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Beyond orthodox semigroups

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ABSTRACT

We introduce the notions of a *generalised category* and of an *inductive generalised category over a band*. Our purpose is to describe a class of semigroups which we name *weakly B-orthodox*. In doing so we produce a new approach to characterising orthodox semigroups, by using *inductive generalised groupoids*. Here B denotes a band of idempotents; we note that if B is a semilattice then a weakly B-orthodox semigroup is exactly an Ehresmann semigroup. Weakly B-orthodox semigroups are analogues of orthodox semigroups, where the relations $\widetilde{\mathcal{R}}_B$ and $\widetilde{\mathcal{L}}_B$ play the role that $\mathcal{R}$ and $\mathcal{L}$ take in the regular case. We show that the category of weakly B-orthodox semigroups and admissible morphisms is isomorphic to the category of inductive generalised categories over bands and pseudo-functors. Our approach is influenced by Nambooripad's work on the connection between biordered sets and regular semigroups. However, there are significant differences in strategy, the first being the introduction of generalised categories and the second being that it is more convenient to consider (generalised) categories equipped with pre-orders, rather than with partial orders. Our work may be regarded as extending a result of Lawson for Ehresmann semigroups.

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Introduction

Our aim in this article is to introduce *generalised categories* and to use these to study *weakly B-orthodox semigroups*, a wide class containing all orthodox semigroups and all abundant semigroups with a band of idempotents. Our motivation is to build on the Ehresmann–Schein–Nambooripad (ESN) Theorem below, and its many extensions due to Armstrong [1,2], Lawson [8], Meakin [9,10] and Nambooripad [11–13].

Theorem A *(ESN Theorem). The category of inverse semigroups and morphisms is isomorphic to the category of inductive$_1$ groupoids and inductive$_1$ functors.*

An inductive$_1$ groupoid is a groupoid equipped with a partial order possessing restrictions and co-restrictions, and the set of idempotents forming a semilattice under the partial ordering. The subscript is used to distinguish this meaning of the word 'inductive' from both Ehresmann's use and a generalised definition which will occur below.

Inverse semigroups are precisely regular semigroups in which the idempotents form a semilattice. Consequently, we can regard the set of idempotents of a regular semigroup as a generalisation of a semilattice. This idea is precisely described in the definition of a regular biordered set, introduced by Nambooripad [11]. In that article, Nambooripad defined an inductive$_2$ groupoid to be a functorially ordered groupoid equipped with the structure of a regular biordered set on its identities, which is compatible with the ordered groupoid structure. This leads to a generalisation of Theorem A from a semilattice to a regular biordered set.

Theorem B. *(See Nambooripad [11].) The category of regular semigroups and morphisms is equivalent to the category of inductive$_2$ groupoids and inductive$_2$ functors.*

Note that for a technical reason, 'isomorphic' in Theorem A has been replaced by 'equivalent' in Theorem B. Of course, Theorem B may be specialised to orthodox semigroups. Theorem B was extended by Armstrong [1] from regular to concordant semigroups, replacing ordered groupoids by more general kinds of ordered categories.

Theorem C. *(See Armstrong [1].) The category of concordant semigroups and good morphisms is equivalent to the category of inductive$_2$ cancellative categories and inductive$_2$ functors.*

A concordant semigroup is an abundant semigroup with a regular biordered set of idempotents and satisfying the extra condition of being idempotent-connected (IC), which is a condition of a standard type that gives some control over the position of idempotents in products of elements of a semigroup.

Theorem A was generalised in a different direction to Ehresmann semigroups by Lawson [8]. His use of two partial orders on an Ehresmann semigroup is an important observation for the ideas discussed in this paper.

Theorem D. *(See Lawson [8].) The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.*

Ehresmann semigroups have a semilattice of idempotents, need not be regular or even abundant, need not satisfy an (IC) condition, and indeed need not be restriction semigroups. Lawson overcomes the lack of an (IC) condition by using two partial order relations. Our aim is to extend Lawson's result to the class of weakly B-orthodox semigroups, which extend the class of Ehresmann semigroups by replacing semilattices by bands. To this end we *could* use triples such as in [1], and this is the approach the second author takes in [15]. However, we take the opportunity to introduce *generalised categories* which we believe provide a clearer framework for this problem. Briefly, from a regular (concordant) semigroup one can produce a certain ordered category and then endow the category with a so-called pseudo-product. Unfortunately this need not produce the original semigroup: to do so requires factoring by a congruence. Our use of generalised categories circumvents this latter inconvenience. A further point is that we *could* use partial orders on a semigroup as standard in this area, but to do so would be rather clumsy. It turns out that without the (IC) condition and without the idempotents forming a semilattice, pre-orders provide the most elegant approach.

The structure of the paper is as follows. In Section 1 we give some basic definitions and make some elementary observations concerning weakly U-abundant semigroups, where U is a subset of idempotents of a semigroup. Of particular significance to us are the relations $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ on a semigroup S; if $U = E(S)$ and S is regular, then $\widetilde{\mathcal{L}}_U = \mathcal{L}$ and $\widetilde{\mathcal{R}}_U = \mathcal{R}$. In Section 2, we define generalised categories, inductive generalised categories over bands, and pseudo-functors. We show that the class of inductive generalised categories over bands and pseudo-functors forms a category. Section 3 constructs a weakly B-orthodox semigroup from an inductive generalised category over a band B.

Section 4 gives our main theorem, which is an analogue of Theorems A, B, C and D, connecting the category of weakly B-orthodox semigroups and that of inductive generalised categories over bands. We turn our attention to some special cases in Section 5, including orthodox semigroups, and in particular recover Theorem D. In Section 6, we change our angle a little to discuss the trace of weakly B-orthodox semigroups. Finally, in Section 7 we give an example of a weakly B-orthodox semigroup built from a monoid acting on the left and right of a band B subject to some compatibility conditions. This construction is inspired by that of the free ample monoid [3]. This paper may be regarded as the first step in describing weakly U-abundant semigroups where U is a regular biordered set, in terms of (generalised) ordered categories.

1. Preliminaries

In this section we list the notation and background results necessary for the rest of the paper. Further details of the relations defined below can be found in [5].

Let S be a semigroup. We denote as usual its set of idempotents by $E(S)$. Consider a non-empty subset $U \subseteq E(S)$; we will call it the set of *distinguished idempotents*. The relation $\leqslant_{\widetilde{\mathcal{L}}_U}$ on S is defined by the rule that for all $a, b \in S$, $a \leqslant_{\widetilde{\mathcal{L}}_U} b$ if and only if

$$\{e \in U\colon be = b\} \subseteq \{e \in U\colon ae = a\}.$$

It is clear that $\leqslant_{\widetilde{\mathcal{L}}_U}$ is a pre-order. We denote the associated equivalence relation by $\widetilde{\mathcal{L}}_U$, so that for $a, b \in S$, $a\,\widetilde{\mathcal{L}}_U\,b$ if and only if

$$\{e \in U\colon ae = a\} = \{e \in U\colon be = b\}.$$

It is easy to see that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$ and $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}_U$ if S is regular and $U = E(S)$. Moreover, $\mathcal{L}$ and $\mathcal{L}^*$ are always right compatible, although the same need not be true for $\widetilde{\mathcal{L}}_U$. The last fact is shown by a very simple example: the null semigroup of two elements with an adjoined identity.

Notice that for $e, f \in U$, $e \leqslant_{\widetilde{\mathcal{L}}_U} f$ if and only if $e \leqslant_{\mathcal{L}} f$, so that $e\,\widetilde{\mathcal{L}}_U\,f$ if and only if $e\,\mathcal{L}\,f$. Another useful observation is that if $a \in S$ and $e \in U$, then $a\,\widetilde{\mathcal{L}}_U\,e$ if and only if $ae = a$ and for all $f \in U$, $af = a$ implies that $ef = e$.

The relations $\leqslant_{\widetilde{\mathcal{R}}_U}$ and $\widetilde{\mathcal{R}}_U$ are the left-right duals of $\leqslant_{\widetilde{\mathcal{L}}_U}$ and $\widetilde{\mathcal{L}}_U$.

In a manner analogous to the definition of an abundant semigroup, S is said to be *weakly U-abundant* if every $\widetilde{\mathcal{L}}_U$-class and every $\widetilde{\mathcal{R}}_U$-class contains an idempotent of U. If S is such a semigroup and $a \in S$, then we follow usual practice and denote idempotents in the $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class of a by a^* and $a^\dagger$, respectively. Note that there need not be a unique choice for a^* and $a^\dagger$ unless U is a semilattice. The $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class containing a will be denoted by $\widetilde{L}_{U,a}$ and $\widetilde{R}_{U,a}$, respectively, abbreviated as $\widetilde{L}_a$ and $\widetilde{R}_a$.

We will be interested in semigroups S in which the relation $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence. In this case, we say that S satisfies the *Congruence Condition* (C) (with respect to U).

It is easy to see that morphisms between semigroups preserve Green's relations. They need not, however, preserve $\mathcal{L}^*$ and $\mathcal{R}^*$, nor $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$. With this in mind we define the notion of admissible morphisms.

Let S and T be semigroups with distinguished subsets of idempotents U and V, respectively, and let $\phi : S \to T$ be a morphism. Then ϕ is said to be (U, V)*-admissible* if for any $a, b \in S$,

$$a\,\widetilde{\mathcal{L}}_U\,b \quad \text{implies} \quad a\phi\,\widetilde{\mathcal{L}}_V\,b\phi,$$
$$a\,\widetilde{\mathcal{R}}_U\,b \quad \text{implies} \quad a\phi\,\widetilde{\mathcal{R}}_V\,b\phi,$$

and $U\phi \subseteq V$. Briefly, we will refer to the notion of being (U, V)-admissible as *admissible*, where no ambiguity can occur.

Lemma 1.1. *(See [14].) Let S, T be semigroups with distinguished subsets of idempotents U, V respectively. Suppose that S is weakly U-abundant, and let $\phi : S \to T$ be a morphism. Then ϕ is admissible if and only if $U\phi \subseteq V$ and for any $a \in S$ there exist idempotents $f \in \widetilde{L}_a \cap U$ and $e \in \widetilde{R}_a \cap U$ such that $a\phi\,\widetilde{\mathcal{L}}_V\,f\phi$ and $a\phi\,\widetilde{\mathcal{R}}_V\,e\phi$.*

We recall that an *orthodox semigroup* is a regular semigroup S such that $E(S)$ is a band. Consequently, a weakly B-abundant semigroup is said to be *weakly B-orthodox* if it has (C) and B is a band. This terminology, based on existing convention, needs to be viewed with care: if we talk of a *particular* weakly B-abundant semigroup, then we are referring to a *particular* band B; on the other hand, if we are talking of the *class* of all weakly B-abundant semigroups, the B varies over *all* bands. We say that a weakly B-orthodox semigroup satisfies (WIC) if for all $x \in S$, $x^\dagger$ in B (x^* in B) and $e \in B$ ($f \in B$) with $e \leqslant x^\dagger$ ($f \leqslant x^*$) there exists $g \in B$ ($h \in B$) with $ex = xg$ ($xf = hx$). It is easy to see that if S is orthodox, then S satisfies (WIC) (with respect to $E(S) = B$). For, if $x \in S$ and $e \in B$ ($f \in B$) with $e \leqslant x^\dagger$ ($f \leqslant x^*$), then $ex = xx'ex$ ($xf = xfx''x$), for some inverse x' (x'') of x. It is clear that the collection of weakly B-orthodox semigroups and admissible morphisms forms a category, which we denote by $\mathcal{WO}$.

For convenience we will make the convention that B will always denote a band. Green's relations and their associated pre-orders will always refer to B, unless stated otherwise. In particular, if S is weakly B-orthodox and $e \in B$, then R_e (L_e) denote the $\mathcal{R}$-class ($\mathcal{L}$-class) of e *in* B.

