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Pullback trajectory attractors for evolution equations and application to 3D incompressible non-Newtonian fluid

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Abstract
This paper studies the pullback asymptotic behaviour of trajectories for evolution equations. We first combine the idea of trajectory attractor and pullback attractor to formulate a new type of attractor called pullback trajectory attractor. Then we prove a sufficient condition for the existence of a pullback trajectory attractor for the translation cocycle defined on the united trajectory space of the evolution equations. Finally, we take a three-dimensional incompressible non-Newtonian fluid as the applied example and prove its pullback trajectory asymptotic smoothing effect.

Mathematics Subject Classification: 35B41; 35Q35; 76D03

1. Introduction

The attractor is an important concept in the study of evolution equations because it could provide some useful information about the asymptotic behaviour of solutions. There are many works concerning this subject, see, e.g., [3, 24, 29, 35, 43, 46, 47]. With the development of non-autonomous and random dynamical systems, a new type of attractor, called pullback (or cocycle) attractor, was formulated and investigated (see e.g. [25, 26, 31, 32, 36, 37]). Factually, the theory of pullback attractor has proved very useful in the understanding of the dynamics

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of non-autonomous and random systems, including those with delays (see e.g. [1, 9, 14–16, 19, 21, 22, 44, 45]). One can refer to Boukrouche *et al* [9], Caraballo *et al* [14, 15], Caraballo and Langa [16, 17], Caraballo *et al* [20], Cheban *et al* [22], Kloeden and Schmalfuss [31, 32] Langa and Schmalfuss [36], Langa *et al* [37], etc.

In many evolution equations, possible absence of uniqueness of solutions makes some difficulties in the study of the asymptotic behaviour of solutions. For example, we all know that the uniqueness of weak solutions to the basic boundary value problem for three-dimensional (3D) Navier–Stokes equations still remains unproved. Hence, one cannot use the classical methods based on the analysis of the global attractor (or kernel sections, uniform attractor) of the corresponding semigroup (or process) to discuss the behaviour of solutions to these systems.

There are three methods to overcome the difficulties associated with possible non-uniqueness of solutions in the study of dynamical systems. The first one is generalized semi flow which was formulated by Ball [2]. The second one is multi-valued dynamical systems, one can refer to Melnik and Valero [40, 41] for multi-valued semi flows, Kapustyan and Valero [33], Caraballo *et al* [13, 20], Wang and Zhou [50] for multi-valued process (or semiprocess) and Caraballo *et al* [10–12] for multi-valued random dynamical systems. Caraballo *et al* [18] gave a detailed comparison between the theories for generalized semi flow and multi-valued dynamical systems. The third one is trajectory attractor. The definition of trajectory attractor was initially developed to overcome difficulties related to possible non-uniqueness of weak solutions for the 3D Navier–Stokes equations (see e.g. [23, 24, 49]). Later, the theory of trajectory attractor proved very useful for other models whose solution corresponding to each initial state can be non-unique. As will be seen below, the works of Chepyzhov and Vishik [23, 24, 49] greatly influence the presentation of this paper.

Although the above three methods use different concepts to describe the asymptotic behaviour of the systems with possible non-uniqueness of solutions, in fact they deal with very similar problems in which the dynamics is governed by a collection of possible solutions through each initial condition.

The major motivation and original goal of this paper is to study the pullback asymptotic behaviour of solutions for the following non-autonomous 3D incompressible non-Newtonian fluid:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot \tau(e(u)) + g(x,t), \qquad x = (x_1, x_2, x_3) \in \Omega, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^3$, the unknown vector function $u = u(x,t) = (u^{(1)}, u^{(2)}, u^{(3)})$ denotes the velocity of the fluid, $g(x,t) = g(t) = (g^{(1)}, g^{(2)}, g^{(3)})$ is the time-dependent external force function, and the scalar function p represents the pressure. Equations (1.1) and (1.2) describe the motion of an isothermal incompressible viscous fluid, where $\tau(e(u)) = (\tau_{ij}(e(u)))_{3\times 3}$, which is usually called the extra stress tensor of the fluid, is a matrix of order 3×3 and

$$\tau_{ij}(e(u)) = 2\mu_0(\epsilon + |e|^2)^{-\alpha/2} e_{ij} - 2\mu_1 \Delta e_{ij}, \qquad i, j = 1, 2, 3, \tag{1.3}$$

$$e_{ij} = e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad |e|^2 = \sum_{i,j=1}^{3} |e_{ij}|^2,$$

and $\mu_0, \mu_1, \alpha, \epsilon$ are parameters which generally depend on the temperature and pressure. Here we assume μ_0, μ_1, ϵ and α are positive constants. In (1.3) if $\tau_{ij}(e(u))$ linearly depends on $e_{ij}(u)$ then we say the corresponding fluid is a Newtonian one. Generally speaking, gases, water, motor oil, alcohols and simple hydrocarbon compounds tend to be Newtonian fluids and their motions can be described by Navier–Stokes equations. If the relation between $\tau_{ij}(e(u))$ and $e_{ij}(u)$ is nonlinear, then the fluid is called non-Newtonian. For instance, molten plastics, polymer solutions and paints tend to be non-Newtonian fluids. One can refer to [5–7, 34, 39, 42] and the references therein for detailed physical significance. There are many works concerning the unique existence, regularity and asymptotic behaviour of solutions to equations (1.1)–(1.3) or its associated version (see e.g. [4, 7, 8, 27, 28, 30, 34, 39, 42] and [51–54]).

We will first combine the idea of Caraballo *et al* [14, 15, 19] for pullback attractor and Chepyzhov and Vishik [23, 24, 49] for trajectory attractor to formulate a new type of attractor, called *pullback trajectory attractor*, for evolution equations. Then we prove a sufficient condition for the existence of a pullback trajectory attractor for the translation cocycle on an abstract united trajectory space. The definition of pullback trajectory attractor, which is formulated in terms of a θ-cocycle map ϕ on a united trajectory space driven by a group θ on a time symbol space (Σ, σ), provides a family of time-dependent compact sets that pullback attract bounded sets in the united trajectory space and satisfy a cocycle invariance property. This concept can be defined for evolution equations with possible non-uniqueness of solutions as long as the existence can be ensured. The central idea of this concept originates from the above mentioned three methods. At the same time, it contains the characteristic of pullback attractor. It is indeed a technique combination of the trajectory attractor and pullback attractor. Similar to the theory of pullback attractor pointed out by Caraballo *et al* [14], we expect that the concept of pullback trajectory attractor enables one to treat more general non-autonomous terms and will work under random environments as well.

To illustrate the applications of pullback trajectory attractor, we take the initial boundary value problem associated with non-Newtonian fluid equations (1.1)–(1.3) as an example. We first construct the united trajectory space $\mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ and prove the existence of a compact pullback trajectory attractor $\{\mathcal{A}^{\mathrm{tr}}_g\}_{g\in\mathcal{H}(g_0)} = \mathcal{A}^{\mathrm{tr}}_{\mathcal{H}(g_0)} \subset L^\infty(\mathbb{R}_+; H) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta})$. Then we construct the regular united trajectory space $\mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ and prove the existence of a compact pullback trajectory attractor $\{\mathscr{A}^{\mathrm{tr}}_g\}_{g\in\mathcal{H}(g_0)} = \mathscr{A}^{\mathrm{tr}}_{\mathcal{H}(g_0)} \subset L^\infty(\mathbb{R}_+; V) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$. Meanwhile, we establish that $\mathcal{A}^{\mathrm{tr}}_g = \mathscr{A}^{\mathrm{tr}}_g$ for each $g \in \mathcal{H}(g_0)$. This regularity of the pullback trajectory attractor reveals the pullback trajectory asymptotic smoothing effect of the 3D fluid in the following sense: the trajectories issued from $u_0 \in H$ belong to $L^\infty(\mathbb{R}_+; H) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta})$, and (under the pullback acting of the translation cocycle) eventually belong to $L^\infty(\mathbb{R}_+; V) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$ after long enough time, which is the starting point for our interest in the problem.

Compared with the work of [53], here we only require $\alpha > 0$ because we only need the existence of solutions and the uniqueness of solutions does not play an important role in our discussion. While the uniqueness of solutions is a building block of the definition for the corresponding cocycle on the state space and hence [53] takes $\alpha \in (0, 1)$.

We should point out that Chepyzhov and Vishik [24] formulated the concept of united trajectory space and uniform trajectory attractor for non-autonomous equations, where the natural translation semigroup $\{T(t)\}_{t\geqslant 0}$ was considered to act on the united trajectory space and the uniform (with respect to $\sigma \in \Sigma$) trajectory attractor *forward* attracts uniformly (with respect to $\sigma \in \Sigma$) any bounded set of the united trajectory space. Compared with the uniform trajectory attractor for the translation semigroup $\{T(t)\}_{t\geqslant 0}$, the pullback trajectory attractor here considers, in the *pullback* attracting property (see definition 2.4) of the translation cocycle $\phi(t, \theta_{-t}(\sigma), \cdot)$, the state of the trajectories as time $t \to +\infty$ when the initial time $-t$ of the symbol $\theta_{-t}(\sigma)$ goes to $-\infty$.

It is worth emphasizing that Caraballo *et al* [13] studied the pullback attractor of non-autonomous and stochastic multi-valued dynamical systems, where the definition of multi-valued dynamical process (MDP) generalizes the notion of process corresponding to a non-autonomous equation whose Cauchy problem is uniquely solvable. The MDP was defined as a two-parameter family of multi-valued maps. The attraction of any bounded set of the phase space to the attractor is uniform with respect to the first parameter, whereas the rate of attraction and attractor itself can depend on the second one. Moreover, the trajectories of the MDP can be unbounded in time. We note that the pullback trajectory attractor attracts the bounded trajectories in $\mathcal{T}_\sigma^{\text{tr}}$ of the equation with symbol $\sigma \in \Sigma$. At the same time, the attraction manner of the pullback trajectory attractor differs from that of the pullback attractor in [13], where the attractor attracts the solutions of the systems from $-\infty$: the initial state of time goes to $-\infty$ and the final time remains fixed.

