\to 0,\quad n\to\infty. \tag{3.14}$$

Using (V_2), we have $C_3 > 0$, such that

$$V(u) \le -\frac{C_3}{|u|^\alpha}, \quad \forall\, |u| > 0. \tag{3.15}$$

Therefore, $h - V(u_n) > 0$ when n is large enough, it follows that

$$\begin{aligned} f_\varepsilon(u_n) &= \frac{1}{2}\int_0^1 |\dot u_n|^2 dt \int_0^1 \left(h - V(u_n) + \frac{\varepsilon}{|u_n|^\gamma}\right) dt \\ &\ge \frac{1}{2}\int_0^1 |\dot u_n|^2 dt \int_0^1 \frac{\varepsilon}{|u_n|^\gamma} dt \\ &\ge \frac{\varepsilon}{2}\int_0^1 |\dot u_n|^2 dt \|u_n\|_\infty^{-\gamma}. \end{aligned} \tag{3.16}$$

Then by Sobolev's inequality, (3.14) and $\gamma > 2$, we have

$$f_\varepsilon(u_n) \ge 6\varepsilon \|u_n\|_\infty^{2-\gamma} \to +\infty, \quad n \to \infty.$$

(2) If $u \not\equiv$ constant, by the weakly lower-semi-continuity property for norm, we have

$$\liminf_{n\to\infty} \int_0^1 |\dot u_n|^2 dt \ge \int_0^1 |\dot u|^2 dt > 0. \tag{3.17}$$

Since V_ε satisfies Gordon's Strong Force condition, then by Lemma 2.6 and (3.17), we have

$$f_\varepsilon(u_n) \to +\infty, \quad n \to \infty.$$ □

Lemma 3.4 *V_ε satisfies Gordon's strong force condition.*

Proof Let

$$\overline{V} = \frac{-1}{\lambda |u|^\lambda}, \quad 0 < \lambda < \frac{\gamma - 2}{2}.$$

Then

$$\lim_{|u|\to 0} \overline{V} = -\infty. \tag{3.17}$$

By calculation, we obtain

$$|\overline{V}'|^2 = \frac{1}{|u|^{2\lambda+2}}.$$

Note that

$$-V_\varepsilon(u) = -V(u) + \frac{\varepsilon}{|u|^\gamma} \ge \frac{\varepsilon}{|u|^\gamma}. \tag{3.18}$$

Since

$$\frac{\varepsilon}{|u|^\gamma} \ge \frac{1}{|u|^{2\lambda+2}}, \quad \forall\, \varepsilon > 0, \tag{3.19}$$

when $|u|$ is small enough, so there exists a neighborhood $\mathcal{N}$ of 0 such that $-V_\varepsilon \ge |\overline{V}'|^2, \forall\, u \in \mathcal{N}\setminus\{0\}$. Therefore, V_ε satisfies Gordon's strong force condition. □

Combining (3.4), Lemmas 3.3 and 3.4, we deduce $u_n \rightharpoonup u \in \Lambda_0$.

Step 3 We prove that $u_n \to u$ strongly.

By $u_n \rightharpoonup u \in \Lambda_0$ and the compact embedding theorem, we have

$$\max_{0\le t\le 1} |u_n(t) - u(t)| \to 0.$$

By the continuity of $V_\varepsilon,\ V'_\varepsilon$ and the inner product $\langle\cdot\rangle$, we have the uniformly convergent for $0 \le t \le 1$,

$$\begin{aligned} &V_\epsilon(u_n) \to V_\varepsilon(u),\\ &W_\varepsilon(u_n) \to W_\varepsilon(u),\\ &\langle V'_\varepsilon(u_n), u_n\rangle \to \langle V'_\varepsilon(u), u\rangle. \end{aligned} \tag{3.21}$$