Lemma 1.2. *Let S be a weakly B-orthodox semigroup. For any $x, y \in S$ we have $(yx)^* \leqslant_{\mathcal{L}} x^*$ and $(xy)^\dagger \leqslant_{\mathcal{R}} x^\dagger$.*

Proof. Let $x, y \in S$. Clearly $(yx)^*x^*\,\widetilde{\mathcal{L}}_B\,yxx^* = yx$. Thus

$$(yx)^*\,\mathcal{L}\,(yx)^*x^* \leqslant_{\mathcal{L}} x^*.$$

Dually, we obtain that $(xy)^\dagger \leqslant_{\mathcal{R}} x^\dagger$. □

Lemma 1.3. *Let S be a weakly B-orthodox semigroup. For any $x \in S$ and $e, f, g, h \in B$,*

(i) *if $e \leqslant_{\mathcal{R}} g$ or $e \leqslant_{\mathcal{L}} g$ and $g\,\mathcal{R}\,x^\dagger$, then $ex\,\widetilde{\mathcal{R}}_B\,e$;*
(ii) *if $f \leqslant_{\mathcal{L}} h$ or $f \leqslant_{\mathcal{R}} h$ and $h\,\mathcal{L}\,x^*$, then $xf\,\widetilde{\mathcal{L}}_B\,f$.*

Proof. To prove (i), suppose that $e \leqslant_{\mathcal{R}} g\,\mathcal{R}\,x^\dagger$, then $ex\,\widetilde{\mathcal{R}}_B\,ex^\dagger\,\widetilde{\mathcal{R}}_B\,eg\,\mathcal{R}\,e$, otherwise, $e \leqslant_{\mathcal{L}} g$, and so $ex\,\widetilde{\mathcal{R}}_B\,eg = e$. By a similar argument, we can show that (ii) holds. □

We now present pre-orderings on a weakly B-orthodox semigroup, which can be considered as an analogue of the orderings on Ehresmann semigroups studied by Lawson [8].

Let S be a weakly B-orthodox semigroup. We define relations $\leqslant_r$ and $\leqslant_l$ by the rule that for any $x, y \in S$,

$$x \leqslant_r y \quad \text{if and only if} \quad x = ey \quad \text{for some } e \in B,$$

and

$$x \leqslant_l y \quad \text{if and only if} \quad x = yf \quad \text{for some } f \in B.$$

Since B is a band, the following lemma is clear.

Lemma 1.4. *On a weakly B-orthodox semigroup S, the relations $\leqslant_r$ and $\leqslant_l$ given above are pre-orderings.*

The next lemma is an immediate consequence of Lemma 1.2.

Lemma 1.5. *Let S be a weakly B-orthodox semigroup and x, y be elements of S.*

(i) *If $x \leqslant_r y$, then $x^* \leqslant_{\mathcal{L}} y^*$.*
(ii) *If $x \leqslant_l y$, then $x^\dagger \leqslant_{\mathcal{R}} y^\dagger$.*

We remark that if B is a semilattice, then the above relations become partial orders. For if $x \leqslant_r y \leqslant_r x$ and $x = ey$ and $y = fx$ for some $e, f \in B$, then from $ex = x$ we deduce that $ey = efx = fex = fx = y$. Thus $x = y$.

The astute reader will point out that Armstrong uses partial orderings in [1]. Indeed a weakly B-orthodox semigroup possesses a pair of partial orders, as we now demonstrate. However, for our purpose, pre-orders are more convenient.

Let S be weakly B-orthodox. We define relations $\leqslant'_r$ and $\leqslant'_l$ on S by the rule that for $x, y \in S$,

$$x \leqslant'_r y \quad \text{if and only if} \quad x = ey \quad \text{for some } e \in B \text{ and } x^\dagger \leqslant_{\mathcal{R}} y^\dagger;$$

and

$$x \leqslant'_l y \quad \text{if and only if} \quad x = yf \quad \text{for some } f \in B \text{ and } x^* \leqslant_{\mathcal{L}} y^*.$$

Lemma 1.6. *Let S be a weakly B-orthodox semigroup. Then $\leqslant'_r$ and $\leqslant'_l$ are partial orders on S. If in addition S satisfies condition (WIC), then $\leqslant'_l = \leqslant'_r$.*

Proof. It is clear that $\leqslant'_r$ is reflexive and transitive. If $x \leqslant'_r y \leqslant'_r x$ and $x = ey$ and $y = fx$ where $e, f \in B$ and $x^\dagger \leqslant_{\mathcal{R}} y^\dagger \leqslant_{\mathcal{R}} x^\dagger$, then clearly $x \widetilde{\mathcal{R}}_B y$. Hence from $ex = x$ we deduce that $x = ey = y$. Thus $\leqslant'_r$ is a partial order; dually for $\leqslant'_l$.

Suppose now that S has (WIC) and $x \leqslant'_r y$. Then $x = ey$ for some $e \in B$ and $x^\dagger \leqslant_{\mathcal{R}} y^\dagger$. We have $x = y^\dagger x = y^\dagger e y^\dagger y = yf$ for some $f \in B$, since $y^\dagger e y^\dagger \leqslant y^\dagger$. Clearly as $x = ey$ we have $x^* \leqslant_{\mathcal{L}} y^*$. Hence $x \leqslant'_l y$. Dually, $\leqslant'_l \subseteq \leqslant'_r$, so that the two relations coincide. □

We note that the concordant semigroups studied in [1] satisfy (WIC). We call $\leqslant_r$ and $\leqslant_l$ ($\leqslant'_r$ and $\leqslant'_l$) the *natural pre-orders* (*natural partial orders*) of a weakly B-orthodox semigroup S.

We remark that if E is a semilattice, then a weakly E-orthodox semigroup is an *Ehresmann semigroup* (with distinguished semilattice E). It is easy to see that in such a semigroup, every $\widetilde{\mathcal{R}}_E$-class and every $\widetilde{\mathcal{L}}_E$-class contains a unique idempotent of E. Thus in an Ehresmann semigroup, we have two unary operations given by $a \mapsto a^\dagger$ and $a \mapsto a^*$. We may therefore regard Ehresmann semigroups as algebras with signature $(2, 1, 1)$; as such, they form a variety $\mathcal{E}$, which is generated by the quasi-variety of adequate semigroups [6]. The corresponding result in the one-side case may be found in [4] or [7].

Lemma 1.7. *Let S be an Ehresmann semigroup with distinguished semilattice E. Then $\leqslant_r = \leqslant'_r$ and $\leqslant_l = \leqslant'_l$, so that $\leqslant_r$ and $\leqslant_l$ are partial orders.*

Proof. We have already remarked that $\leqslant_r'$ and $\leqslant_l'$ are partial orders. Further, we notice that if $x \leqslant_r y$, where $x = ey$ for some $e \in E$, then $x^\dagger = (ey)^\dagger = ey^\dagger = y^\dagger e$, so that $x^\dagger \leqslant y^\dagger$ and $x \leqslant_r' y$; dually, $\leqslant_l = \leqslant_l'$. $\square$

2. Inductive generalised categories

Let I, R, L and D be disjoint sets and let p denote a collection of four (well-defined) onto maps:

$$I \twoheadrightarrow R, \quad I \twoheadrightarrow L, \quad R \twoheadrightarrow D \quad \text{and} \quad L \twoheadrightarrow D,$$
$$i \mapsto R_i, \quad i \mapsto L_i, \quad R_i \mapsto D_i, \qquad L_i \mapsto D_i$$

such that

$$\begin{tikzcd} & I \arrow[dl] \arrow[dr] & \\ R \arrow[dr] & & L \arrow[dl] \\ & D & \end{tikzcd}$$

commutes. We denote this configuration by (I, R, L, D, p) and refer to it as a *context*.

We pause to give our motivating example. Let B be a band and p denote the natural maps:

$$B \twoheadrightarrow B/\mathcal{R}, \quad B \twoheadrightarrow B/\mathcal{L}, \quad B/\mathcal{R} \twoheadrightarrow B/\mathcal{D} \quad \text{and} \quad B/\mathcal{L} \twoheadrightarrow B/\mathcal{D}.$$

Then $(B, B/\mathcal{R}, B/\mathcal{L}, B/\mathcal{D}, p)$ is a context. Of course, if B is a semilattice, then all of Green's relations are trivial and the p-maps are essentially the identity maps.

Definition 2.1. A *generalised category* P over a context (I, R, L, D, p) consists of

(GC1) a class $\mathrm{ob}(P)$ of objects $R \,\dot{\cup}\, L$;
(GC2) a class $\hom(P)$ of morphisms between the objects. Each morphism x has a unique domain $\mathbf{d}(x) \in R$ and codomain $\mathbf{r}(x) \in L$. Denote the hom-class of all morphisms from $R_i \in R$ to $L_j \in L$ by $\hom(R_i, L_j)$;
(GC3) if $R_i, R_k \in R$ and $L_j, L_h \in L$ with $D_j = D_k$, then there is a binary operation

$$\hom(R_i, L_j) \times \hom(R_k, L_h) \to \hom(R_i, L_h), \quad (x, y) \mapsto x \cdot y$$

called *composition of morphisms* such that if $x \in \hom(R_i, L_j)$, $y \in \hom(R_k, L_h)$, and $z \in \hom(R_m, L_n)$, where $D_j = D_k$ and $D_h = D_m$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(GC4) for each $i \in I$, there exists a distinguished morphism, again denoted by i, such that $i \in \hom(R_i, L_i)$ and if $\mathbf{d}(x) = R_i$ and $\mathbf{r}(y) = L_i$, then $i \cdot x = x$ and $y \cdot i = y$.

Let P be a generalised category over a context (I, R, L, D, p). Following the usual convention when building categories from semigroups, we may identify $\hom(P)$ with P. If B is a band and P is a generalised category over $(B, B/\mathcal{R}, B/\mathcal{L}, B/\mathcal{D}, p)$, where p denotes the natural maps, then we say simply P is a *generalised category over* B.

Our notion of generalised category is motivated by that of the 'trace product' of a weakly B-orthodox semigroup. We explain this in Section 6 but comment briefly here on the special case of a band.

We have seen that if B is a band, then $(B, B/\mathcal{R}, B/\mathcal{L}, B/\mathcal{D}, p)$ is a context. Define a generalised category P over B by putting $\hom(P) = B$ and for $e \in B$, put $\mathbf{d}(e) = R_e$ and $\mathbf{r}(e) = L_e$. Let the partial binary operation be given by $e \cdot f = ef$, where $e \cdot f$ exists. Note the latter is true if and only if $D_e = D_f$. Thus the effect of our generalised category is to restrict the multiplication in B to that within its $\mathcal{D}$-classes.

We now focus on generalised categories over a band B, in general more extensive than the example above, making use of the natural partial order on $B/\mathcal{R}$ and $B/\mathcal{L}$. Note that if $e \in B$ then by (GC4) we have that $e \in \hom(R_e, L_e)$, so that $\mathbf{d}(e) = R_e$ and $\mathbf{r}(e) = L_e$.

We build on Definition 2.1 to define an inductive generalised category over B, which is an analogue of inductive$_2$ groupoids [11] and inductive$_2$ cancellative categories [1]. We will see that the elements of our inductive generalised category may be pre-ordered or partially ordered, in two ways, reflecting the approach of [8].

Definition 2.2. Let P be a generalised category over a band B. Then P is an *inductive* generalised category if the following conditions and the duals (I1)°, (I2)°, and (I3)° of (I1), (I2) and (I3) hold:

(I1) if $x \in P$ and $e, u \in B$ with $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$, then there exists an element ${}_e|x$ in P, called the *restriction* of x to e, such that $e \in \mathbf{d}({}_e|x)$ and $\mathbf{r}({}_e|x) \leqslant_{\mathcal{L}} \mathbf{r}(x)$; in particular, if $e \in \mathbf{d}(x)$, then ${}_e|x = x$;
(I2) if $x \in P$ and $e, f, g, u \in B$ with $e \leqslant_{\mathcal{L}} g \,\mathcal{R}\, f \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$, then ${}_{ef}|x = {}_e|({}_f|x)$;
(I3) if $x, y \in P$ and $e, u \in B$ with $x \cdot y$ defined in P and $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$, then ${}_e|(x \cdot y) = ({}_e|x) \cdot ({}_f|y)$, where $f \in \mathbf{r}({}_e|x)$;
(I4) if $x, y \in P$ and $e_1, e_2, f_1, f_2 \in B$ with $e_1, e_2 \in \mathbf{r}(x)$ and $f_1, f_2 \in \mathbf{d}(y)$, then $x|_{e_1 f_1} \cdot {}_{e_1 f_1}|y = x|_{e_2 f_2} \cdot {}_{e_2 f_2}|y$;
(I5) if $x \in P$ and $e, f, u, v, g, h \in B$ with $g \in \mathbf{r}(x)$, $h \in \mathbf{d}(x)$, $u \in \mathbf{d}(x|_{gf})$ and $v \in \mathbf{r}({}_{eh}|x)$, then ${}_{eu}|(x|_{gf}) = ({}_{eh}|x)|_{vf}$;
(I6) if $e, g, h, u, v \in B$ are such that $u \leqslant_{\mathcal{R}} g \,\mathcal{L}\, e$ and $v \leqslant_{\mathcal{L}} h \,\mathcal{R}\, e$, then $e|_u = eu$ and ${}_v|e = ve$.