The paper is organized as follows. In section 2 we first formulate the concept of pullback trajectory attractor and some related notation for abstract evolution equations. Then we prove a sufficient condition for the existence of a pullback trajectory attractor. In section 3, we first introduce some operators to put the initial boundary value problem associated with equations (1.1)–(1.3) into an abstract Cauchy problem and then prove the existence of a compact pullback trajectory attractor $\mathcal{A}_{\mathcal{H}(g_0)}^{\text{tr}} = \{\mathcal{A}_g^{\text{tr}}\}_{g\in\mathcal{H}(g_0)} \subset L^\infty(\mathbb{R}_+; H) \cap \mathcal{C}_{\text{loc}}(\mathbb{R}_+; H^{-\eta})$. In section 4, we first construct the regular united trajectory space and prove the existence of a compact pullback trajectory attractor $\mathscr{A}_{\mathcal{H}(g_0)}^{\text{tr}} = \{\mathscr{A}_g^{\text{tr}}\}_{g\in\mathcal{H}(g_0)} \subset L^\infty(\mathbb{R}_+; V) \cap \mathcal{C}_{\text{loc}}(\mathbb{R}_+; H^{2-\eta})$. Then we establish that $\mathcal{A}_g^{\text{tr}} = \mathscr{A}_g^{\text{tr}}$ for each $g \in \mathcal{H}(g_0)$.

2. Preliminaries

In this section, we first introduce some notation related to the concept of pullback trajectory attractor and to the natural translation cocycle for an abstract evolution equation. Then we establish a sufficient condition for the existence of a compact pullback trajectory attractor.

Consider the following abstract evolution equation

$$\frac{\partial u}{\partial t} = F_{\sigma(t)}(u), \qquad t \in \mathbb{R}_+. \tag{2.1}$$

For each $t \in \mathbb{R}_+$, we are given an operator $F_{\sigma(t)}(\cdot) : X \longmapsto Y$, where X, Y are Banach spaces such that $X \subseteq Y$ ($X = Y$ is also possible). Here the functional parameter $\sigma(t)$ is the time symbol of equation (2.1), which reflects the dependence on time of the equation. We assume that the values of $\sigma(t)$ belong to some Banach space Σ. (σ, Σ) will be called the time symbol space of equation (2.1). We also assume that there exists a group $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ acting on (σ, Σ) and satisfying

$$\theta_{t+s} = \theta_t\theta_s, \qquad \theta_0 = \text{Identity operator on } \Sigma, \qquad \theta_t\Sigma = \Sigma, \qquad \forall\, t \in \mathbb{R}. \tag{2.2}$$

Proposition 2.1 (Lions–Magenes [38]). *Suppose that $f(\cdot) \in L^\infty(0, T; X)$ and the function $f(t)$ is weakly continuous in Y, i.e., for any function $\psi \in Y^*$ (the dual space of Y) the function $\langle f(t), \psi\rangle \in \mathcal{C}([0, T])$. Then $f(t) \in X$ for all $t \in [0, T]$ and the function $f(t)$ is weakly continuous in X.*

We shall study equation (2.1) with various symbols $\sigma \in \Sigma$. We suppose that for each $\sigma \in \Sigma$, equation (2.1) admits at least one solution. We next treat the solutions $u(t) \in \mathcal{C}(\mathbb{R}_+; Y) \cap L^\infty(\mathbb{R}_+; X)$ of equation (2.1) with $\sigma \in \Sigma$ as a whole. By proposition 2.1, we see that if $u(t) \in \mathcal{C}(\mathbb{R}_+; Y) \cap L^\infty(\mathbb{R}_+; X)$, then $u(t) \in X$ for all $t \in \mathbb{R}_+$.

Definition 2.1. *For each $\sigma \in \Sigma$, $\mathcal{T}_\sigma^{\text{tr}}$ (the trajectory space of equation (2.1) with symbol σ) denotes the set of all solutions of equation (2.1) belonging to $\mathcal{C}(\mathbb{R}_+; Y) \cap L^\infty(\mathbb{R}_+; X)$. $\mathcal{T}_\Sigma^{\text{tr}} = \bigcup_{\sigma\in\Sigma} \mathcal{T}_\sigma^{\text{tr}}$ is the united trajectory space of equation (2.1).*

Now it is necessary for us to introduce a topology in the space $\mathcal{T}_\Sigma^{\rm tr}$. We first introduce a topology in $\mathcal{C}(\mathbb{R}_+; Y)$. The sequence $\{f_n(s)\} \subset \mathcal{C}(\mathbb{R}_+; Y)$ converges to a function $f(s) \in \mathcal{C}(\mathbb{R}_+; Y)$ means for any $T > 0$ there holds

$$\max_{s\in[0,T]} \|f_n(s) - f(s)\|_Y \longrightarrow 0 \quad \text{as } n \to \infty.$$

We denote the topology introduced in $\mathcal{C}(\mathbb{R}_+; Y)$ by $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$. One can see that $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$ is a Fréchet space. Moreover, the topological space $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$ induces a topology in $\mathcal{T}_\Sigma^{\rm tr}$. We also need to define bounded sets in $\mathcal{T}_\Sigma^{\rm tr}$. For this purpose, we adapt the norm of the space $L^\infty(\mathbb{R}_+; X)$. The set $\mathcal{B} \subset \mathcal{T}_\Sigma^{\rm tr}$ is said to be bounded if it is bounded with respect to the norm of $L^\infty(\mathbb{R}_+; X)$, in other words, there exists a number $r = r(\mathcal{B})$ such that

$$\|u(\cdot)\|_{L^\infty(\mathbb{R}_+;X)} = \operatorname*{esssup}_{s\geqslant 0} \|u(s)\|_X \leqslant r, \qquad \forall\, u(\cdot) \in \mathcal{B}.$$

Definition 2.2. *A θ cocycle on $\mathbb{R}_+ \times \Sigma \times \mathcal{T}_\Sigma^{\rm tr}$ is a family of maps $\phi(t, \sigma, \cdot) : \mathcal{T}_\sigma^{\rm tr} \longmapsto \mathcal{T}_{\theta_t(\sigma)}^{\rm tr}$, $(t, \sigma) \in \mathbb{R}_+ \times \Sigma$, satisfying*

(I) $\phi(0, \sigma, u) = u$ for all $(\sigma, u) \in \Sigma \times \mathcal{T}_\sigma^{\rm tr}$,

(II) $\phi(t + s, \sigma, u) = \phi(t, \theta_s(\sigma), \phi(s, \sigma, u))$, $\forall\, t, s \in \mathbb{R}_+$, $(\sigma, u) \in \Sigma \times \mathcal{T}_\sigma^{\rm tr}$. The θ-cocycle ϕ is said to be continuous if for all $(t, \sigma) \in \mathbb{R}_+ \times \Sigma$, the mapping $\phi(t, \sigma, \cdot) : \mathcal{T}_\sigma^{\rm tr} \longmapsto \mathcal{T}_{\theta_t(\sigma)}^{\rm tr}$ is continuous.

In the following, we use Π_+ to denote the restriction operator (with respect to time variable) to the interval $\mathbb{R}_+ = [0, +\infty)$. Analogously, Π_T stands for the restriction operator to the interval $[0, T]$. For example, if $u(\cdot) \in \mathcal{C}(\mathbb{R}_+; Y) \cap L^\infty(\mathbb{R}_+; X)$, then $\Pi_T u(\cdot) \in \mathcal{C}([0, T]; Y) \cap L^\infty(0, T; X)$; $\Pi_T u(t) = u(t)$ if $t \in [0, T]$.

Definition 2.3. *A family of sets $\{\mathscr{P}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_\Sigma^{\rm tr}$ with $\mathscr{P}_\sigma \subset \mathcal{T}_\sigma^{\rm tr}$ for each $\sigma \in \Sigma$ is called a pullback trajectory absorbing set for the cocycle ϕ if for each $\sigma \in \Sigma$ and any family of bounded sets $\{\mathscr{B}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_\Sigma^{\rm tr}$ with $\mathscr{B}_\sigma \subset \mathcal{T}_\sigma^{\rm tr}$, there exists a time $t(\sigma, \mathscr{B}_{\theta_{-t}(\sigma)}, \mathscr{P}_\sigma)$ such that $\phi(t, \theta_{-t}(\sigma), \mathscr{B}_{\theta_{-t}(\sigma)}) \subset \mathscr{P}_\sigma$ for all $t \geqslant t(\sigma, \mathscr{B}_{\theta_{-t}(\sigma)}, \mathscr{P}_\sigma)$. If moreover $\cup_{\sigma\in\Sigma}\mathscr{P}_\sigma$ is compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$ and for each $\sigma \in \Sigma$, $\mathscr{P}_\sigma$ is bounded in $L^\infty(\mathbb{R}_+; X)$, then ϕ is said to be pullback trajectory compact in $\mathcal{T}_\Sigma^{\rm tr}$.*

Definition 2.4. *A family of non-empty compact sets $\{\mathscr{A}_\sigma^{\rm tr}\}_{\sigma\in\Sigma} \subseteq \mathcal{T}_\Sigma^{\rm tr}$ is called a compact pullback trajectory attractor of the θ-cocycle ϕ with respect to the topology $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$ if $\mathscr{A}_\sigma^{\rm tr} \subset \mathcal{T}_\sigma^{\rm tr}$ for each $\sigma \in \Sigma$ and*

(i) compactness: for each $\sigma \in \Sigma$, $\mathscr{A}_\sigma^{\rm tr}$ is compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$ and bounded in $L^\infty(\mathbb{R}_+; X)$;

(ii) ϕ-invariance: $\phi(t, \sigma, \mathscr{A}_\sigma^{\rm tr}) = \mathscr{A}_{\theta_t(\sigma)}^{\rm tr}$, $\forall\, t \in \mathbb{R}_+, \sigma \in \Sigma$;

(iii) pullback attracting property: for any family of sets $\{\mathscr{B}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_\Sigma^{\rm tr}$ which satisfies: for each $\sigma \in \Sigma$, the set $\mathscr{B}_\sigma \subset \mathcal{T}_\sigma^{\rm tr}$ and is bounded in $L^\infty(\mathbb{R}_+; X)$, there holds

$$\lim_{t\to+\infty} \operatorname{dist}_{\mathcal{C}([0,T];Y)} \left(\Pi_T \phi(t, \theta_{-t}(\sigma), \mathscr{B}_{\theta_{-t}(\sigma)}), \Pi_T \mathscr{A}_\sigma^{\rm tr}\right) = 0, \qquad \forall\, T > 0, \tag{2.3}$$

where $\operatorname{dist}_{\mathcal{C}([0,T];Y)}(X_1, X_2) = \sup_{y_1\in X_1} \inf_{y_2\in X_2} \max_{t\in[0,T]} |y_1(t) - y_2(t)|$ stands for the Hausdorff semidistance from the set $X_1 \subset \mathcal{C}([0, T]; Y)$ to the set $X_2 \subset \mathcal{C}([0, T]; Y)$;

(iv) minimality: If $\{\mathscr{E}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_\Sigma^{\rm tr}$ is a family of non-empty compact sets that satisfy (iii), then $\mathscr{A}_\sigma \subseteq \mathscr{E}_\sigma$ for each $\sigma \in \Sigma$.