From Step 2, we know $u \in \Lambda_0$, so $\|u\| = \int_0^1 |\dot{u}|^2 dt > 0$, otherwise $u \equiv 0 \in \partial\Lambda_0$ by $u(t+1/2) = -u(t)$. Then by $u_n \rightharpoonup u$ and the weakly lower-semi-continuous property of the norm, we have

$$\liminf_{n\to\infty} \|u_n\| \ge \|u\| > 0. \tag{3.22}$$

By (3.12), we have

$$\langle f'_\varepsilon(u_n), u_n\rangle = \|u_n\|^2 \int_0^1 \left[h - V_\varepsilon(u_n) - \frac{1}{2}\langle V'_\varepsilon(u_n), u_n\rangle\right] dt \to 0. \tag{3.23}$$

Letting $n \to \infty$ in (3.23), by (3.21) and (3.22), we have

$$\int_0^1 (h - V_\varepsilon(u))dt = \frac{1}{2}\int_0^1 \langle V'_\varepsilon(u), u\rangle dt > 0. \tag{3.24}$$

From (3.4), we deduce $f'_\varepsilon(u_n) \to 0$, then $\langle f'_\varepsilon(u_n), v\rangle \to 0, \forall\, v \in H^1$, that is,

$$\int_0^1 \langle \dot{u}_n, \dot{v}\rangle dt \int_0^1 (h - V_\varepsilon(u_n))dt - \frac{1}{2}\|u_n\|^2 \int_0^1 \langle V'_\varepsilon(u_n), v\rangle dt \to 0, \quad \forall\, v \in H^1. \tag{3.25}$$

Taking $v = u$ in (3.25), we get

$$\lim_{n\to\infty} \int_0^1 \langle \dot{u}_n, \dot{u}\rangle dt = \lim_{n\to\infty} \|u_n\|^2. \tag{3.26}$$

By $u_n \rightharpoonup u$, we have

$$\lim_{n\to\infty} \int_0^1 \langle \dot{u}_n, \dot{u}\rangle dt = \int_0^1 |\dot{u}|^2 dt = \|u\|^2. \tag{3.27}$$

From (3.26) and (3.27), it follows that

$$\begin{aligned} \|u_n - u\|^2 = \int_0^1 |\dot{u}_n - \dot{u}|^2 dt &= \int_0^1 (|\dot{u}_n|^2 - 2\langle \dot{u}_n, \dot{u}\rangle + |\dot{u}|^2) dt\\ &\to \|u\|^2 - 2\|u\|^2 + \|u\|^2 = 0. \end{aligned} \tag{3.28}$$

That is, $u_n \to u$ strongly in H^1. □

Lemma 3.5 *f_ε satisfies the condition* (AR$_1$) *in the Mountain Pass Lemma.*

Proof By (3.9), we have $C_4 > 0$ such that $-V_\varepsilon(u) \ge \frac{C_4}{|u|^\alpha}$, so we have

$$\begin{aligned} f_\varepsilon(u) &= \frac{1}{2}\|u\|^2 \int_0^1 (h - V_\varepsilon(u))\, dt\\ &= \frac{h}{2}\|u\|^2 - \frac{1}{2}\|u\|^2 \int_0^1 V_\varepsilon(u) dt\\ &\ge \frac{h}{2}\|u\|^2 + \frac{C_4}{2}\|u\|^2 \|u\|_\infty^{-\alpha}. \end{aligned}$$

Then by Sobolev's inequality, we have $C_5 > 0$ such that

$$f_\varepsilon(u) \geq \frac{h}{2}\|u\|^2 + \frac{C_5}{2}\|u\|^{2-\alpha}.$$

Since $0 < \alpha < 2$, we can choose $\|u\| = \rho$ small enough such that $\frac{h}{2}\rho^2 + \frac{C_5}{2}\rho^{2-\alpha} = \beta > 0$. Hence

$$f_\varepsilon(u) \geq \beta > 0, \quad \forall \|u\| = \rho. \qquad \square$$

Lemma 3.6 *$\exists u_0 \in \Lambda_0$ with $\|u_0\| < \rho$ s.t. $f_\varepsilon(u_0) < \beta$.*