We make some comments on the above definition. In (I3) let $\mathbf{r}(x) = L_v$ and $\mathbf{d}(y) = R_w$. Since there exists $x \cdot y$ we know that $v \,\mathcal{D}\, w$ so we have $\mathbf{r}(x) = L_{wv}$ and $\mathbf{d}(y) = R_{wv}$. Hence by (I1), $f \in \mathbf{r}({}_e|x) \leqslant_{\mathcal{L}} L_{wv}$ and $wv \in \mathbf{d}(y)$, so that ${}_f|y$ exists and $\mathbf{d}({}_f|y) = R_f$. Hence $({}_e|x) \cdot ({}_f|y)$ exists. To simplify the term "$x \cdot y$ exists" may use the expression "$\exists x \cdot y$" or "$x \cdot y$ is defined".

Suppose now that P is a generalised category over a band B. We remarked above that if $e \in B$, then $\mathbf{d}(e) = R_e$ and $\mathbf{r}(e) = L_e$, so that if also $f \in B$ then $\exists e \cdot f$ if and only if $e \,\mathcal{D}\, f$. In this case, clearly $e \in \mathbf{d}(e)$, $e \in \mathbf{d}(e \cdot f)$ and by (I1), ${}_e|e = e$, so that $e \in \mathbf{r}({}_e|e)$. Using (I1), (I3), (I6) and (GC4) we have

$$e \cdot f =_e |(e \cdot f) = ({}_e|e) \cdot ({}_e|f) = e \cdot ef = ef.$$

We pause to introduce a pair of pre-orderings on an inductive generalised category P over a band B deduced from Definition 2.2. We make use of the restriction and co-restriction of P to define relations $\leqslant_r$ and $\leqslant_l$ by the rule that for any $x, y \in P$,

$$x \leqslant_r y \quad \text{if and only if} \quad x = {}_e|y \quad \text{for some } e \in B,$$

and

$$x \leqslant_l y \quad \text{if and only if} \quad x = y|_f \quad \text{for some } f \in B.$$

Lemma 2.3. *The relations $\leqslant_r$ and $\leqslant_l$ are pre-orderings on P.*

Proof. To prove that $\leqslant_r$ is a pre-ordering on P, we first observe that $\leqslant_r$ is reflexive by (I1). It is necessary to show that $\leqslant_r$ is transitive. Assume that $x, y, z \in P$ with $x \leqslant_r y$ and $y \leqslant_r z$. Then there exist $e, f \in B$ such that $x = {}_e|y$ and $y = {}_f|z$. For ${}_e|y$ and ${}_f|z$ to exist we have $e \leqslant_{\mathcal{L}} g \in \mathbf{d}(y) = R_f$ and $f \leqslant_{\mathcal{L}} h \in \mathbf{d}(z)$. From (I2), $x = {}_e|({}_f|z) = {}_{ef}|z$. Hence $x \leqslant_r z$.

By the dual argument, we show that $\leqslant_l$ is a pre-ordering on P. $\square$

The reader might notice that previous articles have used partial orders rather than pre-orders. For our purpose, pre-orders are easier to use, but the partial orders are still there, as we now show.

We define $\leqslant_r'$ and $\leqslant_l'$ on P by the rule that

$$x \leqslant_r' y \quad \text{if and only if} \quad x = {}_e|y \quad \text{for some } e \leqslant u \in \mathbf{d}(y),$$

and

$$x \leqslant_l' y \quad \text{if and only if} \quad x = y|_f \quad \text{for some } f \leqslant v \in \mathbf{r}(y).$$

Lemma 2.4. *The relations $\leqslant_r'$ and $\leqslant_l'$ are partial orders on P.*

Proof. As in Lemma 2.3, $\leqslant_r'$ is reflexive. If $x \leqslant_r' y$ and $y \leqslant_r' z$ then with e, f as in Lemma 2.3, we have $e \leqslant g$ and $f \leqslant h$. Certainly, $x = {}_{ef}|z$ and $efh = ef$, as $f \leqslant h$. Also, $e \leqslant g \mathcal{R} f \leqslant h$, so $hef = ef$. Hence $ef \leqslant h \in \mathbf{d}(z)$.

Finally, suppose that $x \leqslant_r' y \leqslant_r' x$. Then $x = {}_e|y$ and $y = {}_f|x$ for some $e \leqslant u \in \mathbf{d}(y)$ and $f \leqslant v \in \mathbf{d}(x)$. We have $e \leqslant u \mathcal{R} f$ and $f \leqslant v \mathcal{R} e$, so that $e \mathcal{R} f$ and $\mathbf{d}(x) = \mathbf{d}(y)$. Now $x = {}_e|y = y$, by (I1). $\square$

We say that $\leqslant_r$ and $\leqslant_l$ are the *natural pre-orders associated with* P and $\leqslant_r'$ and $\leqslant_l'$ are the *natural partial orders associated with* P.

We end this section by showing that the class of inductive generalised categories over bands forms a category, together with certain maps referred to as pseudo-functors. They appear in the next definition.

Definition 2.5. Let P_1 and P_2 be inductive generalised categories over bands B_1 and B_2, respectively. A *pseudo-functor* F from P_1 to P_2 is a pair of maps, both denoted F, from B_1 to B_2 and from P_1 to P_2, such that the following conditions and the dual $(F2)^\circ$ of (F2) hold:

(F1) the map F is a morphism from B_1 to B_2;
(F2) if $e \in B_1$ and $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$ in P_1, then $F({}_e|x) = {}_{F(e)}|F(x)$;
(F3) if $\exists x \cdot y$ in P_1 then $\exists F(x) \cdot F(y)$ in P_2, and $F(x \cdot y) = F(x) \cdot F(y)$.

To see that (F2) makes sense, suppose that $u \in B_1$, $x \in P_1$ with $u \in \mathbf{d}(x)$. Then $R_u = \mathbf{d}(x)$ so that $\exists u \cdot x$ and $u \cdot x = x$. By (F3), $\exists F(u) \cdot F(x)$ and $F(u) \cdot F(x) = F(x)$. Hence $\mathbf{d}(F(x)) = \mathbf{d}(F(u)) = R_{F(u)}$, as $F(u) \in B_2$. Suppose also that $e \in B_1$, with $e \leqslant_{\mathcal{L}} u$ in B_1. Using (F1), we have $F(e) \leqslant_{\mathcal{L}} F(u) \in \mathbf{d}(F(x))$ in P_2, and so $\exists_{F(e)}|F(x)$. Notice that we can define F on $\mathrm{ob}(P_1)$ by putting $F(R_e) = R_{F(e)}$ and $F(L_e) = L_{F(e)}$.

From the comments above, it is easy to check that Lemma 2.6 holds.

Lemma 2.6. *Let P_1, P_2 and P_3 be inductive generalised categories over B_1, B_2 and B_3, respectively, and let $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ be pseudo-functors. Then $F_2F_1 : P_1 \to P_3$ is a pseudo-functor.*

The next observation follows immediately.

Lemma 2.7. *The class of inductive generalised categories over bands, together with pseudo-functors, forms a category.*

We refer to the category in the above lemma as $\mathcal{IGC}$.

3. Construction

Our primary interest in this section will be a construction of a weakly B-orthodox semigroup, built from an inductive generalised category over B.

Let P be an inductive generalised category over a band B. We define the pseudo-product $\otimes$ on P by

$$x \otimes y = (x|_{ef}) \cdot ({}_{ef}|y),$$

where $e \in \mathbf{r}(x)$, $f \in \mathbf{d}(y)$. It follows from (I4) that the pseudo-product is independent of the choices of e and f and thus is well-defined. We will denote the set P, together with the pseudo-product $\otimes$, by $\mathbf{S}(P)$.

We pause to present our initial idea which follows Armstrong's steps, using the notion of sandwich set, simplifying a little here as our set of idempotents forms a band. We may define a pseudo-product $\otimes'$ on P by the rule that for any $x, y \in P$,

$$x \otimes' y = (x|_{efe}) \cdot ({}_{fef}|y),$$

where $e \in \mathbf{r}(x)$ and $f \in \mathbf{d}(x)$. In that case, condition (I5) is not enough to guarantee that $\otimes'$ is associative in P. To achieve this it is necessary to add a stronger condition in place of (I5), which effectively says that $e \otimes' (x \otimes' y) = (e \otimes' x) \otimes' y$ for any $x, y \in P$ and $e \in B$. This appears to us too contrived. Keeping this in mind we use the pseudo-product $\otimes$ defined as above.

We now present a series of lemmas related to P, which will help us to show our main result at the end of this section.

Lemma 3.1. *If $x, y \in P$ with $\exists x \cdot y$, then $x \otimes y = x \cdot y$.*

Proof. If $\exists x \cdot y$ then $\mathbf{r}(x) = L_e$ and $\mathbf{d}(y) = R_f$ say, where $e \mathcal{D} f$. Then $\mathbf{r}(x) = L_{fe}$ and $\mathbf{d}(y) = R_{fe}$, so $x \otimes y = (x|_{fefe}) \cdot ({}_{fefe}|y) = (x|_{fe}) \cdot ({}_{fe}|y) = x \cdot y$ by (I1). $\square$

Lemma 3.2. *If $e, f \in B$ then $e \otimes f = ef$.*

Proof. We have

$$\begin{aligned} e \otimes f &= (e|_{ef}) \cdot ({}_{ef}|f) \\ &= eef \cdot eff \quad \big(\text{by (I6)}\big) \\ &= ef \cdot ef = ef \quad \big(\text{by (GC4)}\big). \qquad \square \end{aligned}$$

Consequently, B forms the same band under $\otimes$ and the original multiplication.

Lemma 3.3. *If $x \in P$ and $e, f, u \in B$ with $u \mathcal{D} e \leqslant_{\mathcal{L}} f \in \mathbf{d}(x)$ then $u \cdot ({}_e|x) = {}_{ue}|x$.*

Proof. Since $u \mathcal{D} e$, we deduce that

$$\begin{aligned} u \cdot {}_e|x &= u \otimes {}_e|x \quad (\text{Lemma 3.1}) \\ &= (u|_{ue}) \cdot \big({}_{ue}|({}_e|x)\big) \\ &= ue \cdot ({}_{ue}|x) \quad \big(\text{by (I6), (I2)}\big) \\ &= {}_{ue}|x \quad \big(\text{by (GC4)}\big). \qquad \square \end{aligned}$$

Lemma 3.4. *The set* $\mathbf{S}(P)$ *forms a semigroup under the operation* $\otimes$.

