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Quantum Correlations Reduce Classical Correlations with Ancillary Systems *

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We illustrate the dichotomy of classical/quantum correlations by virtue of monogamy. More precisely, we show that correlations in a bipartite state are classical if and only if each party of the state can be perfectly correlated with other ancillary systems. In particular, this means that if there are quantum correlations between two parties, then the classical (as well as quantum) correlating capabilities of the two parties with other systems have to be strictly reduced.

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A general observation in quantum information theory is that while classical correlations can be shared by many parties, quantum correlations usually exhibit monogamy in the sense that they cannot be shared.[1−10] In this context, the central concern is to what extent the correlations between two parties a and b impose limitations on their correlating capabilities with other systems. The so-called monogamy of entanglement states that if a and b are entangled, then their entanglement capabilities with other systems are severely restricted. However, this entanglement monogamy leaves open whether parties a and b can still have *classical* correlations with other systems. In this Letter, we establish that if a and b are entangled, then this entanglement also puts limitations on their *classical* correlating capabilities with other systems. In particular, in such a situation, a and b both cannot have *perfect* correlations with any other systems. More precisely, if a and b are perfectly correlated with certain other systems, then a and b cannot have any quantum correlations (including entanglement) between themselves. This may be interpreted as a kind of monogamy of hybrid correlations (classical and quantum).

Entanglement is a particular kind of quantum correlation, and the notion of quantum correlations is more general than that of entanglement. To make this more precise, we recall the classical/quantum dichotomy for bipartite correlations,[11] which is quite different from the standard separability/entanglement paradigm introduced by Werner.[12] A bipartite state ρ^{ab} shared by parties a and b is classical (with respect to correlations) if it is left invariant by certain local von Neumann measurement.[11] The correlations therein are then called classical and such a quantum state can be essentially identified with a classical bivariate probability distribution.[11,13] Here ρ^{ab} is classical if and only if it has the following representation

$$\rho^{ab} = \sum_{ij} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|.$$

Here $\{p_{ij}\}$ is a classical bivariate probability distribution, $\{|i\rangle\}$ and $\{|j\rangle\}$ are orthonormal sets for parties a and b, respectively. If ρ^{ab} cannot be represented as the above form, then it possesses quantum correlations. Thus, here "classical" refers to the correlations in ρ^{ab} and is with respect to some particular basis. Clearly, classical bipartite states are separable, but the converse is not true since many separable states still possess quantum correlations. However, classicality and separability are intrinsically related to each other.[14] Piani *et al.* have established a remarkable characterization of classical correlations via the operational task of local broadcasting,[13] that is, classical correlations are the only correlations that can be locally broadcast. The result of this study is also motivated by Refs. [15,16], and may be interpreted as a characterization of classical bipartite states in terms of monogamy of hybrid correlations.

Now, let us make precise the notion of perfect correlations between two parties. Consider a bipartite state $\rho^{a'a}$ shared by parties a' and a with the same system dimension. By a local von Neumann measurement $\Pi^{a'} = \{\Pi_i^{a'}\}$ on party a', we mean a family of one-dimensional orthogonal projections such that $\sum_i \Pi_i^{a'} = \mathbf{1}^{a'}$ (identity operator). Here "local" is used to emphasize that we will apply such a measurement locally on the bipartite state $\rho^{a'a}$. If there exist local von Neumann measurements $\Pi^{a'} = \{\Pi_{i'}^{a'}\}$ and $\Pi^a = \{\Pi_i^a\}$ for parties a' and a, respectively, such that the bivariate probability distribution $p_{i'i}^{a'a} := \mathrm{tr}(\Pi_{i'}^{a'} \otimes \Pi_i^a)\rho^{a'a}(\Pi_{i'}^{a'} \otimes \Pi_i^a)$ satisfies $p_{i'i}^{a'a} = p_i^a \delta_{i'i}$, then we say that a can be perfectly correlated with a'. Here $p_i^a := \sum_{i'} p_{i'i}^{a'a}$. The operational meaning for perfect