In definition 2.4, properties (i) and (ii) are generalizations of the compactness and invariance properties of the kernel sections for process (see e.g. [49]). The pullback attracting property (iii) differs from the attraction manner of the kernel sections. Indeed, the above definition of pullback trajectory attractor combines the properties of pullback attractor (generalization of kernel sections, see e.g. [13]) and trajectory attractor (for the equations with possible absence of uniqueness of solutions).

Theorem 2.1. *Let $\phi(t,\sigma,\cdot)$ be a continuous cocycle and the group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on (σ,Σ) satisfies (2.2). If $\phi(t,\sigma,\cdot)$ is pullback trajectory compact in $\mathcal{T}^{\rm tr}_\Sigma$, then $\phi(t,\sigma,\cdot)$ possesses a compact pullback trajectory attractor $\mathscr{A}^{\rm tr}=\{\mathscr{A}^{\rm tr}_\sigma\}_{\sigma\in\Sigma}\subset\mathcal{T}^{\rm tr}_\Sigma$ given by*

$$\mathscr{A}^{\rm tr}_\sigma=\omega_\sigma(\mathscr{P}),\quad \sigma\in\Sigma,$$

where $\mathscr{P}=\{\mathscr{P}_\sigma\}_{\sigma\in\Sigma}$ is the pullback trajectory absorbing set and

$$\omega_\sigma(\mathscr{P})=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t,\theta_{-t}(\sigma),\mathscr{P}_{\theta_{-t}(\sigma)})}$$

(the bar means taking closure in $\mathcal{C}_{\rm loc}(\mathbb{R}_+;Y)$) denotes the pullback ω-limit set of $\mathscr{P}$.

Remark 2.1. Since ϕ is pullback trajectory compact in $\mathcal{T}^{\rm tr}_\Sigma$, $\mathscr{P}_\sigma$ is bounded in $L^\infty(\mathbb{R}_+;X)$ and compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+;Y)$. Thus there is a time $t(\sigma,\mathscr{P}_{\theta_{-t}(\sigma)},\mathscr{P}_\sigma)$ such that $\phi(t,\theta_{-t}(\sigma),\mathscr{P}_{\theta_{-t}(\sigma)})\in\mathscr{P}_\sigma$ for all $t\geqslant t(\sigma,\mathscr{P}_{\theta_{-t}(\sigma)},\mathscr{P}_\sigma)$. Hence

$$\begin{aligned}\omega_\sigma(\mathscr{P})&=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t,\theta_{-t}(\sigma),\mathscr{P}_{\theta_{-t}(\sigma)})}\\&\subseteq\bigcap_{s\geqslant t(\sigma,\mathscr{P}_{\theta_{-t}(\sigma)},\mathscr{P}_\sigma)}\overline{\bigcup_{t\geqslant s}\phi(t,\theta_{-t}(\sigma),\mathscr{P}_{\theta_{-t}(\sigma)})}\\&\subseteq\overline{\mathscr{P}_\sigma}=\mathscr{P}_\sigma\subset\mathcal{T}^{\rm tr}_\sigma.\end{aligned}$$

Proof. The idea of the proof of theorem 2.1 is similar to that of Chepyzhov and Vishik [24] and Temam [47]. For completeness, we present the detailed proof and divide it into four steps.

Step One. We prove that for each fixed $t_0\in\mathbb{R}$, there holds

$$u\in\omega_{\theta_{t_0}(\sigma)}(\mathscr{P})\Longleftrightarrow\begin{cases}\text{For any neighbourhood }\mathcal{N}(u)\text{ (with respect to the topology of}\\ \mathcal{C}_{\rm loc}(\mathbb{R}_+;Y))\text{ of the point }u\text{ there are two sequences}\\ \{u_n\}\subset\mathscr{P}_{\theta_{-t_n+t_0}(\sigma)}\text{ and }\{t_n\}\subset\mathbb{R}_+\text{ with }t_n\to+\infty\\ \text{ as }n\to\infty\text{ such that }\phi(t_n,\theta_{-t_n}(\theta_{t_0}(\sigma)),u_n)\in\mathcal{N}(u).\end{cases}\tag{2.4}$$

In fact, if $u\in\omega_{\theta_{t_0}(\sigma)}(\mathscr{P})$, then for any $s\geqslant 0$, u is a point of tangency of the set $\bigcup_{t\geqslant s}\phi(t,\theta_{-t}(\theta_{t_0}(\sigma)),\mathscr{P}_{\theta_{-t+t_0}(\sigma)})$. Therefore, any neighbourhood $\mathcal{N}(u)$ of u contains a point from $\bigcup_{t\geqslant s}\phi(t,\theta_{-t}(\theta_{t_0}(\sigma)),\mathscr{P}_{\theta_{-t+t_0}(\sigma)})$ for any $s\geqslant 0$. Hence, there exist two sequences $\{u_n\}\subset\mathscr{P}_{\theta_{-t_n+t_0}(\sigma)}$ and $\{t_n\}\subset\mathbb{R}_+$ with $t_n\to+\infty$ as $n\to\infty$ such that $\phi(t_n,\theta_{-t_n}(\theta_{t_0}(\sigma)),u_n)\in\mathcal{N}(u)$. Conversely, assume that for any neighbourhood $\mathcal{N}(u)$ of u there exist two sequences $\{u_n\}\subset\mathscr{P}_{\theta_{-t_n+t_0}(\sigma)}$ and $\{t_n\}\subset\mathbb{R}_+$ with $t_n\to+\infty$ as $n\to\infty$ such that $\phi(t_n,\theta_{-t_n}(\theta_{t_0}(\sigma)),u_n)\in\mathcal{N}(u)$. Since $\phi(t_n,\theta_{-t_n}(\theta_{t_0}(\sigma)),\mathscr{P}_{\theta_{-t_n+t_0}(\sigma)})\subseteq\bigcup_{t\geqslant s}\phi(t,\theta_{-t}(\theta_{t_0}(\sigma)),\mathscr{P}_{\theta_{-t+t_0}(\sigma)})$ when $s\leqslant t_n$, u is a point of tangency of

$$\bigcup_{t\geqslant s}\phi(t,\theta_{-t}(\theta_{t_0}(\sigma)),\mathscr{P}_{\theta_{-t+t_0}(\sigma)})$$

for any $s\geqslant 0$ and whence $u\in\omega_{\theta_{t_0}(\sigma)}(\mathscr{P})$.

Step Two. We prove that for each $\sigma\in\Sigma$, $\omega_\sigma(\mathscr{P})$ is not empty and pullback attracts any family of bounded sets $\{\mathscr{B}_\sigma\}_{\sigma\in\Sigma}\subset\mathcal{T}^{\rm tr}_\Sigma$ with $\mathscr{B}_\sigma\subset\mathcal{T}^{\rm tr}_\sigma$.

Let $\{u_n\}\subset\mathscr{P}_{\theta_{-t_n}(\sigma)}$ and $\{t_n\}\subset\mathbb{R}_+$ with $t_n\to+\infty$ as $n\to\infty$. Set $v_n=\phi(t_n,\theta_{-t_n}(\sigma),u_n)$. If the sequence $\{v_n\}$ has infinite different points, we consider the set $\tilde{\mathscr{P}}=\{v_n\}\bigcup(\cup_{\sigma\in\Sigma}\mathscr{P}_\sigma)$ and prove that $\tilde{\mathscr{P}}$ is relatively compact. Let $\{\mathcal{V}_n\}$ be any covering of $\tilde{\mathscr{P}}$. Obviously, it covers $\cup_{\sigma\in\Sigma}\mathscr{P}_\sigma$. Note that $\cup_{\sigma\in\Sigma}\mathscr{P}_\sigma$ is compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+;Y)$. Thus there is a finite subcovering $\{\mathcal{V}_n\}_{n=1}^{N_0}$ of $\cup_{\sigma\in\Sigma}\mathscr{P}_\sigma$. Let $\mathcal{V}=\bigcup_{n=1}^{N_0}\mathcal{V}_n$. By the pullback absorbing property of $\mathscr{P}_\sigma$, we see that there exists $N_1 \in \mathbb{N}$ such that $v_n = \phi(t_n, \theta_{-t_n}(\sigma), u_n) \in \mathscr{P}_\sigma \subset \cup_{\sigma\in\Sigma}\mathscr{P}_\sigma$ for $n \geqslant N_1$. Thus $\mathcal{V}$ covers $\tilde{\mathscr{P}} \setminus \{v_1, v_2, \cdots, v_{N_1}\}$. Adding the finite number of open sets that cover the finite set $\{v_1, v_2, \cdots, v_{N_1}\}$, we get the finite subcovering of $\tilde{\mathscr{P}}$. Hence $\tilde{\mathscr{P}}$ is relatively compact, which implies that the set $\{v_n\}$ has a limit point. Let v be a limit point of $\{v_n\}$. We now show that $v \in \omega_\sigma(\mathscr{P})$. Let $\mathcal{N}(v)$ be any neighbourhood of v, then there exists some $n_1 \in \mathbb{N}$ such that $v_{n_1} \in \mathcal{N}(v)$, $v_{n_1} \neq v$. Note that $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$ is a Hausdorff space. Hence there is a neighbourhood $\mathcal{N}_1(v) \subset \mathcal{N}(v)$ of v such that $v_{n_1} \notin \mathcal{N}_1(v)$. Analogously, there exists some $n_2 \in \mathbb{N}$ such that $v_{n_2} \in \mathcal{N}_1(v)$, $v_{n_2} \neq v$. Also there is a neighbourhood $\mathcal{N}_2(v) \subset \mathcal{N}(v)$ of v such that $v_{n_2} \notin \mathcal{N}_2(v)$. By this procedure, we get a subsequence $\{v_{n_i}\} \subset \mathcal{N}(v)$, that is, $\phi(t_{n_i}, \theta_{-t_{n_i}}(\sigma), u_{n_i}) \in \mathcal{N}(v)$ with $\{u_{n_i}\} \subset \mathscr{P}_{\theta_{-t_{n_i}}(\sigma)}$ and $t_{n_i} \to +\infty$ as $n_i \to \infty$. Using (2.4), we get $v \in \omega_\sigma(\mathscr{P})$. If $\{v_n\}$ is a finite set, then for some v, let $v_n = v$ infinitely many times, i.e. $\phi(t_{n_i}, \theta_{-t_{n_i}}(\sigma), u_{n_i}) = v_{n_i} = v$ for some subsequence $\{n_i\}$ and so $v \in \omega_\sigma(\mathscr{P})$. By the above facts, we see that $\omega_\sigma(\mathscr{P})$ is not empty.