Proof. It is sufficient to show that $\mathbf{S}(P)$ is associative. Suppose that $x, y, z \in P$ with $x^* \in \mathbf{r}(x)$, $y^\dagger \in \mathbf{d}(y)$, $y^* \in \mathbf{r}(y)$ and $z^\dagger \in \mathbf{d}(z)$. Then

$$\begin{aligned} x \otimes (y \otimes z) &= x \otimes \big((y|_{y^*z^\dagger}) \cdot ({}_{y^*z^\dagger}|z)\big) \\ &= (x|_{x^*u}) \cdot \big({}_{x^*u}|\big((y|_{y^*z^\dagger}) \cdot ({}_{y^*z^\dagger}|z)\big)\big) \quad \big(u \in \mathbf{d}(y|_{y^*z^\dagger})\big) \\ &= (x|_{x^*u}) \cdot \big({}_{x^*u}|(y|_{y^*z^\dagger})\big) \cdot \big({}_v|({}_{y^*z^\dagger}|z)\big) \quad \big(v \in \mathbf{r}\big({}_{x^*u}|(y|_{y^*z^\dagger})\big),\ \text{by (I3)}\big). \end{aligned}$$

Notice, by (I1), that $v \leqslant_{\mathcal{L}} y^*z^\dagger \in \mathbf{r}(y|_{y^*z^\dagger})$ and by (I5), that

$${}_{x^*u}|(y|_{y^*z^\dagger}) = ({}_{x^*y^\dagger}|y)|_{gz^\dagger},$$

where $g \in \mathbf{r}({}_{x^*y^\dagger}|y)$, and so $v \,\mathcal{L}\, gz^\dagger$ and $x^*u \in \mathbf{d}(({}_{x^*y^\dagger}|y)|_{gz^\dagger})$. Thus

$$\begin{aligned} x \otimes (y \otimes z) &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot \big({}_v|({}_{y^*z^\dagger}|z)\big) \\ &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{vy^*z^\dagger}|z) \quad \big(v \leqslant_{\mathcal{L}} y^*z^\dagger,\ \text{by (I2)}\big) \\ &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_v|z) \quad \big(v \leqslant_{\mathcal{L}} y^*z^\dagger\big) \\ &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{vgz^\dagger}|z) \quad \big(v \,\mathcal{L}\, gz^\dagger\big) \\ &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot \big(v \cdot ({}_{gz^\dagger}|z)\big) \quad (\text{Lemma 3.3}) \\ &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger} \cdot v\big) \cdot ({}_{gz^\dagger}|z) \\ &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \quad \big(v \,\mathcal{L}\, gz^\dagger,\ \text{by (GC4)}\big). \end{aligned}$$

Due to the dual of (I1), $u \in \mathbf{d}(y|_{y^*z^\dagger}) \leqslant_{\mathcal{R}} \mathbf{d}(y)$, whence $x^*y^\dagger x^*u = x^*y^\dagger x^*y^\dagger u = x^*y^\dagger u = x^*u$. So

$$\begin{aligned} x \otimes (y \otimes z) &= (x|_{x^*y^\dagger x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \\ &= ((x|_{x^*y^\dagger})|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \quad \big(\text{by (I2)}^\circ,\ \text{since } x^*u \leqslant_{\mathcal{R}} x^*y^\dagger\big) \\ &= \big(\big((x|_{x^*y^\dagger}) \cdot ({}_{x^*y^\dagger}|y)\big)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \quad \big(x^*u \in \mathbf{d}\big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big)\big) \\ &= \big((x \otimes y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \\ &= (x \otimes y) \otimes z. \qquad \square \end{aligned}$$

The following lemma shows that $\mathbf{S}(P)$ is a weakly B-abundant semigroup.

Lemma 3.5. *Let* $x \in \mathbf{S}(P)$, $e \in \mathbf{r}(x)$ *and* $g \in \mathbf{d}(x)$. *Then* $g \,\widetilde{\mathcal{R}}_B\, x \,\widetilde{\mathcal{L}}_B\, e$ *in* $\mathbf{S}(P)$.

Proof. By Lemma 3.1, we obtain that $x \otimes e = x \cdot e = x$. Suppose that $k \in B$ and $x \otimes k = x$. Then

$$\begin{aligned} x \otimes k &= (x|_{ek}) \cdot ({}_{ek}|k) \\ &= (x|_{ek}) \cdot ek \quad \big(\text{by (I6)}\big) \\ &= x|_{ek} \quad \big(\text{by (GC4)}\big). \end{aligned}$$

Thus $x = x|_{ek}$, which implies that $ek \in \mathbf{r}(x)$, and so $e\,\mathcal{L}\,ek$. It follows that

$$\begin{aligned} e \otimes k &= ek \quad (\text{Lemma 3.2}) \\ &= eek \\ &= e \quad (e\,\mathcal{L}\,ek). \end{aligned}$$

Consequently, $x\,\widetilde{\mathcal{L}}_B\,e$.

Similarly, we can show that $x\,\widetilde{\mathcal{R}}_B\,g$. $\Box$

As an application of Lemma 3.5, we give a concrete description of relations $\leqslant_{\widetilde{\mathcal{R}}_B}$ and $\leqslant_{\widetilde{\mathcal{L}}_B}$ on $\mathbf{S}(P)$ as follows.

Lemma 3.6. *For any $x, y \in \mathbf{S}(P)$,*

(i) $x \leqslant_{\widetilde{\mathcal{R}}_B} y$ *if and only if* $\mathbf{d}(x) \leqslant_{\mathcal{R}} \mathbf{d}(y)$;
(ii) $x \leqslant_{\widetilde{\mathcal{L}}_B} y$ *if and only if* $\mathbf{r}(x) \leqslant_{\mathcal{L}} \mathbf{r}(y)$.

Proof. We prove (i). Let $x, y \in P$ and let $\mathbf{d}(x) = R_e$ and $\mathbf{d}(y) = R_f$. Then

$$\begin{aligned} x \leqslant_{\widetilde{\mathcal{R}}_B} y \quad \text{in } \mathbf{S}(P) \quad &\Leftrightarrow \quad e \leqslant_{\widetilde{\mathcal{R}}_B} f \quad \text{in } \mathbf{S}(P) \quad (\text{Lemma 3.5}) \\ &\Leftrightarrow \quad e \leqslant_{\mathcal{R}} f \quad \text{in } B \\ &\Leftrightarrow \quad R_e \leqslant_{\mathcal{R}} R_f \\ &\Leftrightarrow \quad \mathbf{d}(x) \leqslant_{\mathcal{R}} \mathbf{d}(y). \qquad \Box \end{aligned}$$

Now let us sum up results related to $\mathbf{S}(P)$ in the following theorem:

Theorem 3.7. *If P is an inductive generalised category over B, then $(\mathbf{S}(P), \otimes)$ is a weakly B-orthodox semigroup. Further, the natural pre-orders and partial orders in P and $\mathbf{S}(P)$ coincide.*

Proof. We first show that $(\mathbf{S}(P), \otimes)$ has (C). Suppose that $x, y, z \in \mathbf{S}(P)$ and $x\,\widetilde{\mathcal{R}}_B\,y$. It follows from Lemma 3.6 that $\mathbf{d}(x) = \mathbf{d}(y)$. We deduce that $z \otimes x = (z|_{ve}) \cdot ({}_{ve}|x)$ and $z \otimes y = (z|_{ve}) \cdot ({}_{ve}|y)$, where $v \in \mathbf{r}(z)$ and $e \in \mathbf{d}(x) = \mathbf{d}(y)$. Hence $\mathbf{d}(z \otimes x) = \mathbf{d}(z|_{ve}) = \mathbf{d}(z \otimes y)$. By Lemma 3.6, $z \otimes x\,\widetilde{\mathcal{R}}_B\,z \otimes y$. Dually, we can show that $\widetilde{\mathcal{L}}_B$ is a right congruence.

Let $x, y \in P$ and suppose that $x \leqslant_r y$ in P. Then $x = {}_e|y$ for some $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(y)$. Hence

$$e \otimes y = e|_{eu} \cdot {}_{eu}|y = eu \cdot {}_{eu}|y = {}_{eu}|y = {}_e|y = x,$$

so that $x \leqslant_r y$ in $\mathbf{S}(P)$.

If in addition we have $e \leqslant u$, so that $x \leqslant'_r y$ in P, then from $x = {}_e|y$ we have $\mathbf{d}(x) = R_e$ and $x^{\dagger} \leqslant_{\mathcal{R}} y^{\dagger}$, by Lemma 3.6, so $x \leqslant'_r y$ in $\mathbf{S}(P)$.

Conversely, if $x \leqslant_r y$ in $\mathbf{S}(P)$, then $x = f \otimes y$ for some $f \in B$. Hence,

$$x = f \otimes y = f|_{fy^{\dagger}} \cdot {}_{fy^{\dagger}}|y = fy^{\dagger} \cdot {}_{fy^{\dagger}}|y = {}_{fy^{\dagger}}|y,$$

so that $x \leqslant_r y$ in P.

Further, if $x \leqslant'_r y$ in $\mathbf{S}(P)$, then we have $x^{\dagger} \leqslant_{\mathcal{R}} y^{\dagger}$, so that $\mathbf{d}(x) \leqslant_{\mathcal{R}} \mathbf{d}(y)$, that is, $fy^{\dagger} \leqslant_{\mathcal{R}} y^{\dagger}$. Clearly then $fy^{\dagger} \leqslant y^{\dagger}$, so that $x \leqslant'_r y$ in P.

The dual result holds for $\leqslant_l$ and $\leqslant'_l$. $\Box$

We can obtain an admissible morphism between weakly B-orthodox semigroups from a pseudo-functor between inductive generalised categories over bands. This is made more precise in the following lemma.

Lemma 3.8. *Let $F : P_1 \to P_2$ be a pseudo-functor between inductive generalised categories P_1 and P_2, where P_1 and P_2 are over bands B_1 and B_2, respectively. Then the map $\mathbf{S}(F) : \mathbf{S}(P_1) \to \mathbf{S}(P_2)$ defined by the rule that $\mathbf{S}(F)(x) = F(x)$, where $x \in \mathbf{S}(P_1)$, is an admissible morphism; moreover, if $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ are pseudo-functors, then $\mathbf{S}(F_2 F_1) = \mathbf{S}(F_2)\mathbf{S}(F_1)$.*

Proof. We claim first that $\mathbf{S}(F)$ is a semigroup morphism. Suppose that $x, y \in \mathbf{S}(P_1)$. Then by the definition of $\mathbf{S}(F)$,

$$\begin{aligned}
\mathbf{S}(F)(x \otimes y) &= F(x \otimes y) \\
&= F\big((x|_{fu}) \cdot ({}_{fu}|y)\big) \quad \big(f \in \mathbf{r}(x),\ u \in \mathbf{d}(y)\big) \\
&= F(x|_{fu}) \cdot F({}_{fu}|y) \quad \big(\text{by (F3)}\big) \\
&= \big(F(x)|_{F(fu)}\big) \cdot \big({}_{F(fu)}|F(y)\big) \quad \big(\text{by (F2), (F2)}^{\circ}\big) \\
&= \big(F(x)|_{F(f)F(u)}\big) \cdot \big({}_{F(f)F(u)}|F(y)\big) \quad \big(\text{by (F1)}\big).
\end{aligned}$$

Since $f \in \mathbf{r}(x)$ and $u \in \mathbf{d}(y)$, it follows from the comments succeeding Definition 2.5 that $F(f) \in \mathbf{r}(F(x))$ and $F(u) \in \mathbf{d}(F(y))$. Thus,

$$\mathbf{S}(F)(x \otimes y) = F(x) \otimes F(y) = \mathbf{S}(F)(x) \otimes \mathbf{S}(F)(y).$$

We now show that $\mathbf{S}(F)$ is admissible. Clearly, by (F1), $\mathbf{S}(F)(B_1) \subseteq B_2$. For any $e \in \mathbf{r}(x)$, we have $e\,\widetilde{\mathcal{L}}_{B_1}\,x$ and $F(e) \in \mathbf{r}(F(x))$. Thus, $F(e)\,\widetilde{\mathcal{L}}_{B_2}\,F(x)$, that is, $\mathbf{S}(F)(e)\,\widetilde{\mathcal{L}}_{B_2}\,\mathbf{S}(F)(x)$. By a similar argument, we have that for any $k \in \mathbf{d}(x)$, $\mathbf{S}(F)(k)\,\widetilde{\mathcal{R}}_{B_2}\,\mathbf{S}(F)(x)$. By Lemma 1.1, $\mathbf{S}(F)$ is an admissible morphism between weakly B-orthodox semigroups $\mathbf{S}(P_1)$ and $\mathbf{S}(P_2)$.

The final part of the lemma is clear. □

Theorem 3.7 and Lemma 3.8 show that $\mathbf{S} : \mathcal{IGC} \to \mathcal{WO}$ is a functor.

4. Correspondence

In Section 3, we start with an inductive generalised category over B and construct a weakly B-orthodox semigroup. Our present aim is to prove a converse to this result and thus provide a correspondence between the class of inductive generalised categories over bands and the class of weakly B-orthodox semigroups.