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correlations lies in that when the two parties make their respective local measurements, they obtain the same result. In such a case, the measurement outcomes for party a are indeed perfectly correlated with those for party a' in the sense that they can only take the same label value simultaneously, that is, if party a' obtains an outcome labeled by i in his system, then party a will also obtain an outcome labeled by i, it can never happen that the two parties obtain different outcome labels: Their outcomes are perfectly correlated. Note that according to this definition, if $\{|i\rangle\}$ is an orthonormal set for both a' and a, and if $\rho^{a'a} = |\Psi^{a'a}\rangle\langle\Psi^{a'a}|$ with $|\Psi^{a'a}\rangle = \sum_i \sqrt{\lambda_i}|i\rangle \otimes |i\rangle$ (Schmidt form) is a *pure* state, or if $\rho^{a'a}$ is a classical state of the form $\rho^{a'a} = \sum_i p_i|i\rangle\langle i| \otimes |i\rangle\langle i|$, then a' and a are perfectly correlated when both make measurements along the basis $\{|i\rangle\}$. Consequently, perfect correlations can occur in both classical and quantum scenarios. Now we state the main result.

A bipartite quantum state ρ^{ab}, shared by parties a and b, is classical (with respect to correlations) if and only if there exist two ancillary parties a' and b' such that a can be perfectly correlated with a', and b can be perfectly correlated with b'. Alternatively, if a and b are correlated in a quantum fashion (i.e., ρ^{ab} is not classical), then a and b cannot be both perfectly correlated with any other parties.

To establish the above result, first we suppose that ρ^{ab} is classical, and we will show that both parties a and b can be perfectly correlated with some ancillary parties. By the classicality of ρ^{ab}, there exist local orthonormal sets $\{|i\rangle\}$ and $\{|j\rangle\}$ for parties a and b, respectively, such that $\rho^{ab} = \sum_{ij} p_{ij}|i\rangle\langle i| \otimes |j\rangle\langle j|$. Here $\{p_{ij}\}$ is a bivariate probability distribution. Let $H^{a'}$ and $H^{b'}$ be copies of H^a and H^b, respectively, then we can construct a four-partite state

$$\rho^{a'abb'} := \sum_{ij} p_{ij}|i\rangle\langle i| \otimes |i\rangle\langle i| \otimes |j\rangle\langle j| \otimes |j\rangle\langle j|,$$

and we have $\rho^{a'a} = \mathrm{tr}_{bb'}\rho^{a'abb'} = \sum_i p_i|i\rangle\langle i| \otimes |i\rangle\langle i|$, $\rho^{bb'} = \mathrm{tr}_{a'a}\rho^{a'abb'} = \sum_j q_j|j\rangle\langle j| \otimes |j\rangle\langle j|$. Here $p_i := \sum_j p_{ij}$, $q_j := \sum_i p_{ij}$. If we perform local von Neumann measurements $\Pi^{a'} = \{|i'\rangle\langle i'|\}$ and $\Pi^a = \{|i\rangle\langle i|\}$ on parties a' and a ($\Pi^{a'}$ being a copy of Π^a, i.e., $|i'\rangle\langle i'| = |i\rangle\langle i|$), respectively, we obtain the joint probability distribution $p^{a'a}_{i'i} = p_i\delta_{i'i}$, which means that a can be perfectly correlated with a'. A similar statement holds for party b.

Next we proceed to prove the converse. Suppose that there exists a four-partite state $\rho^{a'abb'}$ with marginals $\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{a'abb'}$, $\rho^{a'a} = \mathrm{tr}_{bb'}\rho^{a'abb'}$, $\rho^{bb'} = \mathrm{tr}_{a'a}\rho^{a'abb'}$, such that a can be perfectly correlated with a', and b can be perfectly correlated with b'. Then there are local von Neumann measurements $\Pi^{a'} = \{\Pi^{a'}_{i'}\}, \Pi^a = \{\Pi^a_i\}$ such that

$$\mathrm{tr}(\Pi^{a'}_{i'} \otimes \Pi^a_i)\rho^{a'a}(\Pi^{a'}_{i'} \otimes \Pi^a_i) = p_i\delta_{i'i}.$$

But due to the projective nature of von Neumann measurements and the cyclic property of trace, we have

$$\mathrm{tr}(\Pi^{a'}_{i'} \otimes \Pi^a_i)\rho^{a'a}(\Pi^{a'}_{i'} \otimes \Pi^a_i) = \mathrm{tr}(\Pi^{a'}_{i'} \otimes \Pi^a_i)\rho^{a'a}.$$

Consequently

$$\mathrm{tr}(\Pi^{a'}_{i'}\otimes\Pi^a_i\otimes\mathbf{1}^b\otimes\mathbf{1}^{b'})\rho^{a'abb'} = \mathrm{tr}(\Pi^{a'}_{i'}\otimes\Pi^a_i)\rho^{a'a} = p_i\delta_{i'i}.$$