We next prove that $\{\omega_\sigma(\mathscr{P})\}_{\sigma\in\Sigma}$ pullback attracts any family of bounded sets $\{\mathscr{B}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_\Sigma^{\rm tr}$ with $\mathscr{B}_\sigma \subset \mathcal{T}_\sigma^{\rm tr}$ for each $\sigma \in \Sigma$. We use the argument of contradiction. Suppose that there are a neighbourhood $\mathcal{N}(\omega_\sigma(\mathscr{P}))$ and a sequence $\{u_n\} \subset \mathscr{B}_{\theta_{-t_n}(\sigma)} \subset \mathcal{T}^{\rm tr}_{\theta_{-t_n}(\sigma)}$, $\{t_n\} \subset \mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that

$$\{v_n\}_{n=1}^{\infty} \bigcap \mathcal{N}(\omega_\sigma(\mathscr{P})) = \varnothing, \qquad \text{where } v_n = \phi(t_n, \theta_{-t_n}(\sigma), u_n). \tag{2.5}$$

Since $v_n \in \mathscr{P}_\sigma \subset \cup_{\sigma\in\Sigma}\mathscr{P}_\sigma$ for n large enough and $\cup_{\sigma\in\Sigma}\mathscr{P}_\sigma$ is compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$, the sequence $\{v_n\}$ possesses a limit point (denoted by) v. By (2.4) we see that $v \in \omega_\sigma(\mathscr{P})$. Thus $\mathcal{N}(\omega_\sigma(\mathcal{P}))$ is also a neighbourhood of v, which contradicts (2.5).

Step Three. We prove that $\omega_\sigma(\mathscr{P})$ is the minimal compact set satisfying: for any family of bounded sets $\{\mathscr{B}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_\Sigma^{\rm tr}$ with $\mathscr{B}_\sigma \subset \mathcal{T}_\sigma^{\rm tr}$ there holds

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{C}([0,T];Y)}\left(\Pi_T\phi(t, \theta_{-t}(\sigma), \mathscr{B}_{\theta_{-t}(\sigma)}), \Pi_T\omega_\sigma(\mathscr{P})\right) = 0, \qquad \forall\, T > 0. \tag{2.6}$$

By the definition of ω-limit set, $\omega_\sigma(\mathscr{P})$ is closed. Remark 2.1 shows that $\omega_\sigma(\mathscr{P}) \subset \mathscr{P}_\sigma$, while the pullback trajectory compactness of the cocycle ϕ implies that $\mathscr{P}_\sigma$ is compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$, thus $\omega_\sigma(\mathscr{P})$ is compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$. We next use the argument of contradiction to prove the minimality of $\omega_\sigma(\mathscr{P})$. Let $\mathscr{P}'$ be a compact set satisfying (2.6). Assume that there exists some $v \in \omega_\sigma(\mathscr{P})$ but $v \notin \mathscr{P}'$. Recall that $\mathcal{C}_{\rm loc}(\mathbb{R}_+; Y)$ is a Hausdorff space. Thus for any $u \in \mathscr{P}'$, there exist two neighbourhoods $\mathcal{N}(u)$ and $\mathcal{N}_u(v)$ of u and v, respectively, such that $\mathcal{N}(u) \cap \mathcal{N}_u(v) = \varnothing$. Clearly, the family of open sets $\{\mathcal{N}(u) : u \in \mathscr{P}'\}$ covers $\mathscr{P}'$. Consider a finite subcovering $\{\mathcal{N}(u_i) : i = 1, 2, \cdots, N_0\}$ of $\mathscr{P}'$. Set $\check{\mathscr{P}} = \bigcup_{i=1}^{N_0}\mathcal{N}(u_i)$, $\hat{\mathscr{P}} = \bigcap_{i=1}^{N_0}\mathcal{N}_{u_i}(v)$. Then $\mathscr{P}' \subseteq \check{\mathscr{P}}$, $v \in \hat{\mathscr{P}}$, $\check{\mathscr{P}} \cap \hat{\mathscr{P}} = \varnothing$. Recall $v \in \omega_\sigma(\mathscr{P})$, and we see that there is a sequence $\{u_n\} \subset \mathscr{P}_{\theta_{-t_n}(\sigma)}$ and $\{t_n\} \subset \mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that $\phi(t_n, \theta_{-t_n}(\sigma), u_n) \in \hat{\mathscr{P}}$ (see (2.4)). At the same time, we follow from the absorbing property of $\mathscr{P}'$ that $\phi(t_n, \theta_{-t_n}(\sigma), u_n) \in \check{\mathscr{P}}$ for t_n large enough, which implies that $\check{\mathscr{P}} \cap \hat{\mathscr{P}} \neq \varnothing$. Therefore, we can claim that if $v \in \omega_\sigma(\mathscr{P})$, then $v \in \mathscr{P}'$, i.e. $\omega_\sigma(\mathscr{P}) \subseteq \mathscr{P}'$. The minimality of $\omega_\sigma(\mathscr{P})$ is proved.

Step Four. We show that for each $\sigma \in \Sigma$,

$$\phi(t, \sigma, \omega_\sigma(\mathscr{P})) = \omega_{\theta_t(\sigma)}(\mathscr{P}), \quad \forall\, t \in \mathbb{R}_+. \tag{2.7}$$

Let $v \in \omega_\sigma(\mathscr{P})$ and given any $t_0 \in \mathbb{R}_+$. Consider any neighbourhood $\mathcal{N}(w)$ of the point $w = \phi(t_0, \sigma, v)$. Recall that the map $\phi(t_0, \sigma, \cdot)$ is continuous from $\mathcal{T}_\sigma^{\rm tr}$ to $\mathcal{T}^{\rm tr}_{\theta_{t_0}(\sigma)}$. Hence there exists a neighbourhood $\mathcal{N}(v)$ of v such that $\phi(t_0, \sigma, \mathcal{N}(v)) \subseteq \mathcal{N}(w)$. For the neighbourhood

$\mathcal{N}(v)$ there are $\{u_n\} \subset \mathscr{P}_{\theta_{-t_n+t_0}(\sigma)}$ and $t_n \subset [t_0, +\infty)$ with $t_n \to +\infty$ as $n \to \infty$ such that $\phi(t_n - t_0, \theta_{-t_n+t_0}(\sigma), u_n) \in \mathcal{N}(v)$. Then

$$\begin{aligned}\phi(t_n, \theta_{-t_n}(\theta_{t_0}(\sigma)), u_n) &= \phi(t_0 + t_n - t_0, \theta_{-t_n+t_0}(\sigma), u_n)\\ &= \phi(t_0, \sigma, \phi(t_n - t_0, \theta_{-t_n+t_0}(\sigma), u_n))\\ &\subseteq \phi(t_0, \sigma, \mathcal{N}(v))\\ &\subseteq \mathcal{N}(w)\end{aligned}$$

and again by (2.4) we get $w = \phi(t_0, \sigma, v) \in \omega_{\theta_{t_0}(\sigma)}(\mathscr{P})$. Thus we have

$$\phi(t_0, \sigma, \omega_\sigma(\mathscr{P})) \subseteq \omega_{\theta_{t_0}(\sigma)}(\mathscr{P}), \qquad \forall\, t_0 \in \mathbb{R}_+. \tag{2.8}$$

For the inverse inclusion relation, we need to check two points. One is that $\phi(t_0, \sigma, \omega_\sigma(\mathscr{P}))$ is compact in $\mathcal{C}_{\text{loc}}(\mathbb{R}_+; Y)$ for each $t_0 \in \mathbb{R}_+$. This point is obvious since for each $t_0 \in \mathbb{R}_+$ and $\sigma \in \Sigma$, the map $\phi(t_0, \sigma, \cdot)$ is continuous from $\mathcal{T}_\sigma^{\text{tr}}$ to $\mathcal{T}_{\theta_{t_0}\sigma}^{\text{tr}}$ and $\omega_\sigma(\mathscr{P})$ is compact in $\mathcal{C}_{\text{loc}}(\mathbb{R}_+; Y)$. The other point is that $\phi(t_0, \sigma, \omega_\sigma(\mathscr{P}))$ pullback attracts any family of bounded sets $\{\mathscr{B}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_\Sigma^{\text{tr}}$ with $\mathscr{B}_\sigma \subset \mathcal{T}_\sigma^{\text{tr}}$. Let $\mathcal{V}$ be any open set that contains $\phi(t_0, \sigma, \omega_\sigma(\mathscr{P}))$. By the pullback attracting property of $\omega_\sigma(\mathscr{P})$, we have $\phi(t - t_0, \theta_{-t+t_0}(\sigma), \mathscr{B}_{\theta_{-t+t_0}(\sigma)}) \subset \omega_\sigma(\mathscr{P})$ for any bounded $\mathscr{B}_{\theta_{-t+t_0}(\sigma)} \subset \mathcal{T}_{\theta_{-t+t_0}(\sigma)}^{\text{tr}}$ provided $t \geqslant t(\sigma, \mathscr{B}_{\theta_{-t+t_0}(\sigma)}, \mathscr{P}_\sigma) + t_0$. Now for each $t_0 \in \mathbb{R}_+$,

$$\begin{aligned}\phi(t, \theta_{-t}(\theta_{t_0}(\sigma)), \mathscr{B}_{\theta_{-t+t_0}(\sigma)}) &= \phi(t_0 + t - t_0, \theta_{-t+t_0}(\sigma), \mathscr{B}_{\theta_{-t+t_0}(\sigma)})\\ &= \phi(t_0, \sigma, \phi(t - t_0, \theta_{-t+t_0}(\sigma), \mathscr{B}_{\theta_{-t+t_0}(\sigma)}))\\ (t \text{ being large enough}) &\subseteq \phi(t_0, \sigma, \omega_\sigma(\mathscr{P})) \subseteq \mathcal{V},\end{aligned}$$

which implies the desired pullback attracting property of $\phi(t_0, \sigma, \omega_\sigma(\mathscr{P}))$. From the above two points and the minimality proved in *step three*, we get

$$\omega_{\theta_{t_0}(\sigma)}(\mathscr{P}) \subseteq \phi(t_0, \sigma, \omega_\sigma(\mathscr{P})), \qquad \forall\, t_0 \in \mathbb{R}_+. \tag{2.9}$$

(2.8) and (2.9) give (2.7). The proof of theorem 2.1 is now complete. □

Remark 2.2. From the above proof, we see that the result of theorem 2.1 does not depend on the concrete form of equation (2.1). It is valid for general evolution equations, including those with delays.