Proof For $\tilde{R} > 0$, we consider

$$f_\varepsilon(\tilde{R}u) = \frac{1}{2}\|\tilde{R}u\|^2 \int_0^1 (h - V_\varepsilon(\tilde{R}u))dt.$$

Using (V_3), we have $C_6 > 0$ such that

$$V(u) \geq -C_6|u|^{-\delta}, \quad \forall\, 0 < |u| \leq r.$$

Then we have

$$f_\varepsilon(\tilde{R}u) \leq \frac{h}{2}\tilde{R}^2\|u\|^2 + C_6\tilde{R}^{2-\delta}\|u\|^2\int_0^1 |u|^{-\delta}dt + \varepsilon C_7\tilde{R}^{2-\gamma}\|u\|^2\int_0^1 |u|^{-\gamma}dt. \tag{3.29}$$

Take $u(t) = \xi\sin(2\pi t) + \eta\cos(2\pi t)$, where $|\xi| = 1, |\eta| = 1, \langle\xi,\eta\rangle = 0$, $\xi, \eta \in \mathbb{R}^n$. Then $|u| = 1$, $\|u\| = 2\pi$, hence

$$\begin{aligned} f_\varepsilon(\tilde{R}u) &\leq 4\pi^2\left(\frac{h}{2}\tilde{R}^2 + C_6\tilde{R}^{2-\delta} + \varepsilon C_7\tilde{R}^{2-\gamma}\right) \\ &\leq 4\pi^2(C_6\tilde{R}^{2-\delta} + \varepsilon C_7\tilde{R}^{2-\gamma}). \end{aligned} \tag{3.30}$$

Since $0 < \delta < 2$, so we can take R_0 small enough such that $4\pi^2 C_6 R_0^{2-\delta} < \beta$.

For the above fixed R_0, we choose $\varepsilon > 0$ small enough such that

$$\varepsilon 4\pi^2 C_7 R_0^{2-\gamma} < \beta - 4\pi^2 C_6 R_0^{2-\delta}. \tag{3.31}$$

In fact, we can choose

$$0 < \varepsilon_0 < \frac{\beta - 4\pi^2 C_6 R_0^{2-\delta}}{4\pi^2 C_7 R_0^{2-\gamma}}. \tag{3.32}$$

Choose R_1 small enough such that $\|R_1 u\| < \rho$, take $R = \min\{R_0, R_1\}$, and let $u_0 = Ru$. Then we have

$$f_\varepsilon(u_0) < \beta, \quad \|u_0\| < \rho, \quad \forall\, 0 < \varepsilon \leq \varepsilon_0. \tag{3.33}$$

Lemma 3.7 *$\exists u_1 \in \Lambda_0$ with $\|u_1\| > \rho$ s.t. $f_\varepsilon(u_1) < 0$.*

Proof Let $R > 0$. We consider

$$f_\varepsilon(Ru) = \frac{1}{2}\|Ru\|^2 \int_0^1 (h - V_\varepsilon(Ru))dt.$$

Taking $u = \xi\sin(2\pi t) + \eta\cos(2\pi t)$, by ($V_4$), it follows that

$$\int_0^1 V_\varepsilon(Ru)dt \to 0, \quad R \to +\infty.$$

So $f_\varepsilon(R_0 u) < 0$, when R_0 is large enough. Choose R_1 large enough such that $\|R_1 u\| > \rho$. Take $R = \max\{R_0, R_1\}$, and let $u_1 = Ru$. Then

$$f_\varepsilon(u_1) < 0 < \beta, \quad \|u_1\| > \rho. \qquad \square$$

From Lemmas 3.2–3.7, we know that $\forall\, 0 < \varepsilon \le \varepsilon_0, f_\epsilon$ satisfies $(\mathrm{AR}_1), (\mathrm{AR}_2), (\mathrm{CPS})_{C_\varepsilon}$ condition, and $f_\varepsilon(u_{\{n,\varepsilon\}}) \to +\infty, \forall\, u_{\{n,\varepsilon\}} \rightharpoonup u_\varepsilon \in \partial\Lambda_0$. Let

$$C_\varepsilon = \inf_{P\in\Gamma_\rho} \max_{0\le\xi\le 1} f(P(\xi)).$$

By Lemma 2.5, we know that $\forall\, 0 < \varepsilon \le \varepsilon_0$, there exists $u_\varepsilon \in \Lambda_0$ such that

$$f'_\varepsilon(u_\varepsilon) = 0, \quad f_\varepsilon(u_\varepsilon) = C_\varepsilon \ge \beta > 0. \tag{3.34}$$

