*Another Definition for Ramsey Numbers**
$R(p, q) = r(p-1, q)$ $(2 \le p \le q)$

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Abstract—**We introduce an idea or a concept of restricted coexistence. By the restricted coexistence, the Ramsey number $R(p,q)$ is defined equivalently as $r(p-1,q)$, that is,**

$$R(p,q) = r(p-1,q),$$

where $r(p-1,q)$ is a least integer that has coexistence restricted to the parameters $p-1,q$ with $q \ge p \ge 2$. From this, some basic properties about Ramsey numbers are obtained, for instance,

$$R(p,q) > R(p-1,q+1),$$

where p,q are integers with $q \ge p \ge 3$, and so on.

Keywords-Ramsey numbers; restricted coexistence

I. INTRODUCTION

In the end-twenties of the 20th century, Frank Ramsey, an English logician, laid the foundation for his well-known theory what is now called Ramsey theory. Its form in 2-color case is only exhibited here.

Ramsey's Theorem [1-4]. For each $p,q \in N$ with $p,q \ge 2$, there exists a least integer $R(p,q)$ (the Ramsey number) such that, no matter how a complete graph $K_{R(p,q)}$ is two-colored, it will contain either a red sub-graph K_p or a blue sub-graph K_q.

We say that an integer n has the Ramsey property for the parameters p,q if $n \ge R(p,q)$.

This theory is regarded as a profound and important result that involves such a problem in how large scale discrete system there exist at least one of desired sub-constructions. It contains the pigeonhole principle as a special case. But very little is known about classical Ramsey numbers' exact values [1-4].

Ramsey numbers will be applied possibly to many aspects [3,5,6], for example, searching data, computing the maximal and unambiguous code alphabet in the confusion graph, security in communication and so on.

The work gives a more explicit definition for Ramsey numbers. From this definition, people can obtain easily such relations as

$$R(p,q) > R(p-1,q+1),$$

where $3 \le p \le q$,

$$r(p,p) = r(p-1,p+1),$$

where $2 \le p$, and so forth.

II. A LEMMA

We begin with a basic fact in order to expound another definition about Ramsey numbers.

Lemma. Let p,q be integers with $q \ge p \ge 2$. Then the number $R(p,q)-1$ is the biggest integer n such that, under the restriction that there is neither a red sub-graph K_p nor a blue sub-graph K_q, no matter how the complete graph K_n is two-colored, it will contain both a red sub-graph K_{p-1} and a blue sub-graph K_{q-1}.

Proof. For $p = 2$, the lemma is trivial (a vertex acts as a red K_1 or a blue K_1).

For $p > 2$, by Ramsey's theorem, the Ramsey number $R(p,q)$ exists. Then there must exist a two-colored complete graph $K_{R(p,q)-1}$ which contains neither a red sub-graph K_p nor a blue sub-graph K_q, and for this case, we will prove that the two-colored complete graph $K_{R(p,q)-1}$ must contain both a red sub-graph K_{p-1} and a blue sub-graph K_{q-1}.

Without loss of generality, suppose that there is no red sub-graph K_{p-1} in the above $K_{R(p,q)-1}$. Then we add a vertex v into the $K_{R(p,q)-1}$ and use red color edges linking v to all vertices of the $K_{R(p,q)-1}$ to construct a two-colored $K_{R(p,q)}$.

On the one hand, since we only use the red color edges, obviously, the two-colored $K_{R(p,q)-1}$ doesn't contain a blue K_q.

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On the other hand, the two-colored $K_{R(p,q)-1}$ doesn't contain a red K_p either, because no red K_{p-1} in the $K_{R(p,q)-1}$ itself and v construct a red K_p. Thus, we get a two-colored complete graph $K_{R(p,q)}$ that contains neither a red sub-graph K_p nor a blue sub-graph K_q. This contradicts the definition of the Ramsey number $R(p,q)$. Namely, we have proved that, under the restriction that there is neither a red K_p nor a blue K_q, a two-colored $K_{R(p,q)-1}$ contains always both a red sub-graph K_{p-1} and a blue sub-graph K_{q-1}.