Let S be a weakly B-orthodox semigroup. We define $C(S)$ to be the set S equipped with the following partial binary operation:

$$x \cdot y = \begin{cases} xy & \text{if } x^* \,\mathcal{D}\, y^\dagger \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S. This is known as the *trace product* and denoted by $C(S) = (S, \cdot)$.

It is immediate that if $e, f \in B$ and $x \in S$ are such that $e\,\widetilde{\mathcal{R}}_B\,x\,\widetilde{\mathcal{L}}_B\,f$ then $e \cdot x = x = x \cdot f$.

We now turn to give a number of basic properties of $C(S)$, which will be found useful in the sequel.

Lemma 4.1. *If $\exists x \cdot y$ in $C(S)$, then $x \widetilde{\mathcal{R}}_B \, xy \, \widetilde{\mathcal{L}}_B \, y$ in S.*

Proof. Suppose that x and y are in S such that $x \cdot y$ is defined in $C(S)$. Then $x^* \mathcal{D} \, y^\dagger$. We assume that $x^* \mathcal{L} \, h \, \mathcal{R} \, y^\dagger$, where $h \in B$. Since $\widetilde{\mathcal{R}}_B$ is a left congruence and $\widetilde{\mathcal{L}}_B$ is a right congruence, it follows that $xy \, \widetilde{\mathcal{R}}_B \, xy^\dagger \, \widetilde{\mathcal{R}}_B \, xh = x$ and dually, $xy \, \widetilde{\mathcal{L}}_B \, x^* y \, \widetilde{\mathcal{L}}_B \, hy = y$. So $x \, \widetilde{\mathcal{R}}_B \, xy \, \widetilde{\mathcal{L}}_B \, y$, as required. $\square$

Lemma 4.2. *If S is a weakly B-orthodox semigroup, then $C(S)$ is a generalised category over B such that $\mathbf{d}(x) = R_{x^\dagger}$ and $\mathbf{r}(x) = L_{x^*}$.*

Proof. We have $x \in \hom(R_e, L_f)$ if and only if $x^\dagger \, \mathcal{R} \, e$ and $x^* \, \mathcal{L} \, f$ in B. If in addition $y \in \hom(R_g, L_h)$, then $\exists x \cdot y$ in $C(S)$ if and only if $x^* \, \mathcal{D} \, y^\dagger$, i.e. $D_f = D_g$. Moreover, if $\exists x \cdot y$, then $x \cdot y \in \hom(R_e, L_h)$ by Lemma 4.1. Clearly condition (GC3) holds.

For any $e \in B$, we take the distinguished morphism e associated to e to be itself, whose domain is R_e and codomain is L_e. Certainly, if $e \in \mathbf{d}(x)$ (resp. $e \in \mathbf{r}(x)$), then e is a left (resp. right) identity of x. Hence, (GC4) holds. $\square$

We build on the above to show that $C(S)$ may be equipped with restrictions and co-restrictions, under which it becomes an inductive generalised category.

For $x \in S$ and $e, f \in B$ with $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$ and $f \leqslant_{\mathcal{R}} v \in \mathbf{r}(x)$,

$$ {}_e|x = ex \quad \text{and} \quad x|_f = xf. $$

Lemma 4.3. *Let S be a weakly B-orthodox semigroup. With the above definition of restriction and co-restriction, $C(S)$ becomes an inductive generalised category over B, which we denote by $\mathbf{C}(S)$. Further, the natural pre-orders and partial orders on S and $\mathbf{C}(S)$ coincide.*

Proof. In view of Lemma 4.2, it remains to show that $C(S)$ with the restriction and co-restriction defined above satisfies conditions (I1) to (I6) and the duals $(\mathrm{I1})^\circ$, $(\mathrm{I2})^\circ$ and $(\mathrm{I3})^\circ$ of (I1), (I2) and (I3).

(I1) If $x \in S$ and $e, u \in B$ with $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$, then ${}_e|x = ex$ and so by Lemmas 1.2 and 1.3, condition (I1) is satisfied.
(I2) Since restriction and co-restriction are given by multiplication in S, it is clear that (I2) and its dual hold.
(I3) Suppose that $x, y \in S$ and $e, u \in B$ with $x \cdot y$ defined in $C(S)$, let $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$ and $f \in \mathbf{r}({}_e|x) = L_{(ex)^*}$. Then ${}_e|(x \cdot y) = exy = exfy = ({}_e|x) \cdot ({}_f|y)$.
(I4) It is routine to check that condition (I4) holds, both products being equal to xy.
(I5) As for (I4) this is again routine, with both sides of the equality we must verify being equal to exf.
(I6) Clearly, it is satisfied by the definitions of the restriction and co-restriction, respectively.

Now, let $x, y \in S$. Then

$$\begin{aligned} x \leqslant_r y \quad \text{in } S \quad &\Leftrightarrow \quad x = ey \quad \text{some } e \in B \\ &\Leftrightarrow \quad x = ey^\dagger y \quad \text{some } e \in B,\ y^\dagger \in \mathbf{d}(y) \\ &\Leftrightarrow \quad x = {}_{ey^\dagger}|y \quad \text{some } e \in B,\ y^\dagger \in \mathbf{d}(y) \\ &\Leftrightarrow \quad x = {}_f|y \quad \text{some } f \in B \text{ with } f \leqslant_{\mathcal{L}} u \in \mathbf{d}(y) \\ &\Leftrightarrow \quad x \leqslant_r y \quad \text{in } \mathbf{C}(S). \end{aligned}$$

In addition, with notation as above, if $x \leqslant_r' y$ in S we have that $x^\dagger \leqslant_{\mathcal{R}} y^\dagger$, so that $x = y^\dagger e y^\dagger y = {}_{y^\dagger e y^\dagger}|y$ and $y^\dagger e y^\dagger \leqslant y^\dagger$, so that $x \leqslant_r' y$ in $\mathbf{C}(S)$. Conversely, if $x \leqslant_r' y$ in $\mathbf{C}(S)$, then $x = {}_g|y$, where $g \leqslant y^\dagger \in \mathbf{d}(y)$. Then $x = gy$ in S, and $x^\dagger \mathcal{R}\, g \leqslant_{\mathcal{R}} y^\dagger$, so that $x \leqslant_r' y$ in S. □

Proposition 4.4. *Let S be a weakly B-orthodox semigroup and P be an inductive generalised category over B. Then $\mathbf{S}(\mathbf{C}(S)) = S$ and $\mathbf{C}(\mathbf{S}(P)) = P$.*

Proof. Let S be a weakly B-orthodox semigroup. It follows from Lemma 4.3 that $\mathbf{C}(S)$ is an inductive generalised category over B with multiplication a restriction of that in S and $\mathbf{d}(x) = R_{x^\dagger}$, $\mathbf{r}(x) = L_{x^*}$, for any $x \in S$, and if $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$ and $f \leqslant_r v \in \mathbf{r}(x)$ then ${}_e|x = ex$ and $x|_f = xf$.

We now construct $\mathbf{S}(\mathbf{C}(S))$, which again has underlying set S, by defining the pseudo-product

$$x \otimes y = (x|_{vg}) \cdot ({}_{vg}|y),$$

where $v \in \mathbf{r}(x) = L_{x^*}$ and $g \in \mathbf{d}(y) = R_{y^\dagger}$. Observe that

$$x \otimes y = (x|_{vg}) \cdot ({}_{vg}|y) = xvgvgy = xvgy = xy,$$

so the operations in S and $\mathbf{S}(\mathbf{C}(S))$ are the same. Moreover, the distinguished bands of S and $\mathbf{S}(\mathbf{C}(S))$ are both B. Hence $S = \mathbf{S}(\mathbf{C}(S))$.

We now focus on the converse. Let P be an inductive generalised category over B with partial binary operation $\cdot$. We establish the weakly B-orthodox semigroup $\mathbf{S}(P)$ by defining the pseudo-product $\otimes$ of Theorem 3.7.

We temporarily use the notation $\odot$ for the partial binary operation in $\mathbf{C}(\mathbf{S}(P))$. For any $x, y \in P$ we have

$$\begin{aligned} \exists x \odot y \quad &\Leftrightarrow \quad x^* \mathcal{D}\, y^\dagger \quad \text{in } \mathbf{S}(P) \\ &\Leftrightarrow \quad e\,\mathcal{D}\, f, \quad \text{where } \mathbf{r}(x) = L_e \text{ and } \mathbf{d}(y) = R_f \\ &\Leftrightarrow \quad \exists x \cdot y \quad \text{in } P. \end{aligned}$$

Further, if $\exists x \odot y$, then by Lemma 3.1,

$$x \odot y = x \otimes y = x \cdot y.$$

For $x \in P$ we have that $\mathbf{d}(x) = R_{x^\dagger}$ in $\mathbf{C}(\mathbf{S}(P))$, where $x\,\widetilde{\mathcal{R}}_B\, x^\dagger$ in $\mathbf{S}(P)$. But, the latter holds if and only if $x^\dagger \in \mathbf{d}(x)$ in P, i.e. $\mathbf{d}(x) = R_{x^\dagger}$ in P. Thus $\mathbf{d}$ in P and $\mathbf{C}(\mathbf{S}(P))$ coincide, and dually for $\mathbf{r}$.

Clearly, the distinguished morphisms in P and $\mathbf{C}(\mathbf{S}(P))$ are the same.

Again as a temporary measure, we use $||$ to denote restriction and co-restriction in $\mathbf{C}(\mathbf{S}(P))$.

Let $x \in P$ and let $e, u \in B$ with $e \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$. Then in $\mathbf{C}(\mathbf{S}(P))$,

$${}_e||x = e \otimes x = e|_{eu} \cdot {}_{eu}|x = {}_{eu}|x = {}_e|x$$

and similarly for co-restrictions. □

We now proceed to establish an isomorphism between the category $\mathcal{IGC}$ of inductive generalised categories over bands and the category $\mathcal{WO}$ of weakly B-orthodox semigroups.

The next lemma demonstrates that an admissible morphism between two weakly B-orthodox semigroups gives rise to a pseudo-functor.

Lemma 4.5. *Let S be a weakly B_1-orthodox semigroup and T be a weakly B_2-orthodox semigroup. Suppose that θ is an admissible morphism. Then the map $\mathbf{C}(\theta) : \mathbf{C}(S) \to \mathbf{C}(T)$ given by the rule that $\mathbf{C}(\theta)(x) = \theta(x)$ for $x \in B$ and $x \in S$ is a pseudo-functor. Further, if $\theta_1 : S \to T$ and $\theta_2 : T \to Q$ are admissible morphisms, then $\mathbf{C}(\theta_2\theta_1) = \mathbf{C}(\theta_2)\mathbf{C}(\theta_1)$.*

Proof. (F1) Since θ is an admissible morphism, it follows that θ is a morphism from B_1 to B_2.

(F2) Suppose that $x \in S$ and $e, f \in B_1$ with $e \leqslant_{\mathcal{L}} f \in \mathbf{d}(x)$. Then ${}_e|x$ is defined and ${}_e|x = ex$. Since θ is admissible, it follows that $\theta(e) \leqslant_{\mathcal{L}} \theta(f)$ and $\theta(f)\, \widetilde{\mathcal{R}}_{B_2}\, \theta(x)$, that is, $\theta(f) \in \mathbf{d}(\theta(x))$, which implies that ${}_{\theta(e)}|\theta(x)$ is defined. Then $\mathbf{C}(\theta)({}_e|x) = \mathbf{C}(\theta)(ex) = \theta(ex) = \theta(e)\theta(x) = {}_{\theta(e)}|\theta(x) = {}_{\mathbf{C}(\theta)(e)}|\mathbf{C}(\theta)(x)$.

(F3) If $\exists x \cdot y$ in $\mathbf{C}(S)$, then $x^* \,\mathcal{D}\, y^\dagger$. Hence there is an $h \in B$ with $x\, \widetilde{\mathcal{L}}_{B_1}\, h\, \widetilde{\mathcal{R}}_{B_1}\, y$. Since θ is admissible, $\theta(x)\, \widetilde{\mathcal{L}}_{B_2}\, \theta(h)\, \widetilde{\mathcal{R}}_{B_2}\, \theta(y)$ and $\theta(h) \in B_2$. Thus $\exists \theta(x) \cdot \theta(y)$ in $\mathbf{C}(T)$. Clearly, if $x \cdot y$ exists, $\theta(x \cdot y) = \theta(xy) = \theta(x)\theta(y) = \theta(x) \cdot \theta(y)$, since θ is a morphism.