Let

$$\omega_\varepsilon^2 = \frac{\int_0^1 (h - V_\varepsilon(u_\varepsilon))dt}{\frac{1}{2}\int_0^1 |\dot{u}_\varepsilon|^2 dt}.$$

Then by Lemma 2.3, $y_\varepsilon = u_\varepsilon(\omega_\varepsilon t)$ satisfies

$$\ddot{y}_\varepsilon + V'_\varepsilon(y_\varepsilon(t)) = 0, \tag{3.35}$$

$$\frac{1}{2}\omega_\varepsilon^2 |\dot{u}_\varepsilon(t)|^2 + V_\varepsilon(u_\varepsilon(t)) = h. \tag{3.36}$$

Next, we show that u_ε converges to some u^* which gives rise to a solution y^* of (1.1)–(1.2).

Lemma 3.8 $\exists\, C_8, C_9 > 0$ *s.t.* $C_8 \le \|u_\varepsilon\| \le C_9$.

Proof Since $u_\varepsilon \in \Lambda_0$, so $\|u_\varepsilon\|^2 = \int_0^1 |\dot{u}_\varepsilon|^2 dt \ne 0$, otherwise $u_\varepsilon(t) \equiv 0 \in \partial\Lambda_0$ by $u(t + 1/2) = -u(t)$. By $\langle f'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle = 0$, we have

$$\|u_\varepsilon\|^2 \int_0^1 \left[h - V_\varepsilon(u_\varepsilon) - \frac{1}{2}\langle V'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle\right] dt = 0.$$

Then

$$h = \int_0^1 \left(V_\varepsilon(u_\varepsilon) + \frac{1}{2}\langle V'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle\right) dt. \tag{3.37}$$

Letting $\gamma \to 2$, we have

$$h = \int_0^1 \left(V(u_\varepsilon) + \frac{1}{2}\langle V'(u_\varepsilon), u_\varepsilon\rangle\right) dt.$$

By (V_3), we get

$$h \le \left(1 - \frac{\delta}{2}\right) \int_0^1 V(u_\varepsilon) dt. \tag{3.38}$$

If $\|u_\varepsilon\| \to 0$, as $\varepsilon \to 0$, then $\|u_\varepsilon\|_\infty \to 0$, from (3.15), we deduce that

$$\int_0^1 V(u_\varepsilon) dt \to -\infty,$$

which is a contradiction with (3.38). So we claim

$$\|u_\varepsilon\| \ge C_8 > 0. \tag{3.39}$$

From (3.34), we know

$$f_\varepsilon(u_\varepsilon) = \inf_{P\in\Gamma_\rho} \max_{0\le\xi\le 1} f_\varepsilon(P(\xi)), \quad \forall\, 0 < \varepsilon \le \varepsilon_0.$$

So we have

$$f_\varepsilon(u_\varepsilon) \le \inf_{P\in\Gamma_\rho}\max_{0\le\xi\le1} f_{\varepsilon_0}(P(\xi)) \le \max_{0\le\xi\le1} f_{\varepsilon_0}(P(\xi)) = C_{10}, \quad \forall\, 0<\varepsilon\le\varepsilon_0.$$

That is,

$$f_\varepsilon(u_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|^2\int_0^1 (h - V_\varepsilon(u_\varepsilon))dt \le C_{10}, \quad \forall\, 0<\varepsilon\le\varepsilon_0. \tag{3.40}$$

By (3.9), we have

$$h = \int_0^1\left(V_\varepsilon(u_\varepsilon) + \frac{1}{2}\langle V_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle\right)dt \ge \left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_0^1\langle V_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle dt.$$