3. United trajectory space and pullback trajectory attractor for 3D incompressible non-Newtonian fluid

From the viewpoint of physics, the initial boundary value problem of (1.1)–(1.3) can be formulated as follows:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot \left(2\mu_0(\epsilon + |e|^2)^{-\alpha/2} e - 2\mu_1 \Delta e\right) + g(x, t), \qquad x \in \Omega, \tag{3.1}$$

$$\nabla \cdot u = 0, \qquad x \in \Omega, \tag{3.2}$$

$$u = 0, \qquad \tau_{ijl}\kappa_j\kappa_l = 0, \qquad x \in \partial\Omega, \tag{3.3}$$

$$u|_{t=0} = u_0, \tag{3.4}$$

where $\tau_{ijl} = 2\mu_1(\partial e_{ij}/\partial x_l)$ $(i, j, l = 1, 2, 3)$ and $\kappa = (\kappa_1, \kappa_2, \kappa_3)$ denotes the exterior unit normal to the boundary $\partial\Omega$. The first condition in (3.3) represents the usual no-slip condition associated with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on $\partial\Omega$; it is a direct consequence of the principle of virtual work. We refer to [5–8, 34, 39, 42] and the references therein for detailed physical background.

3.1. United trajectory space

We next introduce some functional spaces and operators. Set

$$\mathscr{V} = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in (\mathcal{C}_0^\infty(\overline{\Omega}))^3,\ \ \nabla\cdot\varphi = 0 \ \text{ in } \ \Omega,\ \ \varphi = 0 \ \text{ on } \ \partial\Omega\},$$
$$H = \text{closure of } \mathscr{V} \ \text{ in } \ (L^2(\Omega))^3 \ \text{ with norm } \|\cdot\|, \qquad H' = \text{dual space of } H;$$
$$V = \text{closure of } \mathscr{V} \ \text{ in } \ (H^2(\Omega))^3 \ \text{ with norm } \|\cdot\|_V, \qquad V' = \text{dual space of } V.$$

$(\cdot,\cdot)$ denotes the inner product in H and $\langle\cdot,\cdot\rangle$ stands for the dual pairing between V and V'; also we set $H^\eta = (-\Delta)^{-\eta/2}H$ ($\eta \geqslant 0$ and the Laplace operator Δ is taken with zero boundary condition $u|_{\partial\Omega} = 0$) and use $H^{-\eta}$ to denote the dual space of H^η. In the whole paper, we take $0 < \eta \leqslant 2$ and thus the embedding $H \hookrightarrow H^{-\eta}$ is compact.

Write

$$a(u,v) = \sum_{i,j,k=1}^{3}\left(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(v)}{\partial x_k}\right) = \sum_{i,j,k=1}^{3}\int_\Omega \frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(v)}{\partial x_k}\mathrm{d}x, \quad u, v \in V. \tag{3.5}$$

Lemma 3.1 (Bloom and Hao [7]). *There exist two positive constants c_1 and c_2 which depend only on Ω such that*

$$c_1\|u\|_V^2 \leqslant a(u,u) \leqslant c_2\|u\|_V^2, \qquad \forall\, u \in V. \tag{3.6}$$

From the definition of $a(\cdot,\cdot)$ and lemma 3.1 we see that $a(\cdot,\cdot)$ defines a positive definite symmetric bilinear form on V. As a consequence of the Lax–Milgram lemma, we obtain an isometric operator $A \in \mathscr{L}(V, V')$, via

$$\langle Au, v\rangle = a(u,v), \qquad \forall\, u,\ v \in V.$$

Moreover, let $D(A) = \{u \in V : Au \in H\}$, then $D(A)$ is a Hilbert space and A is also an isometry map from $D(A)$ to H. Indeed, $A = P\Delta^2$, where P is the Leray projector from $L^2(\Omega)$ to H. Since by lemma 3.1 there holds

$$c_1\|u\|_V^2 \leqslant a(u,u) = \langle Au, u\rangle = (Au, u) \leqslant \|Au\|\|u\|_V, \qquad \forall\, u \in D(A),$$

we have

$$c_1\|u\|_V \leqslant \|Au\|. \tag{3.7}$$

For brevity, we use $H_0^1(\Omega)$ to denote $(H_0^1(\Omega))^3$ in the following. We also define a continuous trilinear form on $H_0^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$ as follows:

$$b(u,v,w) = \sum_{i,j=1}^{3}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j\,\mathrm{d}x, \qquad u, v, w \in H_0^1(\Omega).$$

Since $V \subset H_0^1(\Omega)$, $b(\cdot,\cdot,\cdot)$ is continuous on $V\times V\times V$ and one can check

$$b(u,v,w) = -b(u,w,v), \quad b(u,v,v) = 0, \qquad \forall\, u, v, w \in V. \tag{3.8}$$

Now for any $u \in V$,

$$\langle B(u), w\rangle = b(u,u,w), \qquad \forall\, w \in V, \tag{3.9}$$

defines a continuous functional $B(u)$ from $V\times V$ to V'. Finally, for $u \in V$, we set

$$\mu(u) = 2\mu_0(\epsilon + |e(u)|^2)^{-\alpha/2}$$

and define $N(u)$ as

$$\langle N(u), v\rangle = \sum_{i,j=1}^{3}\int_\Omega \mu(u)e_{ij}(u)e_{ij}(v)\,\mathrm{d}x, \qquad \forall\, v \in V. \tag{3.10}$$

Then the functional $N(u)$ is continuous from V to V'. When $u \in D(A)$, $N(u)$ can be extended to H via

$$\langle N(u), v\rangle = -\int_\Omega \{\nabla \cdot [\mu(u)e(u)]\} \cdot v \, \mathrm{d}x, \qquad \forall\, v \in H. \tag{3.11}$$

In this paper, we take in (3.1) an external force function $g_0(x,t) = g_0(t) = g_0 \in L^2_b(\mathbb{R}; H)$ and take $(\sigma, \Sigma) = (g, \mathcal{H}(g_0))$ as the time symbol space, where $L^2_b(\mathbb{R}; H)$ denotes the set of functions $g \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfying

$$\|g\|^2_{L^2_b(\mathbb{R};H)} = \|g\|^2_{L^2_b} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(\rho)\|^2 \, \mathrm{d}\rho < +\infty,$$

and $\mathcal{H}(g_0(\cdot)) = \overline{\{g_0(\cdot + s) : s \in \mathbb{R}\}}$ if $g_0(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, where the bar denotes taking closure with the topology $L^2_{\mathrm{loc}}(\mathbb{R}; H)$. Note that we have $g \in L^2_b(\mathbb{R}; H)$ and $\|g\|_{L^2_b} \leqslant \|g_0\|_{L^2_b}$ for all $g \in \mathcal{H}(g_0)$. We shall study the family of equations (3.1)–(3.4) with various external forces $g(\cdot, t) \in \mathcal{H}(g_0)$. Excluding the pressure p, we can express the weak version of problems (3.1)–(3.4) in the solenoidal vector fields as follows (see [7, 51]):

$$\frac{\partial u}{\partial t} + 2\mu_1 A u + B(u) + N(u) = g(x,t), \quad t > 0, \tag{3.12}$$

$$u|_{t=0} = u_0 \in H. \tag{3.13}$$

Lemma 3.2. *If $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$, then Au, $B(u)$ and $N(u)$ all belong to $L^{4/3}(0,T;V')$.*

Proof. For any $v \in L^4(0,T;V)$, applying (3.8) and (3.9), Hölder inequality and Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}
\int_0^T \langle B(u(t)), v(t)\rangle \, \mathrm{d}t &\leqslant \int_0^T \|u(t)\|^2_{L^4(\Omega)} \|\nabla v(t)\| \, \mathrm{d}t \\
&\leqslant \lambda \int_0^T \|u(t)\|^{1/2} \|\nabla u(t)\|^{3/2} \|\nabla v(t)\| \, \mathrm{d}t \\
&\leqslant \lambda \int_0^T \|u(t)\|^{1/2} \|u(t)\|^{3/2}_V \|v(t)\|_V \, \mathrm{d}t \\
&\leqslant \lambda \left(\int_0^T \left(\|u(t)\|^{1/2} \|u(t)\|^{3/2}_V \right)^{4/3} \mathrm{d}t \right)^{3/4} \left(\int_0^T \|v(t)\|^4_V \, \mathrm{d}t \right)^{1/4} \\
&\leqslant \lambda \|u(t)\|^{1/2}_{L^\infty(0,T;H)} \|u(t)\|^{3/2}_{L^2(0,T;V)} \|v(t)\|_{L^4(0,T;V)},
\end{aligned} \tag{3.14}$$

hereafter λ is a constant depending on Ω, but not on u and T. Equation (3.14) implies $B(u) \in L^{4/3}(0,T;V')$. Similarly, for any $v \in L^2(0,T;V)$, we have

$$\begin{aligned}
\int_0^T \langle N(u(t)), v(t)\rangle \, \mathrm{d}t &= \sum_{i,j=1}^3 \int_0^T \int_\Omega \mu(u(t)) e_{ij}(u(t)) e_{ij}(v(t)) \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant 9\mu_0 \epsilon^{-\alpha/2} \int_0^T \|\nabla u(t)\| \|\nabla v(t)\| \, \mathrm{d}t \\
&\leqslant 9\mu_0 \epsilon^{-\alpha/2} \left(\int_0^T \|u(t)\|^2_V \, \mathrm{d}t \right)^{1/2} \left(\int_0^T \|v(t)\|^2_V \, \mathrm{d}t \right)^{1/2} \\
&= 9\mu_0 \epsilon^{-\alpha/2} \|u(t)\|_{L^2(0,T;V)} \|v(t)\|_{L^2(0,T;V)}.
\end{aligned} \tag{3.15}$$

Equation (3.15) implies $N(u) \in L^2(0, T; V') \subset L^{4/3}(0, T; V')$. In addition, if $u \in L^2(0, T; V)$, $Au \in L^2(0, T; V') \subset L^{4/3}(0, T; V')$ is obvious. Combining the above facts, we get the desired result. The proof is complete. □

Lemma 3.3 ([24]). *Let E_1 be a Banach space and $E \hookrightarrow E_0 \subseteq E_1$. Also let the embedding $E \hookrightarrow E_0$ be compact. Set*

$$W_{\infty,p}(0, T; E, E_1) = \left\{\phi(t), t \in [0, T] : \ \phi(t) \in L^\infty(0, T; E), \ \ \phi'(t) \in L^p(0, T; E_1)\right\},$$

where $p > 1$, with norm $\|\phi\|_{W_{\infty,p}} = \operatorname{ess\,sup}\{\|\phi(t)\|_E : t \in [0, T]\} + \left(\int_0^T \|\phi'\|_{E_1}^p\right)^{1/p}$. Then $W_{\infty,p}(0, T; E, E_1) \hookrightarrow \mathcal{C}([0, T]; E_0)$ with compact embedding.