Furthermore, since $R(p,q)-1$ plus 1 is the Ramsey number $R(p,q)$ which must break the restriction condition of containing neither a red sub-graph K_p nor a blue sub-graph K_q, the number $R(p,q)-1$ is such the biggest integer. The lemma holds. #

By the lemma, using the principle of mathematical induction, we immediately know that such a least integer that has 'restricted coexistence' exists.

III. Restricted Coexistence

Now we draw out the idea or concept of restricted coexistence from the above arguments.

Definition. Let p,q be integers and not less than 2. If there exists an integer n such that, under the restriction that there is neither a red sub-graph K_p nor a blue sub-graph K_q, no matter how the complete graph K_n is two-colored, it must contain both a red sub-graph K_{p-1} and a blue sub-graph K_{q-1}, then we say that the integer n has coexistence restricted to the parameters $p-1, q-1$.

The existence of the integer n in the definition, including such the biggest integer and such a least integer, had been verified beforehand by the above lemma and the principle of mathematical induction.

Such a least integer n is denoted by $r(p-1,q-1)$. Particularly, we define $r(1,1)=1$.

Example 1. $r(1,3)=3,\ r(2,4)=9.$

By the definition, the above lemma can be restated as that $R(p,q)-1$ is the biggest integer such that has coexistence restricted to the parameters $p-1, q-1$.

In addition, we know easily that for each integer $q \ge 2$, $r(1,q-1)=R(2,q)-1=q-1$.

IV. Another Definition on Ramsey Numbers

The following theorem throws light upon a deep relationship between a least integer with coexistence restricted to given parameters and a corresponding Ramsey number.

Theorem 1. Let p,q be integers with $q \ge p \ge 2$. Then the following equality holds:

$$R(p,q)=r(p-1,q) \ . \tag{1}$$

Proof. We will prove this equality via mathematical induction.

Firstly, for $p=2$ and each integer q such that $q \ge 2$, $R(2,q)=r(1,q)(=q)$ holds obviously.

Next, we assume that, for arbitrary given integer p with $p \ge 2$ and each integer q such that $q \ge p$, the equality $R(p,q)=r(p-1,q)$ is true.

Finally, using $R(p,q)=r(p-1,q)$ – the inductive hypothesis, we will push out such a result that, for each integer q such that $q \ge p+1$, $R(p+1,q)=r(p,q)$ holds always.

In fact, by the restricted coexistence, since $r(p,q)$, where $q > p \ge 2$, is a least integer that has coexistence restricted to parameters p,q, the integer $r(p,q)-1$ hasn't coexistence restricted to the parameters p,q. This means that, under the restriction that there is neither a red sub-graph K_{p+1} nor a blue sub-graph K_{q+1}, a 2-colored complete graph $K_{r(p,q)-1}$ need not contain always both a blue K_q and a red K_p. Therefore, $r(p,q)$ is only regarded as, in two possible ways, either such a least integer n that, under the restriction that there is no red K_{p+1}, a blue K_q appears unavoidably in a 2-colored K_n (we know easily that it is equivalent to a least integer n that, under the restriction that there is no blue K_q, a red K_{p+1} appears always in a two-colored K_n)or, such a least integer n such that, under the restriction that there is no blue K_{q+1}, a red K_p appears unavoidably in a 2-colored K_n (equivalent to a least integer n that, under the restriction that there is no red K_p, a blue K_{q+1} appears unavoidably in a two-colored K_n).

On the other hand, by Ramsey number's definition, we see that, for arbitrary given integers p,q but not less than 2, actually, the Ramsey number $R(p,q)$ is a least integer n that, under the restriction that there is no red K_p, a blue K_q appears unavoidably in a 2-colored K_n (we know easily that it is equivalent to a least integer n that, under the restriction that there is no blue K_q, a red K_p appears unavoidably in a two-colored K_n).