It is routine to see that $\mathbf{C}(\theta_2\theta_1) = \mathbf{C}(\theta_2)\mathbf{C}(\theta_1)$. $\square$

The following result is easy to see, given Lemmas 4.5 and 3.8.

Lemma 4.6. *Let $\theta : S \to T$ be an admissible morphism of weakly B-orthodox semigroups, and $F : P_1 \to P_2$ be a pseudo-functor of inductive generalised categories over bands. Then $\mathbf{S}(\mathbf{C}(\theta)) = \theta$ and $\mathbf{C}(\mathbf{S}(F)) = F$.*

Lemmas 4.3 and 4.5 show that $\mathbf{C} : \mathcal{WO} \to \mathcal{IGC}$ is a functor and Proposition 4.4 and Lemma 4.6 give that $\mathbf{S}$ and $\mathbf{C}$ are mutually inverse. Hence we deduce our main result.

Theorem 4.7. *The category $\mathcal{WO}$ of weakly B-orthodox semigroups and admissible morphisms is isomorphic to the category $\mathcal{IGC}$ of inductive generalised categories over bands and pseudo-functors.*

5. Special cases

In this section, we concentrate on some special kinds of weakly B-orthodox semigroups. We now present a lemma which will be used in our first two cases.

Lemma 5.1. *Let S be a weakly B-orthodox semigroup. Suppose that for any $x \in E(S)$ and $e, f \in B$ with $e\, \widetilde{\mathcal{R}}_B\, x\, \widetilde{\mathcal{L}}_B\, f$ we have $e\, \mathcal{R}^*\, x\, \mathcal{L}^*\, f$. Then $B = E(S)$.*

Proof. Let $x \in E(S)$ and choose $x^\dagger, x^* \in B$. Then $x^\dagger\, \mathcal{R}^*\, x\, \mathcal{L}^*\, x^*$ by assumption. From $x^2 = x$ we have $x^*x = x^*$ and so $x^*\, \mathcal{R}\, x^*x^\dagger$ (in B). Dually, $x^*x^\dagger\, \mathcal{L}\, x^\dagger$. Thus, $x^\dagger\, \mathcal{D}\, x^*$ and so $x\, \mathcal{H}^*\, x^\dagger x^*$, giving $x = x^\dagger x^*$ as any H^*-class contains at most one idempotent. $\square$

Note that Lemma 5.1 can be translated into a corresponding statement concerning inductive generalised categories over bands.

An (inductive) generalised category P is an (*inductive*) *generalised groupoid* if for all $x \in P$ with $\mathbf{d}(x) = R_e$ and $\mathbf{r}(x) = L_f$, there exists $y \in P$ with $\mathbf{d}(y) = R_f$ and $\mathbf{r}(y) = L_e$ such that $e = x \cdot y$ and $y \cdot x = f$.

Corollary 5.2. *The category of orthodox semigroups and morphisms is isomorphic to the category of inductive generalised groupoids over bands and pseudo-functors.*

Proof. Let S be an orthodox semigroup with $B = E(S)$. Suppose that $x \in \mathbf{C}(S)$ with $\mathbf{d}(x) = R_e$ and $\mathbf{r}(x) = L_f$. Since $\mathcal{R} = \widetilde{\mathcal{R}}_B$ and $\mathcal{L} = \widetilde{\mathcal{L}}_B$, we have that $e\, \mathcal{R}\, x\, \mathcal{L}\, f$. It follows from the fact that S is regular that there exists $y \in S$ with $e = xy$ and $yx = f$. We have that $e\, \mathcal{L}\, y\, \mathcal{R}\, f$ and so $\mathbf{d}(y) = R_f$ and $\mathbf{r}(y) = L_e$ and the products $x \cdot y$, $y \cdot x$ exist in $\mathbf{C}(S)$. Moreover, $x \cdot y = xy = e$ and $y \cdot x = yx = f$.

Conversely, let P be an inductive generalised groupoid over B. Suppose that $x \in P$ and $\mathbf{d}(x) = R_e$, $\mathbf{r}(x) = L_f$. Then there exists $y \in P$ with $\mathbf{d}(y) = R_f$ and $\mathbf{r}(y) = L_e$ such that $f = y \cdot x$ and $e = x \cdot y$. It follows that $x \otimes y \otimes x = (x \cdot y) \otimes x = e \otimes x = e \cdot x = x$. Thus, $\mathbf{S}(P)$ is regular. In addition, as $e = x \cdot y = x \otimes y$ and $x = e \otimes x$, we have that $e\,\mathcal{R}\,x$ in $\mathbf{S}(P)$. Dually, $f\,\mathcal{L}\,x$ in $\mathbf{S}(P)$. By Lemma 5.1, we have that $E(\mathbf{S}(P)) = B$. Hence, $\mathbf{S}(P)$ is an orthodox semigroup. $\square$

Now, we focus on the class of abundant semigroups. We replace the distinguished set of idempotents B by the whole set of idempotents and use relations $\mathcal{R}^*$ and $\mathcal{L}^*$ instead of $\widetilde{\mathcal{R}}_B$ and $\widetilde{\mathcal{L}}_B$ in the definition of weakly B-orthodox semigroups. We thus obtain the class of abundant semigroups whose set of idempotents forms a band. An admissible morphism in this context is more usually referred to as a *good* morphism. We define an inductive generalised category P over a band B to be *abundant* if it satisfies the following condition and its dual (I7)°:

(I7) if $e, f, g \in B$ and $x, y, z \in P$ are such that $e, f \leqslant_{\mathcal{L}} g \in \mathbf{d}(x)$, $e \in \mathbf{r}(y)$, $f \in \mathbf{r}(z)$ and $y \cdot {}_e|x = z \cdot {}_f|x$, then $y = z$.

Corollary 5.3. *The category of abundant semigroups whose set of idempotents forms a band and good morphisms is isomorphic to the category of abundant inductive generalised categories over bands and pseudo-functors.*

Proof. Let P be an abundant inductive generalised category over a band B. Suppose that $x \in P$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$. We know that $e\,\widetilde{\mathcal{R}}_B\,x$ in $\mathbf{S}(P)$, so that $e \otimes x = x$. Assume that $y, z \in P$ with $y \otimes x = z \otimes x$, giving that $(y|_{y^*e}) \cdot ({}_{y^*e}|x) = (z|_{z^*e}) \cdot ({}_{z^*e}|x)$, where $y^* \in \mathbf{r}(y)$ and $z^* \in \mathbf{r}(z)$. By (I7), we obtain that $y|_{y^*e} = z|_{z^*e}$. Thus, $y^*e\,\mathcal{L}\,z^*e$ in B. We have

$$\begin{aligned} y \otimes e &= y|_{y^*e} \cdot {}_{y^*e}|e \\ &= z|_{z^*e} \cdot y^*e \quad \big(\text{by (I6)}\big) \\ &= z|_{z^*e} \quad \big(y^*e\,\mathcal{L}\,z^*e\big) \\ &= z|_{z^*e} \cdot {}_{z^*e}|e \\ &= z \otimes e. \end{aligned}$$

This is enough to show that $e\,\mathcal{R}^*\,x$. Dually, we have that $f\,\mathcal{L}^*\,x$.

In view of Lemma 5.1, we have that $E(\mathbf{S}(P)) = B$.

Conversely, let S be an abundant semigroup with $E(S) = B$. It follows that $\mathcal{R}^* = \widetilde{\mathcal{R}}_B$ and $\mathcal{L}^* = \widetilde{\mathcal{L}}_B$. In view of Lemma 4.3, it is sufficient to claim that $\mathbf{C}(S)$ satisfies conditions (I7), and dually, (I7)°. Assume that $e, f, g \in B$ and $x, y, z \in P$ are such that $e, f \leqslant_{\mathcal{L}} g \in \mathbf{d}(x)$, $e \in \mathbf{r}(y)$, $f \in \mathbf{r}(z)$ and $y \cdot {}_e|x = z \cdot {}_f|x$. It follows that $yex = zfx$. Since $g \in \mathbf{d}(x)$, that is, $g\,\mathcal{R}^*\,x$ in S, we have that $yeg = zfg$, that is, $ye = zf$, as $e, f \leqslant_{\mathcal{L}} g$. Hence, $y = z$, as required. Dually, (I7)° holds. $\square$

We now discuss Ehresmann semigroups. Let S be an Ehresmann semigroup with distinguished semilattice E. We mentioned in Lemma 1.7 that $\leqslant_r = \leqslant_r'$ and $\leqslant_l = \leqslant_l'$.

Let P be an inductive generalised category over E. The context

$$(E, E/\mathcal{R}, E/\mathcal{L}, E/\mathcal{D}, p)$$

is essentially four copies of E equipped with the identity map. We therefore identify E with $E/\mathcal{R}$, $E/\mathcal{L}$ and $E/\mathcal{D}$ and note that P becomes a category in the usual sense. Notice that as $P = \mathbf{C}(\mathbf{S}(P))$, we have that $\leqslant_r = \leqslant_r'$, $\leqslant_l = \leqslant_l'$ and $\leqslant_r$ and $\leqslant_l$ are partial orders on P.

For easy reference, we say that a category $\mathcal{C}$ with a partial ordering $\leqslant$ is *ordered* if it satisfies the following conditions:

(OC1) if $x, y \in \mathcal{C}$ with $x \leqslant y$, then $\mathbf{r}(x) \leqslant \mathbf{r}(y)$ and $\mathbf{d}(x) \leqslant \mathbf{d}(y)$;
(OC2) if $\mathbf{r}(x) = \mathbf{r}(y)$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $x \leqslant y$, then $x = y$;
(OC3) if $x' \leqslant x$, $y' \leqslant y$ and both $x' \cdot y'$ and $x \cdot y$ exist, then $x' \cdot y' \leqslant x \cdot y$.

Further, an ordered category C with set of identities E has *restrictions* if for any $x \in C$ and $e \in E$ with $e \leqslant \mathbf{d}(x)$, there exists a unique element ${}_e|x$ such that ${}_e|x \leqslant x$ and $\mathbf{d}({}_e|x) = e$. To possess *co-restrictions* has the dual definition.

Lemma 5.4. *An inductive generalised category P over a semilattice E with $\leqslant_r$ forms an ordered category with restriction.*

Proof. From comments above P is a category (with the appropriate identifications) and $(P, \leqslant_r)$ is a poset.

(OC1) Suppose that $x, y \in P$ with $x \leqslant_r y$. Then there exists $e \in E$ such that $e \leqslant \mathbf{d}(y)$ and $x = {}_e|y$. Thus, $\mathbf{d}(x) = e \leqslant \mathbf{d}(y)$ and $\mathbf{r}(x) \leqslant \mathbf{r}(y)$ by (I1).

(OC2) Suppose that $x, y \in P$ with $\mathbf{r}(x) = \mathbf{r}(y)$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $x \leqslant_r y$. Then there exists $e \in E$ such that $e \leqslant \mathbf{d}(y)$ and $x = {}_e|y$. Certainly, $\mathbf{d}(x) = e$ and so $e = \mathbf{d}(y)$, whence from (I1), $x = y$.

(OC3) If $x' \leqslant_r x$ and $y' \leqslant_r y$, and both $x' \cdot y'$ and $x \cdot y$ exist, then there exist $e, f \in E$ such that $e \leqslant \mathbf{d}(x)$, $f \leqslant \mathbf{d}(y)$, $x' = {}_e|x$ and $y' = {}_f|y$. Thus, we have that $\mathbf{r}({}_e|x) = \mathbf{r}(x') = \mathbf{d}(y') = \mathbf{d}({}_f|y) = f$ and so $x' \cdot y' = ({}_e|x) \cdot ({}_f|y) = {}_e|(x \cdot y)$ by (I3). Hence, $x' \cdot y' \leqslant_r x \cdot y$.