In particular, for $i' \neq i$, we have

$$(\Pi^{a'}_{i'} \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'} = 0.$$

From this we have

$$\begin{aligned}
&(\mathbf{1}^{a'} \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= \Bigg(\Bigg(\sum_{i'} \Pi^{a'}_{i'}\Bigg) \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'}\Bigg)\rho^{a'abb'} \\
&= (\Pi^{a'}_i \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'}. \qquad (1)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&(\Pi^{a'}_i \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= \Bigg(\Pi^{a'}_i \otimes \Bigg(\sum_j \Pi^a_j\Bigg) \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'}\Bigg)\rho^{a'abb'} \\
&= (\Pi^{a'}_i \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'}. \qquad (2)
\end{aligned}$$

Comparing Eqs. (1) and (2), we have

$$\begin{aligned}
&(\mathbf{1}^{a'} \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= (\Pi^{a'}_i \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'}. \qquad (3)
\end{aligned}$$

Taking adjoint of the above equation, we obtain

$$\begin{aligned}
&\rho^{a'abb'}(\mathbf{1}^{a'} \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'}) \\
&= \rho^{a'abb'}(\Pi^{a'}_i \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'}). \qquad (4)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&(\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \Pi^b_j \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= (\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \Pi^{b'}_j)\rho^{a'abb'}, \qquad (5) \\
&\rho^{a'abb'}(\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \Pi^b_j \otimes \mathbf{1}^{b'}) \\
&= \rho^{a'abb'}(\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \Pi^{b'}_j). \qquad (6)
\end{aligned}$$

From Eqs. (3)–(6), we obtain

$$\begin{aligned}
&(\mathbf{1}^{a'} \otimes \Pi^a_i \otimes \Pi^b_j \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= (\mathbf{1}^{a'} \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})(\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \Pi^b_j \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= (\mathbf{1}^{a'} \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})(\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \Pi^{b'}_j)\rho^{a'abb'} \\
&= (\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \Pi^{b'}_j)(\mathbf{1}^{a'} \otimes \Pi^a_i \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= (\mathbf{1}^{a'} \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \Pi^{b'}_j)(\Pi^{a'}_i \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \mathbf{1}^{b'})\rho^{a'abb'} \\
&= (\Pi^{a'}_i \otimes \mathbf{1}^a \otimes \mathbf{1}^b \otimes \Pi^{b'}_j)\rho^{a'abb'}.
\end{aligned}$$

Consequently, we have proved that

$$(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'})\rho^{a'abb'} = (\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'})\rho^{a'abb'}, \tag{7}$$

and by taking the adjoint, we have

$$\rho^{a'abb'}(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'}) = \rho^{a'abb'}(\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}). \tag{8}$$

For any observables A and B for parties a and b, respectively, putting

$$X = \mathbf{1}^{a'}\otimes A\otimes B\otimes\mathbf{1}^{b'}$$

and noting that X commutes with $\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}$, we have

$$\begin{aligned}
&\mathrm{tr}(A\otimes B)\rho^{ab} = \mathrm{tr}X\rho^{a'abb'}\\
=&\,\mathrm{tr}X\Big(\Big(\sum_i\Pi_i^{a'}\Big)\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Big(\sum_j\Pi_j^{b'}\Big)\Big)\rho^{a'abb'}\\
=&\sum_{ij}\mathrm{tr}X(\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}\Big)\rho^{a'abb'}\\
=&\sum_{ij}\mathrm{tr}X\Big(\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}\Big)^2\rho^{a'abb'}\\
=&\sum_{ij}\mathrm{tr}\Big(\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}\Big)\\
&\times X\Big(\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}\Big)\rho^{a'abb'}\\
=&\sum_{ij}\mathrm{tr}\Big(\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}\Big)\\
&\times X\Big(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'}\Big)\rho^{a'abb'}\\
=&\sum_{ij}\mathrm{tr}X\Big(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'}\Big)\\
&\times\rho^{a'abb'}\Big(\Pi_i^{a'}\otimes\mathbf{1}^a\otimes\mathbf{1}^b\otimes\Pi_j^{b'}\Big)\\
=&\sum_{ij}\mathrm{tr}X\Big(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'}\Big)\\
&\times\rho^{a'abb'}\Big(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'}\Big)\\
=&\sum_{ij}\mathrm{tr}\Big(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'}\Big)\\
&\times X\Big(\mathbf{1}^{a'}\otimes\Pi_i^a\otimes\Pi_j^b\otimes\mathbf{1}^{b'}\Big)\rho^{a'abb'}\\
=&\sum_{ij}\mathrm{tr}\Big(\mathbf{1}^{a'}\otimes(\Pi_i^aA\Pi_i^a)\otimes(\Pi_j^bB\Pi_j^b)\otimes\mathbf{1}^{b'}\Big)\rho^{a'abb'}\\
=&\sum_{ij}\mathrm{tr}\Big((\Pi_i^aA\Pi_i^a)\otimes(\Pi_j^bB\Pi_j^b)\Big)\rho^{ab}\\
=&\sum_{ij}\mathrm{tr}(\Pi_i^a\otimes\Pi_j^b)(A\otimes B)(\Pi_i^a\otimes\Pi_j^b)\rho^{ab}\\
=&\sum_{ij}\mathrm{tr}(A\otimes B)(\Pi_i^a\otimes\Pi_j^b)\rho^{ab}(\Pi_i^a\otimes\Pi_j^b)\\
=&\,\mathrm{tr}(A\otimes B)\sum_{ij}(\Pi_i^a\otimes\Pi_j^b)\rho^{ab}(\Pi_i^a\otimes\Pi_j^b).
\end{aligned}$$