So if we understand the $r(p,q)$ $(q > p \ge 2)$ in the first way, that is, $r(p,q)$ $(q > p \ge 2)$ is regarded as such a least integer n that, under the restriction that there is no red K_{p+1}, a blue K_q appears unavoidably in a 2-colored K_n (equivalent to a least integer n that, under the restriction that there is no blue K_q, a red K_{p+1} appears unavoidably in a 2-colored K_n), then we get the desired equality $R(p+1,q)=r(p,q)$ at once and complete the induction. Thus the theorem is true.

But if we understand the $r(p,q)$ $(q>p\geq 2)$ in the second way, that is, it is a least integer n that, under the restriction that there is no blue K_{q+1}, a red K_p appears unavoidably in a 2-colored K_n (equivalently, a least integer n that, under the restriction that there is no red K_p, a blue K_{q+1} appears unavoidably in a 2-colored K_n), then we will infer a contradiction:

Actually, by the second understanding and the above arguments, we know that $R(p,q+1)=r(p,q)$, where $q>p\geq 2$. Again from the inductive hypothesis, it follows that $R(p,q+1)=r(p-1,q+1)$, where $q>p\geq 2$. Hence, we have

$$r(p,q)=R(p,q+1)=r(p-1,q+1).$$

Similarly, by the second understanding once more, $r(p-1,q+1)$ $(q>p\geq 2)$ can also be regarded as a least integer n that, under the restriction that there is no blue K_{q+2}, a red K_{p-1} appears unavoidably in a 2-colored K_n (or equivalently, a least integer n that, under the restriction that there is no red K_{p-1}, a blue K_{q+2} appears unavoidably in a 2-colored K_n).

Otherwise, why do we understand always the $r(p,q)$ with $q>p\geq 2$ in the second way but not the first way? That is , if $r(p-1,q+1)$ $(q>p\geq 2)$ can be accepted as the above first understanding, then the $r(p,q)$ can also be understood in the first way, and the proof will be over at once.

Thus, by the second understanding, again from the inductive hypothesis, we also have

$$\begin{aligned} r(p,q) &= R(p,q+1) = r(p-1,q+1) \\ &= R(p-1,q+2) = r(p-2,q+2). \end{aligned}$$

Going on to do as the above reasoning, we will get an equal-sign sequence

$$\begin{aligned} r(p,q) &= R(p,q+1) = r(p-1,q+1) \\ &= R(p-1,q+2) = r(p-2,q+2) \\ &= \cdots \\ &= R(2,q+p-1) = r(1,q+p-1) = q+p-1. \end{aligned}$$

That is, for arbitrary given integer $p\geq 2$ and each integer q with $q>p$, equalities

$$\begin{aligned} r(p,q) &= R(p,q+1) = R(p-1,q+2) = \cdots \\ &= R(3,q+p-2) = R(2,q+p-1) = q+p-1 \end{aligned}$$

or, equivalent to

$$\begin{aligned} r(p,q) &= r(p-1,q+1) = \cdots \\ &= r(2,q+p-2) = r(1,q+p-1) = q+p-1 \end{aligned}$$

hold always. But, in fact, this is clearly impossible. This contradiction implies that the second understanding for $r(p,q)$ $(q>p\geq 2)$ is incorrect. Therefore we accept only the $r(p,q)$ $(q>p\geq 2)$ as the first way. Namely, the number $r(p,q)$ $(q>p\geq 2)$ is such a least integer n that, under the restriction that there is no red K_{p+1}, a blue K_q appears unavoidably in a 2-colored K_n , result the desired equality

$$R(p+1,q)=r(p,q) \;\; for \;\; each \;\; pair \;\; q>p\geq 2.$$

The inductive procedure has been finished. Hence, the theorem holds. #

Example 2. $r(2,5)=R(3,5)=14\neq r(3,4)=18$.