Finally, we assume that $x \in P$ and $e \in E$ with $e \leqslant \mathbf{d}(x)$. Then ${}_e|x$ is defined and $\mathbf{d}({}_e|x) = e$. Also, ${}_e|x \leqslant_r x$ by (I1). Further, ${}_e|x$ is unique since if $z \leqslant_r x$ and $e = \mathbf{d}(z)$, then there exists $h \in E$ with $h \leqslant \mathbf{d}(x)$ and $z = {}_h|x$, which gives that $h = \mathbf{d}(z)$. Thus, $e = h$. Hence, $z = {}_e|x$. $\square$

As a dual result of Lemma 5.4, we have the following lemma.

Lemma 5.5. *An inductive generalised category P over a semilattice E with $\leqslant_l$ forms an ordered category with co-restriction.*

Next we show that an inductive generalised category P over a semilattice E is an Ehresmann category as defined in [8].

We recall from [8] that an *Ehresmann category* $C = (C, \cdot, \leqslant_r, \leqslant_l)$ is a category $(C, \cdot)$ with set of identities E, equipped with two relations $\leqslant_l$ and $\leqslant_r$ such that the following conditions, and the duals $(\mathrm{E1})^\circ$ and $(\mathrm{E5})^\circ$ of (E1) and (E5) hold:

(E1) $(C, \cdot, \leqslant_r)$ is an ordered category with restriction;
(E2) if $e, f \in E$, then $e \leqslant_r f \Leftrightarrow e \leqslant_l f$;
(E3) E is a meet semilattice under $\leqslant_r$ (or $\leqslant_l$);
(E4) $\leqslant_r \circ \leqslant_l = \leqslant_l \circ \leqslant_r$;
(E5) if $x \leqslant_r y$ and $f \in E$, then $x|_{\mathbf{r}(x)f} \leqslant_r y|_{\mathbf{r}(y)f}$.

We note that [8] interchanges the symbols $\mathbf{r}$ and $\mathbf{d}$ and the notions of restriction and co-restriction, from the conventions of this paper.

Lemma 5.6. *An inductive generalised category P over a semilattice E with the pair of natural partial orderings $(\leqslant_r, \leqslant_l)$ forms an Ehresmann category.*

Conversely, an Ehresmann category $(C, \cdot, \leqslant_r, \leqslant_l)$ with semilattice of identities E, may be regarded as an inductive generalised category over E with natural partial orderings $(\leqslant_r, \leqslant_l)$.

Proof. Let P be inductive generalised category over a semilattice E. In view of the above discussion, we have claimed that P is a category with set of identities E. By Lemmas 5.4 and 5.5, conditions (E1) and $(\mathrm{E1})^\circ$ are satisfied.

(E2) If $e, f \in E$ and $e \leqslant_r f$, then $e = {}_e|f = ef$ so that we must have $e \leqslant f$. Then $f|_e$ is defined and $f|_e = fe = e$ so that $e \leqslant_l f$. Together with the dual, we have that for $e, f \in E$,

$$e \leqslant_r f \quad \Leftrightarrow \quad e \leqslant_l f \quad \Leftrightarrow \quad e \leqslant f,$$

so that in particular, (E2) holds.

(E3) Clearly, E is a semilattice under $\leqslant_r = \leqslant_l = \leqslant$.

(E4) To show that $\leqslant_r \circ \leqslant_l \subseteq \leqslant_l \circ \leqslant_r$, we assume that $x \leqslant_r \circ \leqslant_l y$. Then there exists $z \in P$ such that $x \leqslant_r z \leqslant_l y$. And so there exist $e, f \in E$ with $\mathbf{d}(x) = e \leqslant \mathbf{d}(z) = u$ and $\mathbf{r}(z) = f \leqslant \mathbf{r}(y) = v$, such that $x = {}_e|z$ and $z = y|_f$. Thus, $x = {}_e|(y|_f) = {}_{eu}|(y|_{vf})$. By (I4), we get that $x = ({}_{eh}|y)|_{gf}$, where $h = \mathbf{d}(y)$ and $g = \mathbf{r}({}_{eh}|y)$. Set $z' = {}_{eh}|y$. Then $x \leqslant_l z'$ and $z' \leqslant_r y$. Consequently, $x \leqslant_l \circ \leqslant_r y$. With the dual, we obtain (E4).

(E5) Suppose that $x, y \in P$ and $f \in E$ with $x \leqslant_r y$. Then there exists $k \in E$ with $k \leqslant \mathbf{d}(y)$ and $x = {}_k|y$. So $x|_{\mathbf{r}(x)f} = ({}_k|y)|_{\mathbf{r}(x)f} = ({}_{k\mathbf{d}(y)}|y)|_{\mathbf{r}(x)f}$. Let $h = \mathbf{d}(y|_{\mathbf{r}(y)f})$. By (I4), we obtain that $({}_{k\mathbf{d}(y)}|y)|_{\mathbf{r}(x)f} = {}_{kh}|(y|_{\mathbf{r}(y)f})$, so that $x|_{\mathbf{r}(x)f} \leqslant_r y|_{\mathbf{r}(y)f}$.

Conversely, let $C = (C, \cdot, \leqslant_r, \leqslant_l)$ be an Ehresmann category with semilattice of identities E. Then $C = (C, \cdot)$ may also be regarded as a generalised category over E.

We let $\leqslant$ denote the restriction of $\leqslant_r$ ($\leqslant_l$) to E. It is clear that the first part of (I1) holds, moreover, by uniqueness of restriction, ${}_e|x = x$ if $e = \mathbf{d}(x)$.

For (I2), if $x \in C$ and $e, f, g, u \in E$, with $e \leqslant_{\mathcal{L}} g \mathcal{R} f \leqslant_{\mathcal{L}} u \in \mathbf{d}(x)$, then this simplifies to $e \leqslant f \leqslant \mathbf{d}(x)$. Now ${}_{ef}|x = {}_e|x \leqslant_r x$ and $\mathbf{d}({}_e|x) = e$; also, ${}_e|({}_f|x) \leqslant_r {}_f|x \leqslant_r x$ and $\mathbf{d}({}_e|({}_f|x)) = e$. By uniqueness of restriction, ${}_{ef}|x = {}_e|({}_f|x)$.

(I3) If $x, y \in C$ with $\exists x \cdot y$, then $\mathbf{r}(x) = \mathbf{d}(y)$. If $e \leqslant \mathbf{d}(x)$, then we have

$${}_e|(x \cdot y) \leqslant_r x \cdot y \quad \text{and} \quad \mathbf{d}({}_e|x \cdot y) = e$$

and also

$$({}_e|x) \cdot ({}_f|y) \leqslant_r x \cdot y \quad \text{and} \quad \mathbf{d}\big(({}_e|x) \cdot ({}_f|y)\big) = e,$$

where $f = \mathbf{r}({}_e|x)$. Hence, ${}_e|(x \cdot y) = ({}_e|x) \cdot ({}_f|y)$.

(I4) This is clear.

(I5) Let $x \in C$ and $e, f, u, v, g, h \in E$ with $g = \mathbf{r}(x)$, $h = \mathbf{d}(x)$, $u = \mathbf{d}(x|_{gf})$ and $v = \mathbf{r}({}_{eh}|x)$. Then $(e \mathbin{\overline{\otimes}} x) \mathbin{\overline{\otimes}} f = e \mathbin{\overline{\otimes}} (x \mathbin{\overline{\otimes}} f)$, where $\overline{\otimes}$ is defined [8], by $x \mathbin{\overline{\otimes}} y = (x|_k) \cdot ({}_k|y)$, where $k = \mathbf{r}(x)\,\mathbf{d}(y)$. As shown in [8], $\overline{\otimes}$ is associative, hence,

$$\begin{aligned}(e \mathbin{\overline{\otimes}} x) \mathbin{\overline{\otimes}} f &= \big((e|_{eh}) \cdot ({}_{eh}|x)\big)|_{vf} \cdot ({}_{vf}|f)\\ &= \big(eh \cdot ({}_{eh}|x)\big)|_{vf} \cdot vf\\ &= ({}_{eh}|x)|_{vf}\end{aligned}$$

and similarly, $e \mathbin{\overline{\otimes}} (x \mathbin{\overline{\otimes}} f) = {}_{eu}|(x|_{gf})$, so we obtain that $({}_{eh}|x)|_{vf} = {}_{eu}|(x|_{gf})$.

(I6) Suppose that $e, g, h, u, v \in E$ are such that $u \leqslant_{\mathcal{R}} g \mathcal{L} e$ and $v \leqslant_{\mathcal{L}} h \mathcal{R} e$, which simplifies to $u \leqslant e$ and $v \leqslant e$. Clearly, $e|_u = u = eu$ and ${}_v|e = v = ve$. $\square$

Let $\mathbf{C} = (C, \cdot, \leqslant_r, \leqslant_l)$ and $\mathbf{D} = (D, \cdot, \leqslant_r, \leqslant_l)$ be Ehresmann categories with semilattices E_C and E_D of identities, respectively. A *strongly ordered functor* [8] $F : \mathbf{C} \to \mathbf{D}$ is a functor which preserves $\leqslant_r$, $\leqslant_l$ and the binary operation of the semilattices. Hence F is a morphism $E_C \to E_D$. As shown in [8], F preserves restrictions and co-restrictions. Thus F is a pseudo-functor in the sense of Definition 2.5.

On the other hand, if $G : \mathbf{C} \to \mathbf{D}$ is a pseudo-functor, then from the comments following Definition 2.5, G is a functor, which by (F1) preserves $\wedge$. Suppose now that $x, y \in \mathbf{C}$ with $x \leqslant_r y$. Then

$x = {}_e|y$ for some $e \in E$, so that by (F2), $G(x) = {}_{G(e)}|G(y)$ so that $G(x) \leqslant_r G(y)$. Dually, G preserves $\leqslant_l$, so that G is a strongly ordered functor. Theorem 4.7, Lemma 4.3 and the comments above now give us Lawson's result from [8], Theorem D.

Corollary 5.7. *(See [6, Theorem 4.24].) The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.*

We now turn to *weakly B-superorthodox semigroups*, which are weakly B-orthodox semigroups such that each $\widetilde{\mathcal{H}}_B$-class contains a distinguished idempotent in B. We say that a generalised category P over a band B is a *super-generalised category* if it is an inductive generalised category and satisfies the following condition:

(I8) if $x \in P$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$, then $e \,\mathcal{D}\, f$.

Corollary 5.8. *The category of weakly B-superorthodox semigroups and admissible morphisms is isomorphic to the category of super-generalised categories over B and pseudo-functors.*

Proof. Let S be a weakly B-superorthodox semigroup. It follows from Lemma 4.3 that it is only necessary to show that $\mathbf{C}(S)$ satisfies condition (I8). Suppose that $x \in S$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$. Then $e \,\widetilde{\mathcal{R}}_B\, x \,\widetilde{\mathcal{L}}_B\, f$ in S. As S is a weakly B-superorthodox semigroup, it follows that there exists $h \in B$ such that $h \,\widetilde{\mathcal{H}}_B\, x$. Thus, $e \,\mathcal{R}\, h \,\mathcal{L}\, f$, which implies that $e \,\mathcal{D}\, f$.

Conversely, let P be a super-generalised category over B. It is sufficient to show that $\mathbf{S}(P)$ is weakly B-superorthodox. Suppose that $x \in P$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$. Then by (I8), $e \,\mathcal{D}\, f$, that is, $e \,\mathcal{R}\, ef \,\mathcal{L}\, f$. As $e \,\widetilde{\mathcal{R}}_B\, x \,\widetilde{\mathcal{L}}_B\, f$ in $\mathbf{S}(P)$, we get that $x \,\widetilde{\mathcal{H}}_B\, ef$. Hence $\mathbf{S}(P)$ is a weakly B-superorthodox semigroup. $\square$

Next, we discuss the class of weakly B-orthodox semigroups which have condition (WIC) mentioned in Section 1. We define an inductive generalised category P over a band B to be *connected* if it satisfies the following condition and its dual (I9)°:

(I9) if $x \in P$ and $e \leqslant u \in \mathbf{d}(x)$ then there exists $f \leqslant v \in \mathbf{r}(x)$ such that ${}_e|x = x|_f$.