We next specify the definition of the solution of (3.12). Given $g \in \mathcal{H}(g_0)$. A function $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ is called a weak solution of (3.12) on the interval $[0, T]$ if u, together with its derivative $\partial_t u$, satisfies (3.12) in the sense of distributions in $\mathcal{D}'(0, T; V')$ (see [48]). We can prove by using the Galerkin method that (3.12) has at least one solution $u(t) \in L^\infty(0, T; H) \cap L^2(0, T; V)$ defined on the interval $[0, T]$ ($\forall T > 0$) and satisfying the following energy inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u(t), u(t)) + 2\mu_1\langle Au(t), u(t)\rangle + \langle N(u(t)), u(t)\rangle \leqslant \langle g, u(t)\rangle, \qquad \forall t \in [0, T], \tag{3.16}$$

in the following sense:

$$-\frac{1}{2}\int_0^T \|u(t)\|^2\psi'(t)\,\mathrm{d}t + 2\mu_1\int_0^T \langle Au(t), u(t)\rangle\psi(t)\,\mathrm{d}t + \int_0^T \langle N(u(t)), u(t)\rangle\psi(t)\,\mathrm{d}t$$
$$\leqslant \int_0^T \langle g, u(t)\rangle\psi(t)\,\mathrm{d}t, \qquad \forall\,\psi(t) \in \mathcal{C}_0^\infty([0, T]), \quad \psi(t) \geqslant 0, \quad \forall\, T \geqslant 0. \tag{3.17}$$

We would like to point out that, by lemma 3.2, the derivative $\partial_t u$ of the weak solution belongs to $L^{4/3}(0, T; V')$. Hence, u is almost everywhere equal to some function from $\mathcal{C}([0, T]; H)$ and the initial condition (3.13) makes sense.

Definition 3.1. *The trajectory space $\mathcal{T}_g^{\mathrm{tr}}$ of equation (3.12) with symbol g consists of functions $u(x, t) \in L^\infty(\mathbb{R}_+; H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; V)$ such that for $\forall\, T > 0$ the function $\Pi_T u(t)$ is a weak solution of (3.12) on $[0, T]$ and $\Pi_T u(t)$ satisfies (3.17). $\mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)} = \bigcup_{g\in\mathcal{H}(g_0)} \mathcal{T}_g^{\mathrm{tr}}$ is called the united trajectory space of equation (3.12).*

Let the group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(g, \mathcal{H}(g_0))$ be defined by

$$\theta_t(g(\cdot)) = g(t + \cdot), \qquad \forall\, t \in \mathbb{R}, \quad \forall\, g(\cdot) \in \mathcal{H}(g_0).$$

Evidently, we have

$$\theta_t\mathcal{H}(g_0) = \mathcal{H}(g_0), \qquad \forall\, t \in \mathbb{R}.$$

We can now define the natural translation θ-cocycle $\phi(t, g, u)$ on $\mathbb{R}_+ \times \mathcal{H}(g_0) \times \mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ as

$$\phi(t, g, u(\cdot)) = u_{\theta_t(g)}(t + \cdot), \qquad (t, g, u(\cdot)) \in \mathbb{R}_+ \times \mathcal{H}(g_0) \times \mathcal{T}_g^{\mathrm{tr}}, \tag{3.18}$$

where $u_{\theta_t(g)}$ denotes the weak solution of (3.12) with symbol $\theta_t(g)$. In fact, we have

$$\phi(0, g, u(\cdot)) = u(\cdot), \qquad \forall\,(g, u) \in \mathcal{H}(g_0) \times \mathcal{T}_g^{\mathrm{tr}},$$

and

$$\begin{aligned}\phi(t + s, g, u(\cdot)) &= u_{\theta_{t+s}(g)}(t + s + \cdot) = \phi(t, \theta_s(g), \phi(s, g, u)) = \phi(t, \theta_s(g), u_{\theta_s(g)}(s + \cdot))\\ &= \phi(t, \theta_s(g), \phi(s, g, u(\cdot))).\end{aligned}$$

Lemma 3.4. *(i) For each $g \in \mathcal{H}(g_0)$ and any $u_0 \in H$, there exists at least one trajectory $u(t) \in \mathcal{T}_g^{\mathrm{tr}} \subset \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ such that $u(0) = u_0$;*

(ii) For any $t \in \mathbb{R}_+$, $g \in \mathcal{H}(g_0)$ and $u \in \mathcal{T}_g^{\mathrm{tr}} \subset \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$, there holds $\phi(t, g, u) \in \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$.

Proof. The proof of (i) can be found in [7]. The assertion of (ii) is obvious because for any $u \in \mathcal{T}_g^{\mathrm{tr}}$, we have by definition that $\phi(t, g, u) = u_{\theta_t(g)}(t + \cdot) \in \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$. □

Evidently, we have $\phi(t, g, \mathcal{T}_g^{\mathrm{tr}}) \subseteq \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ for any $t \in \mathbb{R}_+$ and $g \in \mathcal{H}(g_0)$.

Lemma 3.5. $\mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}} \subseteq \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta}) \cap L^\infty(\mathbb{R}_+; H)$.

Proof. We need to show that for each $g \in \mathcal{H}(g_0)$, $\mathcal{T}_g^{\mathrm{tr}} \subseteq \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta}) \cap L^\infty(\mathbb{R}_+; H)$. Indeed, for any $u(\cdot) \in \mathcal{T}_g^{\mathrm{tr}}$, $u(\cdot) \in L^\infty(\mathbb{R}_+; H)$ is obvious. At the same time, since $u(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}_+; V) \cap L^\infty(\mathbb{R}_+; H)$ and $g \in \mathcal{H}(g_0) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+; H) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+; V')$, we use lemma 3.2 and equation (3.12) to get $\partial_t u(\cdot) \in L^{4/3}_{\mathrm{loc}}(\mathbb{R}_+; V')$. Note that $H \hookrightarrow H^{-\eta} \subseteq V'$ and the embedding $H \hookrightarrow H^{-\eta}$ is compact, we infer from lemma 3.3 that $u(\cdot) \in \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta})$. By the arbitrariness of g and $u(\cdot)$, we end the proof. □

3.2. Pullback trajectory attractor for 3D incompressible non-Newtonian fluid

In this section, we use the result developed in section 2 to prove the existence of a pullback trajectory attractor for the incompressible non-Newtonian fluid.

Definition 3.2. *A family of compact sets $\{\mathcal{A}_g^{\mathrm{tr}}\}_{g\in\mathcal{H}(g_0)} \subseteq \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ is called the compact pullback trajectory attractor of equation (3.12) with respect to the topology $\mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta})$ if $\mathcal{A}_g^{\mathrm{tr}} \subset \mathcal{T}_g^{\mathrm{tr}}$ for each $g \in \mathcal{H}(g_0)$ and*

(1) compactness: $\mathcal{A}_g^{\mathrm{tr}}$ is compact in $\mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta})$ and bounded in $L^\infty(\mathbb{R}_+; H)$;

(2) ϕ-invariance: $\phi(t, g, \mathcal{A}_g^{\mathrm{tr}}) = \mathcal{A}_{\theta_t(g)}^{\mathrm{tr}}$, $\forall t \geqslant 0$;

(3) pullback attracting property: for any family of bounded (in $L^\infty(\mathbb{R}_+; H)$ norm) sets $\{\mathcal{B}_g\}_{g\in\mathcal{H}(g_0)} \subset \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ with $\mathcal{B}_g \subset \mathcal{T}_g^{\mathrm{tr}}$ there holds

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{C}([0,T]; H^{-\eta})} \left(\Pi_T \phi(t, \theta_{-t}(g), \mathcal{B}_{\theta_{-t}(g)}), \Pi_T \mathcal{A}_g^{\mathrm{tr}}\right) = 0, \qquad \forall T > 0; \tag{3.19}$$

(4) minimality: If $\{\mathcal{E}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{T}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ is a family of non-empty compact sets that satisfy (3), then $\mathcal{A}_g \subseteq \mathcal{E}_g$ for each $g \in \mathcal{H}(g_0)$.

Lemma 3.6 ([24]). *Let $y(s), K(s) \in L^1_{\mathrm{loc}}(0, +\infty)$ and*

$$-\int_0^{+\infty} y(s)\psi'(s)\,\mathrm{d}s + \delta \int_0^{+\infty} y(s)\psi(s)\,\mathrm{d}s \leqslant \int_0^{+\infty} K(s)\psi(s)\,\mathrm{d}s$$

hold for any $\psi(s) \in \mathcal{C}_0^\infty(\mathbb{R}_+)$, $\psi(s) \geqslant 0$, where $\delta \in \mathbb{R}$. Then for any $t \geqslant \tau \geqslant 0$ there holds

$$y(t)\mathrm{e}^{\delta t} - y(\tau)^{\delta\tau} \leqslant \int_\tau^t K(s)\mathrm{e}^{\delta s}\,\mathrm{d}s.$$

The following estimate plays an important role in our proof.