V. What does the Equality Mean?

Equality (1) demonstrates that the reason for so-called Ramsey phenomena – that is, no matter how each edge of K_n is colored with one of two colors (red or blue), either a red K_p or a blue K_q appears always in the 2-colored K_n, is that K_n 's order n reaches the value $r(p-1,q)$ of its minimal state under the restricted coexistence, which the two-colored $K_{r(p-1,\,q)}$ is under the restriction that there is neither a red K_p nor a blue K_{q+1}, where p,q are integers such that $q\geq p\geq 2$.

Additionally, its proof implies that

Corollary 1.1. For each $p,q\in N$ with $3\leq p\leq q$, the strict inequality

$$R(p,q)>R(p-1,q+1) \tag{2}$$

holds.

Proof. By Theorem 1, we have

$$R(p,q)=r(p-1,q), R(p-1,q+1)=r(p-2,q+1)$$

where $q\geq p\geq 3$, and thus we just need to prove the following inequality: $r(p-1,q)>r(p-2,q+1)$.

Firstly, by Theorem 1 and the restricted coexistence, we get easily that

$$R(p,q)\geq R(p-1,q+1)$$

where $q\geq p\geq 3$.

Secondly, again by Theorem 1's proof, we know that $r(p-1,q)$ $(q\geq p\geq 3)$ can not be accepted as the second understanding, namely, it is not a least integer n that, under the restriction that there is no blue K_{q+1} , a red K_{p-1} appears unavoidably in a 2-colored K_n. In other words, when

$q \ge p \ge 3$, a least integer n that, under the restriction that there is no blue K_{q+1}, a red K_{p-1} appears unavoidably in a 2-colored K_n, fails to reach such a least integer m that a 2-colored K_m contain always a blue K_q under the restriction that there is no red K_p, namely, a two-colored $K_{r(p-2,\, q+1)}$ need not contain both a red complete sub-graph K_{p-1} and a blue complete sub-graph K_q under the restriction that there is neither a red K_p nor a blue K_{q+1}. Hence, the strict inequality holds. #

Doubtless, the result is equivalent to Theorem 1.

Using Theorem 1, we also find another interesting phenomenon:

Corollary 1. 2. For each $p \in N$ with $p \ge 2$, the equality

$$r(p, p) = r(p-1, p+1) \tag{3}$$

holds.

Proof. By the definition of restricted coexistence, the integer $r(p, p)$, that sets at the left end of the equality, is a least integer with coexistence restricted to parameters: p, p. The existence of the $r(p, p)$ can be verified by that of $R(p+1, p+1) - 1$ that, by the above lemma, is the biggest integer with coexistence restricted to parameters: p, p.

First, we prove the inequality $r(p, p) \ge R(p, p+1)$.

In fact, if there is no red K_p in the two-colored $K_{r(p,\, p)}$, then, by the restricted coexistence, a blue color K_{p+1} must appear in it, and if there is no blue color K_{p+1} in the two-colored $K_{r(p,\, p)}$, then, by the definition of $r(p, p)$, a red color K_p must exist in it. By the symmetry of $r(p, p)$'s parameters, thus we have proved that the $r(p, p)$ has the Ramsey property for the parameters $p, p+1$. The inequality follows from this.

Again by Theorem 1, we have

$$R(p, p+1) = r(p-1, p+1),$$

so that

$$r(p, p) \ge r(p-1, p+1).$$

Conversely, by $r(p, p)$'s definition, the $r(p, p)$ is regarded naturally as a least integer n such that ,under no blue K_{p+1}, a red K_p appears always in a 2-colored. K_n. That means that a 2-colored complete graph $K_{r(p,\, p)-1}$ need not contain a red K_p under no blue K_{p+1}, namely, $r(p, p) - 1$ doesn't have the Ramsey property for the parameters: $p, p+1$ .Thus ,we have