Corollary 5.9. *The category of weakly B-orthodox semigroups with (WIC) and admissible morphisms is isomorphic to the category of connected inductive generalised categories over bands and pseudo-functors.*

Proof. Let S be a weakly B-orthodox semigroup with (WIC). In view of Lemma 4.3, it remains to show that $\mathbf{C}(S)$ satisfies conditions (I9) and (I9)°. We will show that (I9) holds, dually, (I9)° holds. Suppose that $x \in S$ and $e \leqslant u \in \mathbf{d}(x)$. Then ${}_e|x = ex$. Since S has (WIC), it follows that there exists $f \in B$ such that $ex = xf$. Then $ex = xvfv$, where $v \in \mathbf{r}(x)$. Thus, $ex = xvfv = x|_{vfv}$.

Conversely, let P be a connected inductive generalised category over a band B. Suppose that $x \in P$ and $e \leqslant u \in \mathbf{d}(x)$. Then it follows from (I9) that there exists $f \leqslant v \in \mathbf{r}(x)$ such that ${}_e|x = x|_f$. Thus $e \otimes x = {}_e|x = x|_f = x \otimes f$. Together with the dual argument, we have shown that $\mathbf{S}(P)$ has (WIC). $\square$

6. Trace of weakly B-orthodox semigroups

First, we define the trace of a weakly B-orthodox semigroup to be $C(S) = (S, \cdot)$, as in Section 4. Remark that $C(S)$ contains $C(B) = (B, \cdot)$ as a substructure, where $C(B)$ is the band B with multiplication restricted to $\mathcal{D}$-classes.

Now let P be any generalised category over B. Define $\odot$ on $P^0 = P \,\dot{\cup}\, \{0\}$ by the rule that

$$x \odot y = \begin{cases} x \cdot y & \text{if } \exists x \cdot y \text{ in } P \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.1. *The set $(P^0, \odot)$ is a semigroup containing a band $(B^0, \odot)$ as a subsemigroup, where $(B^0, \odot)$ is the 0-direct union of the $\mathcal{D}$-classes of B. Further, P^0 is primitive weakly B^0-orthodox, in the sense that distinguished idempotents are all primitive in B^0.*

Proof. Let $x, y, z \in P^0$. If any of x, y, z is 0, then clearly $x \odot (y \odot z) = (x \odot y) \odot z = 0$. Suppose that $x, y, z \in P$. Then

$$\begin{aligned} x \odot (y \odot z) &= \begin{cases} x \odot (y \cdot z) & \text{if } \exists y \cdot z \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x \cdot (y \cdot z) & \text{if } \exists y \cdot z \text{ and } \exists x \cdot (y \cdot z) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x \cdot (y \cdot z) & \text{if } \exists y \cdot z \text{ and } \exists x \cdot y \\ 0 & \text{otherwise} \end{cases} \\ &= (x \odot y) \odot z \end{aligned}$$

for reasons of symmetry. Clearly B^0 is a subsemigroup of P^0.

Let $x \in P^0$. If $x = 0$, then $x \widetilde{R}_{B^0} 0$. If $x \in P$, then choosing $e \in \mathbf{d}(x)$ we have $\exists e \cdot x$ and $e \cdot x = x$, so that $e \odot x = x$. If $f \in B^0$ and $f \odot x = x$, then clearly $f \in B$ and $\exists f \cdot x$ with $f \cdot x = x$. Hence $R_f = \mathbf{d}(x) = R_e$ so that $e \mathcal{R} f$ and $f \odot e = e$. Hence $x \widetilde{\mathcal{R}}_{B^0} e$ and it follows that P^0 is weakly B^0-abundant.

Notice that $x \widetilde{\mathcal{R}}_{B^0} f$ where $f \in B$ if and only if $\mathbf{d}(x) = R_f$. If follows that $x \widetilde{\mathcal{R}}_{B^0} y$ if and only if $\mathbf{d}(x) = \mathbf{d}(y)$. Thus for any $z \in P$, $z \odot x = 0$ if and only if $z \odot y = 0$, and if $z \odot x \neq 0$, then $\mathbf{d}(z \odot x) = \mathbf{d}(z) = \mathbf{d}(z \odot y)$. It is clear that (C) holds and P^0 is weakly B^0-orthodox. It is immediate that P^0 is B^0-primitive. $\square$

Let S be weakly B-orthodox. From Lemma 4.2, $C(S) = (S, \cdot)$ is an inductive generalised category over B. Then $C(S)^0$ is a primitive weakly B^0-abundant semigroup; $C(S)^0$ is also sometimes called the *trace* of S. From Lemma 4.3, $C(S)$, and with a little adjustment, $C(S) \cup \{0\}$, can be endowed with an inductive structure from which we can recover S.

The natural partial orders in any primitive weakly B-orthodox semigroup with 0 are trivial, in the following sense:

Lemma 6.2. *Let S be a primitive weakly B-orthodox semigroup with 0, where $0 \in B$. Then B is a 0-disjoint union of $\mathcal{D}$-classes. If $x, y \in S$, then $x \leqslant'_r y$ if and only if $x = 0$ or $x = y$, and dually for $\leqslant'_l$.*

Proof. We know that B is a semilattice Y of $\mathcal{D}$-classes D_α, $\alpha \in Y$. We must have that Y contains a zero τ and $D_\tau = \{0\}$. If $\tau < \alpha < \beta$, let $e \in D_\alpha$ and $f \in D_\beta$. Then $fef \in D_\alpha$ and $0 < fef < f$, a contradiction. It follows that B is a 0-disjoint union of its $\mathcal{D}$-classes.

If $x \neq 0$ and $x \leqslant'_r y$, then $x = ey$ for some $e \in B$ and $x^\dagger \leqslant_{\mathcal{R}} y^\dagger$. Thus $x^\dagger y^\dagger \leqslant y^\dagger$ so that $x^\dagger y^\dagger = y^\dagger$. Also, $x^\dagger \leqslant_{\mathcal{R}} e$ so that similarly, $x^\dagger e = x^\dagger$. Now $x = ey = x^\dagger ey = x^\dagger y = x^\dagger y^\dagger y = y^\dagger y = y$. $\square$

7. Example

This section is concerned with the promised example. We show how a weakly B-orthodox semigroup may be naturally obtained from a monoid acting via morphisms on the left and right of a band with identity. This construction is reminiscent of that underlying the free ample monoid, and we believe will be of subsequent use.

Let B be a band with 1 and let T be a monoid acting on the left and right of B by $\cdot$ and $\circ$ via morphisms such that

$$(t \cdot g) \circ t = (1 \circ t)g \quad \text{and} \quad t \cdot (g \circ t) = g(t \cdot 1),$$

for all $g \in B$ and $t \in T$.

We note that as T acts by morphisms, if $e, f \in B$ with $e \leqslant_{\mathcal{L}} f$, then for any $t \in T$, $t \cdot e = t \cdot ef = (t \cdot e)(t \cdot f) \leqslant_{\mathcal{L}} t \cdot f$, so that $\cdot$ preserves $\leqslant_{\mathcal{L}}$. Dually, $\circ$ preserves $\leqslant_{\mathcal{R}}$.

Let $S = B *_1 T = \{(e,t)\colon\ e \leqslant_{\mathcal{L}} t \cdot 1\} \subseteq B \times T$ with semidirect product multiplication, i.e.

$$(e,t)(f,s) = \big(e(t \cdot f), ts\big).$$

Now if $e \leqslant_{\mathcal{L}} t \cdot 1$ and $f \leqslant_{\mathcal{L}} s \cdot 1$, then $t \cdot f \leqslant_{\mathcal{L}} t \cdot (s \cdot 1) = ts \cdot 1$, and so $e(t \cdot f) \leqslant_{\mathcal{L}} ts \cdot 1$. Thus S is closed, and consequently, it is a semigroup.

We now obtain a series of lemmas to verify that S constructed above is a weakly $\overline{B}$-orthodox semigroup, where $\overline{B} = \{(e,1)\colon\ e \in B\}$.

Lemma 7.1. *The set $\overline{B} = \{(e,1)\colon\ e \in B\}$ is isomorphic to B.*

Proof. Let $e, f \in B$. Then $e \leqslant_{\mathcal{L}} 1_B = 1_T \cdot 1_B$ and $(e,1)(f,1) = (e(1 \cdot f), 1) = (ef, 1)$, whence it follows that $\overline{B}$ is a band isomorphic to B. $\square$

Lemma 7.2. *For any $(e,t) \in S$, $(e,t)\, \widetilde{\mathcal{R}}_{\overline{B}}\, (e,1)$.*

Proof. Let $(e,t) \in S$. Then $(e,1)(e,t) = (e(1 \cdot e), t) = (e,t)$ and if $(f,1)(e,t) = (e,t)$, then $(fe,t) = (e,t)$, so $fe = e$ and $(f,1)(e,1) = (e,1)$. Thus, $(e,t)\, \widetilde{\mathcal{R}}_{\overline{B}}\, (e,1)$. $\square$

Let $(e,t), (f,s) \in S$. By Lemmas 7.1 and 7.2,

$$(e,t)\, \widetilde{\mathcal{R}}_{\overline{B}}\, (f,s) \quad \Leftrightarrow \quad e \,\mathcal{R}\, f.$$

Lemma 7.3. *For any $(e,t) \in S$, $(e,t)\, \widetilde{\mathcal{L}}_{\overline{B}}\, (e \circ t, 1)$.*

Proof. Let $(e,t) \in S$. Then

$$\begin{aligned}(e,t)(e \circ t, 1) &= \big(e\big(t \cdot (e \circ t)\big), t\big) \\ &= \big(e\big(e(t \cdot 1)\big), t\big) \\ &= (e,t) \quad (e \leqslant_{\mathcal{L}} t \cdot 1).\end{aligned}$$

Further, if $(e,t)(f,1) = (e,t)$, then $e(t \cdot f) = e$. Now

$$\begin{aligned}e \circ t &= \big(e(t \cdot f)\big) \circ t = (e \circ t)\big((t \cdot f) \circ t\big) \\ &= (e \circ t)(1 \circ t) f = \big((e1) \circ t\big) f \\ &= (e \circ t) f,\end{aligned}$$

so $(e \circ t, 1)(f,1) = (e \circ t, 1)$. Thus $(e,t)\, \widetilde{\mathcal{L}}_{\overline{B}}\, (e \circ t, 1)$. $\square$

Again by Lemma 7.1,

$$(e,t)\, \widetilde{\mathcal{L}}_{\overline{B}}\, (f,s) \quad \Leftrightarrow \quad e \circ t \,\mathcal{L}\, f \circ s \quad \text{in } B.$$

Lemma 7.4. *The semigroup S is weakly $\overline{B}$-orthodox, where $\overline{B} = \{(e,1)\colon\ e \in B\}$.*

Proof. In view of Lemmas 7.1, 7.2 and 7.3, it is sufficient to show that S has (C). Suppose that $(e,t)\,\widetilde{\mathcal{R}}_{\overline{B}}\,(f,s)$ and $(g,u)\in S$. Then $(g,u)(e,t)=(g(u\cdot e),ut)$ and $(g,u)(f,s)=(g(u\cdot f),us)$. As $e\,\mathcal{R}\,f$ we have $u\cdot e\,\mathcal{R}\,u\cdot f$ and then $g(u\cdot e)\,\mathcal{R}\,g(u\cdot f)$, so that $\widetilde{\mathcal{R}}_{\overline{B}}$ is a left congruence.

Now let $(e,t)\,\widetilde{\mathcal{L}}_{\overline{B}}\,(f,s)$ and $(g,u)\in S$. Then $(e,t)(g,u)=(e(t\cdot g),tu)$ and $(f,s)(g,u)=(f(s\cdot g),su)$. We have

$$\begin{aligned}\bigl(e(t\cdot g)\bigr)\circ t &= (e\circ t)\bigl((t\cdot g)\circ t\bigr)\\ &= (e\circ t)(1\circ t)g\\ &= (e\circ t)g\,\mathcal{L}\,(f\circ s)g\\ &= \bigl(f(s\cdot g)\circ s\bigr),\end{aligned}$$

so that $(e(t\cdot g))\circ tu\,\mathcal{L}\,(f(s\cdot g))\circ su$. Thus $\widetilde{\mathcal{L}}_{\overline{B}}$ is a right congruence. Hence S is weakly $\overline{B}$-orthodox. $\square$

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