So

$$\int_0^1\langle V_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle dt \ge \frac{h}{\frac{1}{2}-\frac{1}{\alpha}} > 0. \tag{3.41}$$

Then by (3.37), we obtain

$$\int_0^1 (h - V_\varepsilon(u_\varepsilon))dt \ge \frac{h}{1-\frac{2}{\alpha}}. \tag{3.42}$$

(3.40) and (3.42) imply

$$\|u_\varepsilon\| \le C_9. \tag{3.43}$$

The proof is completed. □

Since E is a reflexive Banach space, by (3.43) and Lemma 2.8, there is a subsequence, still denoted by $\{u_\varepsilon\}$ such that $u_\varepsilon \rightharpoonup u^*$, then by compact embedding theorem, $u_\varepsilon \to u^*$ uniformly.

In the following, we can use almost the same proofs of Ambrosetti–Coti Zelati [7] to get Lemmas 3.9–3.11, but we should remember $\gamma > 2$, so in order to get our result, we need to let $\gamma \to 2$, for the convenience of the readers, we write the complete proofs.

Lemma 3.9 (1) $V(u^*(t)) \not\equiv h$. (2) $u^*(t) \not\equiv 0$.

Proof (1) If not, $V(u^*(t)) \equiv h$, then

$$V(u_\varepsilon(t)) \to V(u^*(t)) \equiv h, \quad \langle V'(u_\varepsilon(t)), u_\varepsilon(t)\rangle \to \langle V'(u^*(t)), u^*(t)\rangle.$$

Since

$$h = \int_0^1\left(V_\varepsilon(u_\varepsilon) + \frac{1}{2}\langle V_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle\right)dt,$$

letting $\gamma \to 2$, we get

$$h = \int_0^1\left(V(u_\varepsilon) + \frac{1}{2}\langle V'(u_\varepsilon), u_\varepsilon\rangle\right)dt.$$

Then letting $\varepsilon \to 0$, we have

$$h = h + \frac{1}{2}\int_0^1\langle V(u^*), u^*\rangle dt.$$

Hence $\langle V(u^*), u^*\rangle = 0$, this is a contradiction with (V_2).

(2) If not, $u^* \equiv 0$, $u_\varepsilon \to 0$ uniformly.

Since

$$h=\int_0^1\left(V_\varepsilon(u_\varepsilon)+\frac{1}{2}\langle V_\varepsilon'(u_\varepsilon),u_\varepsilon\rangle\right)dt,$$

letting $\gamma\to 2$, then by (V_3), we have

$$h=\int_0^1\left(V(u_\varepsilon)+\frac{1}{2}\langle V'(u_\varepsilon),u_\varepsilon\rangle\right)dt\le\left(1-\frac{\delta}{2}\right)\int_0^1 V(u_\varepsilon)dt.$$

So

$$\int_0^1 V(u_\varepsilon)dt\ge\frac{h}{1-\frac{\delta}{2}}. \tag{3.44}$$

On the other hand, since $u_\varepsilon\to 0$ uniformly, by (3.15), we have

$$\int_0^1 V(u_\varepsilon)dt\to-\infty,\quad \varepsilon\to 0, \tag{3.45}$$

which is a contradiction with (3.44). □

Lemma 3.10 *There are numbers $\delta,\Delta>0$, s.t.*

$$\delta\le\omega_\varepsilon\le\Delta. \tag{3.46}$$

Proof From Lemma 3.9, we conclude that there exists a closed interval I such that

$$|I|>0,\ u^*(t)\ne 0,\ V(u^*(t))\ne h,\ \forall\, t\in I. \tag{3.47}$$

Integrating (3.36) on I, we have

$$\frac{1}{2}\omega_\varepsilon^2\int_I|\dot u_\varepsilon|^2dt+\int_I V_\varepsilon(u_\varepsilon)dt=h|I|. \tag{3.48}$$