$$r(p, p) - 1 < R(p, p+1) = r(p-1, p+1).$$

The desired equality holds from the above. #

This corollary means that the Ramsey number $R(p, p+1)$ is regarded as the common value of two distinct minimal states under the restricted coexistence: one is that of parameters: $p-1, p+1$; another is that of parameters p, p. For $r(p, p)$, one form of $R(p, p+1)$, its next Ramsey number is viewed as $R(p+1, p+1)$ $(= r(p, p+1))$; for $r(p-1, p+1)$, another form of $R(p, p+1)$, its next Ramsey number is viewed as $R(p, p+2)$ $(= r(p-1, p+2))$.

Example 3. Let $p = 5$ in Corollary 1.2, and we illustrate the meaning of Equality (3) by the following network:

$$\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\rightarrow & R(5,5) & \rightarrow & R(5,6) & \rightarrow & R(5,7) & \rightarrow \cdots \\
 & \downarrow & \searrow & \downarrow & & \downarrow & \\
 & R(5,6) & & R(6,6) & \rightarrow & R(6,7) & \rightarrow \cdots \\
 & & & \downarrow & \searrow & \downarrow &
\end{array}$$

Figure 1. The local network for $R(p,q)$.

By the above equalities (1) and (3), $R(5,6)$ is not only the value of the minimal state under coexistence restricted to 5, 5 but also the value of the minimal state under coexistence restricted to 4, 6; for the minimal state under coexistence restricted to the first parameter set, the value under its next minimal state is accounted as $R(6,6)$ $(= r(5,6))$, and for the minimal state under coexistence restricted to the second parameter set, the value under its next minimal state is accounted as $R(5,7) (= r(4,7))$.

With the help of the idea of restricted coexistence and Theorem 1, we put forward a useful principle.

Corollary 1.3.(The relative capacity principle) In the contact diagram of two-color Ramsey numbers,

$$\begin{array}{cccccc}
 & \downarrow & & \downarrow & \\
\rightarrow & R(p,q) & \rightarrow & R(p,q+1) & \rightarrow \\
\ddots & \downarrow & & \downarrow & \\
 & \iddots & & \iddots & \\
\rightarrow & R(u,v) & \rightarrow & R(u,v+1) & \rightarrow \\
 & \downarrow & \ddots & \downarrow &
\end{array}$$

Figure 2. A general form of the local network for $R(p,q)$.

for arbitrary given integers p,q,u,v that meet that $2 \le p \le q$ and $p < u \le v+1$, if $R(p,q) \le R(u,v)$, then

$$R(p,q+1) - R(p,q) \le R(u,v+1) - R(u,v). \tag{4}$$

Proof. Without loss effectiveness, we make an example instead of the proof.

We know that $R(4,7) \ge R(3,8) = 28$ by the known data ([4]). That is that the absolute capacity of the Ramsey number for $4,7$ is not less than that of Ramsey number for $3,8$. Again since the minimal parameter 3 of $R(3,8)$ is less than the minimal parameter 4 of $R(4,7)$, by Theorem 1 or Inequality (2), the relative capacity of the Ramsey number for $4,7$ is not less than that of Ramsey number for $3,8$, in other words, the number of vertexes needed to be added into the original state $K_{r(2,8)}$ under coexistence restricted to $2,8$ in order to reach its saturated state $K_{R(3,9)-1}$

doesn't exceed that of vertexes needed to be added into the original state $K_{r(3,7)}$ under coexistence restricted to parameters 3,7 in order to reach its saturated state $K_{R(4,8)-1}$. That is,

$$R(4,8)-R(4,7)\ge R(3,9)-R(3,8)$$

unless Theorem 1 is false .#

Example 4. By the principle and known data ([4]), we can rise up the Ramsey number $R(4,8)$'s lower bound to 57 from the present 56, namely $R(4,8)\ge 57$.