Since the above equations hold for any A and B, we conclude that

$$\rho^{ab} = \sum_{ij}(\Pi_i^a\otimes\Pi_j^b)\rho^{ab}(\Pi_i^a\otimes\Pi_j^b),$$

which implies that ρ^{ab} is a classical state.

The main result qualifies classical correlations between two parties in terms of their correlating capabilities with ancillary systems. In this context, it will be desirable to extend the result in a more quantitative way: How can we quantify the tradeoff relations for the correlations between different parties? Here we recall some relevant measures of correlations and leave the quantitative characterization issue open. Recall that total correlations in ρ^{ab} are quantified by the quantum mutual information $I(\rho^{ab}) := S(\rho^a) + S(\rho^b) - S(\rho^{ab})$, where $S(\rho^a) := -\mathrm{tr}\rho^a\log\rho^a$ is the von Neumann entropy, and an intuitive measure for classical correlations is the observable correlations,[17]

$$C(\rho^{ab}) = \sup I(\Pi(\rho^{ab})),$$

where $\Pi(\rho^{ab}) := \sum_{ij}(\Pi_i^a\otimes\Pi_j^b)\rho^{ab}(\Pi_i^a\otimes\Pi_j^b)$, and the sup is over all local von Neumann measurements $\Pi^a = \{\Pi_i^a\}$ and $\Pi^b = \{\Pi_j^b\}$. By the monotonicity of quantum mutual information under local operations (measurements), we know that $I(\rho^{ab}) \geq C(\rho^{ab})$, and a measure of quantum correlations can be defined as[17]

$$Q(\rho^{ab}) := I(\rho^{ab}) - C(\rho^{ab}).$$

Apparently, if ρ^{ab} is a classical state, then $I(\rho^{ab}) = C(\rho^{ab})$ and $Q(\rho^{ab}) = 0$, thus here all correlations are classical and there are no quantum correlations. On the other hand, if ρ^{ab} is a *pure* state, then $I(\rho^{ab}) = 2S(\rho^a)$ and $C(\rho^{ab}) = Q(\rho^{ab}) = S(\rho^{ab})$, and the total correlations are equally distributed as classical and quantum parts. See Refs. [17,18] for some detailed accounts of the relations between classical and quantum correlations. Now, in terms of the observable correlations $C(\rho^{ab})$ and quantum correlations $Q(\rho^{ab})$, the main result may be phrased as that if $Q(\rho^{ab}) > 0$, then a and b cannot be both perfectly correlated with other parties. The relations for correlations between different parties are worth further investigation.

In summary, we have provided a characterization of classical correlations (and thus also quantum correlations) in bipartite states via monogamy, which states that if both of the two parties can be perfectly correlated with other ancillary systems, then there cannot exist any quantum correlations between them. Put it alternatively, if there are quantum correlations between the two parties, then these quantum correlations will reduce the classical correlating capabilities of both parties with other systems. This stands in sharp contrast to the situation of classical correlations, which can be shared by any number of parties, as well as to the situation of monogamy of entanglement, which involves only quantum correlations.

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