From (3.43), we deduce

$$\int_I|\dot u_\varepsilon|^2dt\le\int_0^1|\dot u_\varepsilon|^2dt\le C_9^2.$$

From (3.34), $h-V_\varepsilon(u_\varepsilon)>0$, then by (3.47), $V_\varepsilon(u_\varepsilon)\to V(u^*)$ uniformly on I and $\int_I(h-V(u^*))dt>0$, it follows that

$$\omega_\varepsilon^2\ge\frac{2\int_I(h-V_\varepsilon(u_\varepsilon))dt}{C_9^2}\to\frac{2\ \int_I(h-V(u^*))dt}{C_9^2}>0. \tag{3.49}$$

Integrating (3.36) on $[0,1]$, we have

$$\frac{1}{2}\omega_\varepsilon^2\int_0^1|\dot u_\varepsilon|^2dt+\int_0^1 V_\varepsilon(u_\varepsilon)dt=h.$$

Then by (3.3), (3.37), (3.39) and (3.40), we have

$$\omega_\varepsilon^2=\frac{4f_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|^4}\le\frac{4C_{10}}{C_8^4}. \tag{3.50}$$

Lemma 3.11 *Suppose that* (V_1)–(V_4) *hold. Then for any $h<0$, u^* is a weak solution of* (1.1)–(1.2).

Proof Let

$$J = \{t \in [0,1] \,|\, u^*(t) = 0\}. \tag{3.51}$$

Integrating (3.36) on J, we have

$$\frac{1}{2}\omega_\varepsilon^2 \int_J |\dot{u}_\varepsilon|^2 dt + \int_J V_\varepsilon(u_\varepsilon)dt = h|J|. \tag{3.52}$$

Combining (3.52), Lemma 3.8 and Lemma 3.10, we obtain

$$\int_J V_\varepsilon(u_\varepsilon)dt = |J|h - \frac{1}{2}\omega_\varepsilon^2 \int_J |\dot{u}_\varepsilon|^2 dt \geq |J|h - \frac{1}{2}\Delta^2 C_9^2. \tag{3.53}$$

But $u_\varepsilon \to 0$ uniformly on J, if J has positive measure, then $\int_J V_\varepsilon(u_\varepsilon)dt \to -\infty$, which is a contradiction with (3.53).

Let $K_n \subset [0,1]\backslash J$ be an increasing sequence of compact sets with

$$\bigcup_{n\geq 1} K_n = [0,1]\backslash J,$$

and set

$$K_n^* = \{u^*(t) \,|\, t \in K_n\}.$$

Each $K_n^* \subset \mathbb{R}^n \setminus \{0\}$ is compact and has a neighborhood $\mathcal{N}_n$ such that $\overline{\mathcal{N}}_n \subset \mathbb{R}^n \setminus \{0\}$. Then $V_\epsilon \to V$ in $C^1(\mathcal{N}_n, \mathbb{R})$, and therefore $V_\varepsilon'(u_\varepsilon(t)) \to V'(u^*(t))$ uniformly on K_n.

Since u_ε satisfies

$$\omega_\varepsilon^2 \ddot{u}_\varepsilon + V_\varepsilon'(u_\varepsilon) = 0,$$

by Lemma 3.10, we have

$$\omega_\varepsilon \to \omega^* \neq 0.$$

It follows that

$$\begin{aligned} &u_\varepsilon \to u^* \quad \text{in } C^2(K_n, \mathbb{R}^n), \\ &{\omega^*}^2 \ddot{u}^* + V'(u^*) = 0 \quad \text{on } K_n. \end{aligned}$$

Since $\bigcup K_n = [0,1] \setminus J$, it follows that

$${\omega^*}^2 \ddot{u}^* + V'(u^*) = 0, \quad \forall\, t \in [0,1] \setminus J,$$

and $y^*(t) = u^*(\omega^* t)$ satisfies

$$\ddot{y}^* + V'(y^*) = 0, \quad \forall\, t \in [0,1] \setminus J.$$

The energy conservation (1.2) on $[0,1] \setminus J$ follows directly from (3.36). □

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