Lemma 3.7. *For any $g \in \mathcal{H}(g_0)$ and any trajectory $u \in \mathcal{T}^{\mathrm{tr}}_{\theta_{-t}(g)} \subset \mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$, there exist positive constants c_8, c_9, R_0 and δ, which are independent of g and u, such that*

$$\begin{aligned}
&\|\phi(t,\theta_{-t}(g),u)\|_{L^\infty(\mathbb{R}_+;H)} + \|\phi(t,\theta_{-t}(g),u)\|_{L^2(0,1;V)} + \|\partial_t\phi(t,\theta_{-t}(g),u)\|_{L^{4/3}(0,1;V')} \\
&\qquad = \operatorname*{esssup}_{s\geqslant 0}\|u(s+t)\| + \left(\int_0^1 \|u(s+t)\|_V^2\,\mathrm{d}s\right)^{1/2} + \left(\int_0^1 \|\partial_t u(t+s)\|_{V'}^{4/3}\,\mathrm{d}s\right)^{3/4} \\
&\qquad = \operatorname*{esssup}_{\rho\geqslant t}\|u(\rho)\| + \left(\int_t^{t+1} \|u(\rho)\|_V^2\,\mathrm{d}\rho\right)^{1/2} + \left(\int_t^{t+1} \|\partial_t u(\rho)\|_{V'}^{4/3}\,\mathrm{d}\rho\right)^{3/4} \\
&\qquad \leqslant c_8\|u\|_{L^\infty(0,1;H)}\mathrm{e}^{-\delta t/2} + c_9\|u\|^2_{L^\infty(0,1;H)}\mathrm{e}^{-\delta t} + R_0, \qquad \forall t \geqslant 0.
\end{aligned} \tag{3.20}$$

Proof. Obviously, we have

$$\|u\|^2 \leqslant \|u\|_V^2, \qquad \forall u \in V. \tag{3.21}$$

Let $u(\cdot) \in \mathcal{T}^{\mathrm{tr}}_{\theta_{-t}(g)}$. From (3.17), lemma 3.1 and the non-negativity of $\langle N(u(\rho)), u(\rho)\rangle$, we see that

$$\begin{aligned}
&-\int_0^{+\infty} \|u(\rho)\|^2\psi'(\rho)\,\mathrm{d}\rho + 2\mu_1 c_1\int_0^{+\infty}\|u(\rho)\|^2\psi(\rho)\,\mathrm{d}\rho \\
&\qquad \leqslant \int_0^{+\infty}\left[\frac{2}{\mu_1 c_1}\|g(\rho-t)\|^2 - 2\mu_1 c_1(\|u(\rho)\|_V^2 - \|u(\rho)\|^2)\right]\psi(\rho)\,\mathrm{d}\rho
\end{aligned} \tag{3.22}$$

holds for $\forall\,\psi(\rho) \in \mathcal{C}_0^\infty(\mathbb{R}_+)$, $\psi(\rho) \geqslant 0$. Applying lemma 3.6 to (3.22) for

$$\delta = 2\mu_1 c_1, \quad y(\rho) = \|u(\rho)\|^2, \quad K(\rho) = \frac{2}{\mu_1 c_1}\|g(\rho-t)\|^2 - 2\mu_1 c_1(\|u(\rho)\|_V^2 - \|u(\rho)\|^2),$$

we get

$$\begin{aligned}
&\|u(t)\|^2\mathrm{e}^{\delta t} - \|u(\tau)\|^2\mathrm{e}^{\delta\tau} + \delta\int_\tau^t \left(\|u(\rho)\|_V^2 - \|u(\rho)\|^2\right)\mathrm{e}^{\delta\rho}\,\mathrm{d}\rho \\
&\qquad \leqslant \frac{4}{\delta}\int_\tau^t \|g(\rho-t)\|^2\mathrm{e}^{\delta\rho}\,\mathrm{d}\rho, \qquad t \geqslant \tau \geqslant 0, \quad \forall g \in \mathcal{H}(g_0),
\end{aligned} \tag{3.23}$$

which, together with (3.21) gives

$$\|u(t)\|^2 \leqslant \|u(\tau)\|^2\mathrm{e}^{-\delta(t-\tau)} + \frac{4}{\delta}\int_\tau^t \|g(\rho-t)\|\mathrm{e}^{-\delta(t-\rho)}\,\mathrm{d}\rho, \qquad \forall t \geqslant \tau \geqslant 0, \quad \forall g \in \mathcal{H}(g_0). \tag{3.24}$$

For the second term on the right-hand side of (3.24), we have

$$\begin{aligned}
&\frac{4}{\delta}\int_\tau^t \mathrm{e}^{-\delta(t-\rho)}\|g(\rho-t)\|^2\,\mathrm{d}\rho \\
&\quad \leqslant \frac{4}{\delta}\left(\int_{t-1}^t \mathrm{e}^{-\delta(t-\rho)}\|g(\rho-t)\|^2\,\mathrm{d}\rho + \int_{t-2}^{t-1}\mathrm{e}^{-\delta(t-\rho)}\|g(\rho-t)\|^2\,\mathrm{d}\rho + \cdots\right) \\
&\quad \leqslant \frac{4}{\delta}\left(\int_{t-1}^t \|g(\rho-t)\|^2\,\mathrm{d}\rho + \mathrm{e}^{-\delta}\int_{t-2}^{t-1}\|g(\rho-t)\|^2\,\mathrm{d}\rho + \mathrm{e}^{-2\delta}\int_{t-3}^{t-2}\|g(\rho-t)\|^2\,\mathrm{d}\rho + \cdots\right) \\
&\quad \leqslant \frac{4}{\delta}\left(1 + \mathrm{e}^{-\delta} + \mathrm{e}^{-2\delta} + \cdots\right)\|g\|^2_{L_b^2} \\
&\quad \leqslant \frac{4}{\delta}\left(1 + \frac{1}{\delta}\right)\|g_0\|^2_{L_b^2}, \qquad \forall t \geqslant \tau \geqslant 0, \quad \forall g \in \mathcal{H}(g_0).
\end{aligned}$$

Hence, from (3.24) we get

$$\|\phi(t,\theta_{-t}(g),u)\|_{L^\infty(\mathbb{R}_+;H)} \leqslant e^{\delta/2}\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + R_1, \qquad \forall\, t\in\mathbb{R}_+, \tag{3.25}$$

where $R_1 = 2\sqrt{\frac{1}{\delta}(1+\frac{1}{\delta})}\|g_0\|_{L^2_b}$ is independent of g and u.

Using u to take dual pairing $\langle\cdot,\cdot\rangle$ with equation (3.12) and integrating with respect to time variable from t to $t+1$, we obtain

$$\begin{aligned}
&\frac{1}{2}(\|u(t+1)\|^2 - \|u(t)\|^2) + 2\mu_1\int_t^{t+1}\langle Au(\rho),u(\rho)\rangle\,d\rho + \int_t^{t+1}\langle N(u(\rho)),u(\rho)\rangle\,d\rho\\
&\quad = \int_t^{t+1}\langle g(\rho-t),u(\rho)\rangle\,d\rho \leqslant \int_t^{t+1}\|g(\rho-t)\|\|u(\rho)\|_V\,d\rho, \qquad \forall\, t\geqslant 0, \quad \forall\, g\in\mathcal{H}(g_0).
\end{aligned} \tag{3.26}$$

Recall that $\int_t^{t+1}\langle N(u(\rho)),u(\rho)\rangle\,d\rho \geqslant 0$. Using Cauchy inequality, we deduce from (3.26) and lemma 3.1 that

$$\begin{aligned}
&\|u(t+1)\|^2 + 2c_1\mu_1\int_t^{t+1}\|u(\rho)\|_V^2\,d\rho\\
&\qquad \leqslant \frac{1}{2c_1\mu_1}\int_t^{t+1}\|g(\rho-t)\|^2\,d\rho + \|u(t)\|^2\\
&\qquad \leqslant \frac{1}{2c_1\mu_1}\|g_0\|_{L_b^2}^2 + \|u(t)\|^2, \qquad \forall\, t\geqslant 0, \quad \forall\, g\in\mathcal{H}(g_0).
\end{aligned} \tag{3.27}$$

It then follows from (3.25) and (3.27) that

$$\begin{aligned}
\|\phi(t,\theta_{-t}(g),u)\|_{L^2(0,1;V)} &= \left(\int_t^{t+1}\|u(\rho)\|_V^2\,d\rho\right)^{1/2}\\
&\leqslant \left(\frac{1}{2\mu_1c_1}\left(\frac{\|g_0\|_{L_b^2}^2}{2\mu_1c_1} + \|u(t)\|^2\right)\right)^{1/2}\\
&\leqslant \frac{\|g_0\|_{L_b^2}}{2\mu_1c_1} + \frac{e^{\delta/2}\|u\|_{L^\infty(0,1;H)}^{-\delta t/2} + R_1}{\sqrt{2\mu_1c_1}}\\
&\doteq c_3\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + R_2, \qquad \forall\, t\geqslant 0, \quad \forall\, g\in\mathcal{H}(g_0),
\end{aligned} \tag{3.28}$$

where both $c_3 = \frac{e^{\delta/2}}{\sqrt{2\mu_1c_1}}$ and $R_2 = \frac{R_1}{\sqrt{2\mu_1c_1}} + \frac{\|g_0\|_{L_b^2}}{2\mu_1c_1}$ are independent of g and u.