VI. EXTENSION

In similar ways to the above definition, lemma, theorem as well as their corollaries, it's not difficult to get the corresponding results for multi-color graphs and hyper-graphs. We outline them without their proofs as follows.

Theorem 2. Let $t, p_1, p_2, \cdots, p_k$ be any monotone increasing sequence with integer value and $k,t\ge 2$. Then there exists a positive integer n such that, under the restriction that there is not a c_1 color $K^t_{p_1}, \cdots,$ a c_{k-1} color $K^t_{p_{k-1}}$ or a c_k color $K^t_{p_k+1}$ ([2,3]), no matter how each t-element subset of a n-element set is colored with any one of k colors, there must coexist a subset of size p_1-1 such that all of its t-element subsets are c_1 color (that is, a c_1 color $K^t_{p_1-1}$), … , a subset of size $p_{k-1}-1$ such that all of its t-element subsets are c_{k-1} color (that is, a c_{k-1} color $K^t_{p_{k-1}-1}$) and a subset of size p_k such that all of its t-element subsets are c_k color (that is, a c_k color $K^t_{p_k}$). Such a least integer n is denoted by

$$r_t(p_1-1, p_2-1, \cdots, p_{k-1}-1, p_k). \tag{5}$$

Further,

$$R_t(p_1, p_2, \cdots p_k) = r_t(p_1-1, p_2-1, \cdots, p_{k-1}-1, p_k), \tag{6}$$

$$R_t(p_1, p_2, \cdots, p_k) > R_t(\ldots, p_i-1, \ldots, p_j+1, \ldots)(p_i \ge t+1), \tag{7}$$

$$r_t(\underbrace{p, p, \cdots, p}_{s+1}) = r_t(p-1, \underbrace{p, \cdots, p}_{s-1}, p+1)(s\ge 1, p\ge t). \tag{8}$$

The theorem is an extension of the two-parameter and $t=2$ forms of the conclusions in Sections II-V respectively. Techniques or ideas of the proof are also the same as them.

Example 5. ([2-4]) $r_2(2,9)=r(2,9)=R(3,9)=36,$

$$r_3(2,2,9)=R_3(3,3,9)=9;\quad R_3(4,5,6)>R_3(3,6,6);$$

$$r_3(3,4,4,5)=r_3(4,4,4,4)=R_3(4,5,5,5).$$

VII. CONCLUSION

By the existence of the minimal coexisting-states under the restriction, we understand fairly the reason for the phenomena described by Ramsey's theorem, namely Ramsey phenomena. The above properties or relations can't be directly obtained from Ramsey's theorem, so the technique or idea of restricted coexistence will be helpful in research on Ramsey theory.

Let us end our arguments by enjoying with everybody two conjectures:

For each $p\in N$ with $p\ge 3$, the equalities

$$R(p+1, p+1) = R(p, p+3)$$

and

$$R(p,p,p) = R(p-1, p+1, p+1) - 1$$

are true, as known instances in case of taking $p=3$, $R(4,4)=R(3,6)=18,\ \ R(3,3,3)=R(2,4,4)-1$ ([3,4]).

Further discussion will be arranged in the future.

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IX. REFERENCES

[1] Ramsey F..P, On a problem of formal logic, Proceedings of the London Mathematical Society, 1930,30,pp.264-286.

[2] Richard A. Brualdi, Introductory Combinatorics, fourth edition , Prentice-Hell, 2004.

[3] Ronald L.Graham, Bruce L.Rothschild,& Joel L.Spencer, Ramsey Theory , second edition, New York: Wiley, 1990.

[4] S.P.Radziszowski, Small Ramsey Numbers, Electronic Jounal Combinatorics, Dynamic Survey 1, 2006.

[5] Lovasz L., On the Shannon Capacity of a Graph, IEEE Trans. Inform. Theory, 17-25(1979),pp1-7.

[6] Biggs N., Algebraic Graph Theory, London: Cambridge University Press, 1993.