Now we derive from (3.14) that

$$\begin{aligned}
&\left(\int_t^{t+1}\|B(u(\rho))\|_{V'}^{4/3}\,d\rho\right)^{3/4}\\
&\qquad \leqslant \lambda\|u(\rho)\|_{L^\infty(t,t+1;H)}^{1/2}\|u(\rho)\|_{L^2(t,t+1;V)}^{3/2}\\
&\qquad = \lambda\|\phi(t,\theta_{-t}(g),u)\|_{L^\infty(\mathbb{R}_+;H)}^{1/2}\|\phi(t,\theta_{-t}(g),u)\|_{L^2(0,1;V)}^{3/2}\\
&\qquad \leqslant \lambda\left(e^{\delta/2}\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + R_1\right)^{1/2}\left(c_3\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + R_2\right)^{3/2}\\
&\qquad \leqslant \lambda\left(c_4^2\|u\|_{L^\infty(0,1;H)}^2e^{-\delta t} + 2c_4R_3\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + R_3^2\right),
\end{aligned} \tag{3.29}$$

where both $c_4 = \max\{e^{\delta/2}, c_3\}$ and $R_3 = \max\{R_1, R_2\}$ are independent of g and u; λ is the same constant as appearing in (3.14). Similarly, by (3.15) we have

$$\left(\int_t^{t+1} \|N(u(s))\|_{V'}^{4/3}\,\mathrm{d}s\right)^{3/4} \leqslant \left(\int_t^{t+1} \|N(u(s))\|_{V'}^{2}\,\mathrm{d}s\right)^{1/2}$$
$$\leqslant 9\mu_0\epsilon^{-\alpha/2}\left(\int_t^{t+1} \|u(s)\|_V^2\,\mathrm{d}s\right)^{1/2} \leqslant 9\mu_0\epsilon^{-\alpha/2}\left(R_2 + c_3\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2}\right)$$
$$\doteq c_5\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + R_4, \tag{3.30}$$

where both $c_5 = 9c_3\mu_0\epsilon^{-\alpha/2}$ and $R_4 = 9\mu_0\epsilon^{-\alpha/2}R_2$ are independent of g and u. Since A is an isometry operator from V to V' and $g \in L_b^2(\mathbb{R}; H) \subset L_b^2(\mathbb{R}; V')$, equation (3.12) implies that

$$\left(\int_t^{t+1} \|\partial_t u(\rho)\|_{V'}^{4/3}\,\mathrm{d}\rho\right)^{3/4}$$
$$\leqslant 2\mu_1\left(\int_t^{t+1} \|Au(\rho)\|_{V'}^{4/3}\,\mathrm{d}\rho\right)^{3/4} + \left(\int_t^{t+1} \|B(u(\rho))\|_{V'}^{4/3}\,\mathrm{d}\rho\right)^{3/4}$$
$$+\left(\int_t^{t+1} \|N(u(\rho))\|_{V'}^{4/3}\,\mathrm{d}\rho\right)^{3/4} + \left(\int_t^{t+1} \|g(\rho-t)\|^{4/3}\,\mathrm{d}\rho\right)^{3/4}$$
$$\leqslant 2\mu_1\left(\int_t^{t+1} \|Au(\rho)\|_{V'}^{2}\,\mathrm{d}\rho\right)^{1/2} + \left(\int_t^{t+1} \|B(u(\rho))\|_{V'}^{4/3}\,\mathrm{d}\rho\right)^{3/4}$$
$$+\left(\int_t^{t+1} \|N(u(\rho))\|_{V'}^{4/3}\,\mathrm{d}\rho\right)^{3/4} + \left(\int_t^{t+1} \|g(\rho-t)\|^{2}\,\mathrm{d}\rho\right)^{1/2}$$
$$\leqslant 2\mu_1\left(\int_t^{t+1} \|u(\rho)\|_{V}^{2}\,\mathrm{d}\rho\right)^{1/2} + \lambda c_4^2\|u\|^2_{L^\infty(0,1;H)}e^{-\delta t} + \lambda R_3^2 + R_4$$
$$+2\lambda c_4 R_3\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + c_5\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + \|g_0\|_{L_b^2}$$
$$\doteq c_6\|u\|^2_{L^\infty(0,1;H)}e^{-\delta t} + c_7\|u\|_{L^\infty(0,1;H)}e^{-\delta t/2} + R_5, \tag{3.31}$$

where both $c_6 = \lambda c_4^2$, $c_7 = 2\mu_1 c_3 + 2\lambda c_4 R_3 + c_5$ and $R_5 = 2\mu_1 R_2 + R_3 + R_4 + \|g_0\|_{L_b^2}$ are independent of g and u. Using (3.25), (3.28) and (3.31), we obtain (3.20). The proof is complete. □

Lemma 3.8. *Let $g_0 \in L_b^2(\mathbb{R}; H)$, then the θ-cocycle $\phi(t, g, \cdot)$ (defined by (3.18)) possesses a pullback trajectory absorbing set $\{\mathcal{P}_g\}_{g\in\mathcal{H}(g_0)} \subset \mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ with $\mathcal{P}_g \subset \mathcal{T}^{\mathrm{tr}}_g$ for each $g \in \mathcal{H}(g_0)$. Moreover, there is a set $\mathcal{P} \subset \mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ such that $\bigcup_{g\in\mathcal{H}(g_0)}\mathcal{P}_g \subset \mathcal{P}$ and $\mathcal{P}$ is bounded in $L^\infty(\mathbb{R}_+; H)$.*

Proof. Set

$$\mathcal{P}_g = \left\{u \in \mathcal{T}^{\mathrm{tr}}_g : \sup_{t\geqslant 0}\{\|u\|_{L^\infty(t,t+1;H)} + \|\partial_t u\|_{L^{4/3}(t,t+1;V')}\} \leqslant 3R_0\right\},$$
$$\mathcal{P} = \left\{u \in \mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)} : \sup_{t\geqslant 0}\{\|u\|_{L^\infty(t,t+1;H)} + \|\partial_t u\|_{L^{4/3}(t,t+1;V')}\} \leqslant 3R_0\right\}, \tag{3.32}$$

where R_0 is the same positive constant as appearing in lemma 3.7. We next prove that $\{\mathcal{P}_g\}_{g\in\mathcal{H}(g_0)}$ is a pullback trajectory absorbing set for the translation cocycle $\phi(t, g, \cdot)$. Indeed, let $\{\mathcal{B}_g\}_{g\in\mathcal{H}(g_0)}$ be a family of bounded (in $L^\infty(\mathbb{R}_+; H)$ norm) sets of $\mathcal{T}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ with $\mathcal{B}_g \subset \mathcal{T}^{\mathrm{tr}}_g$.

Then from (3.20) we see that for $\forall u \in \mathcal{B}_{\theta_{-t}(g)} \subset \mathcal{T}^{\rm tr}_{\theta_{-t}(g)}$, there exists a t_0 such that $c_8\|u\|_{L^\infty(0,1;H)}\mathrm{e}^{-\delta t/2} \leqslant R_0$ and $c_9\|u\|^2_{L^\infty(0,1;H)}\mathrm{e}^{-\delta t} \leqslant R_0$ provided $t \geqslant t_0$. Hence,

$$\|u\|_{L^\infty(t,t+1;H)} + \|\partial_t u\|_{L^{4/3}(t,t+1;V')} \leqslant 3R_0, \qquad \forall t \geqslant t_0, \tag{3.33}$$

which implies that $\phi(t, \theta_{-t}(g), \mathcal{B}_{\theta_{-t}(g)}) \subseteq \mathcal{P}_g$ for $\forall t \geqslant t_0$ and thus $\{\mathcal{P}_g\}_{g\in\mathcal{H}(g_0)}$ is a pullback trajectory absorbing set for $\phi(t, g, \cdot)$ in $\mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$. Obviously, $\mathcal{P}_g$ is bounded in $\mathcal{T}^{\rm tr}_g$ for each $g \in \mathcal{H}(g_0)$. Also, $\bigcup_{g\in\mathcal{H}(g_0)} \mathcal{P}_g \subset \mathcal{P}$ and $\mathcal{P}$ is bounded (in $L^\infty(\mathbb{R}_+; H)$ norm) in $\mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$. The proof is complete. □

Lemma 3.9. *Let $\{u^{(n)}\}$ be a bounded (in the norm of $L^\infty(\mathbb{R}_+; H)$) sequence in $\mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$ and there exists a function $u^* \in \mathcal{C}_{\rm loc}(\mathbb{R}_+; H^{-\eta})$ such that*

$$u^{(n)} \longrightarrow u^* \text{ strongly in } \mathcal{C}_{\rm loc}(\mathbb{R}_+; H^{-\eta}) \text{ as } n \to \infty. \tag{3.34}$$

Then $u^* \in \mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$.

Proof. The proof of this lemma is similar to that of lemma 4.1 in [54]. The difference is that we use the united trajectory space $\mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$ to replace the trajectory space $\mathcal{T}^{\rm tr}$. Since $\{u^{(n)}\} \subset \mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$, we see that there exists a sequence $\{g^{(n)}\} \subset \mathcal{H}(g_0)$ such that

$$\frac{\partial u^{(n)}}{\partial t} + 2\mu_1 A u^{(n)} + B(u^{(n)}) + N(u^{(n)}) = g^{(n)}, \qquad n = 1, 2, \cdots. \tag{3.35}$$

Note that $g^{(n)} \subset \mathcal{H}(g_0) \subset L^2_{\rm loc}(\mathbb{R}; H)$ is bounded in $L^2(0, T; H)$ and $L^2(0, T; H)$ is a reflexive Banach space for each $T > 0$. Whence, there is a $g \in \mathcal{H}(g_0)$ such that

$$g^{(n)} \rightharpoonup g \text{ weakly in } L^2(0, T; H) \subset L^2(0, T; V') \subset L^{4/3}(0, T; V') \text{ as } n \to \infty.$$

The rest of the proof is essentially the same as that of lemma 4.1 in [54] and we omit it here. □

We now can state the main result of this section.

Theorem 3.1. *Let $g_0 \in L^2_b(\mathbb{R}; H)$. Then equation (3.12) possesses a compact pullback trajectory attractor in $\mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$: $\mathcal{A}^{\rm tr} = \{\mathcal{A}^{\rm tr}_g\}_{g\in\mathcal{H}(g_0)} = \{\omega_g(\mathcal{P})\}_{g\in\mathcal{H}(g_0)}$.*

Proof. According to theorem 2.1 and lemma 3.8, we only need to prove that the set $\mathcal{P}$ constructed by (3.32) is compact in $\mathcal{C}_{\rm loc}(\mathbb{R}_+; H^{-\eta})$. Indeed, from (3.20) one sees that $\Pi_T\mathcal{P}$ is bounded in $W_{\infty,4/3}(0, T; H, V')$ and thus $\Pi_T\mathcal{P}$ is relatively compact in $\mathcal{C}([0, T]; H^{-\eta})$ (thanks to lemma 3.3). Hence, we only need to show that $\Pi_T\mathcal{P}$ is closed in $\mathcal{C}([0, T]; H^{-\eta})$ for any $T > 0$. Assume that $\{u_n\} \subset \mathcal{P}$ and $\Pi_T u_n \longrightarrow \Pi_T u$ strongly in $\mathcal{C}([0, T]; H^{-\eta})$ as $n \to \infty$. Since $\{u_n\}$ is bounded in $L^\infty(\mathbb{R}_+; H)$, applying lemma 3.9 we see $u \in \mathcal{T}^{\rm tr}_{\mathcal{H}(g_0)}$. Moreover, in the detailed proof of lemma 3.9 we know $\partial_t\Pi_T u_n \rightharpoonup \partial_t\Pi_T u$ weakly in $L^{4/3}(0, T; V')$ and $u_n \rightharpoonup u$ weakly star in $L^\infty(\mathbb{R}_+; H)$ as $n \to \infty$. Thus we obtain

$$\begin{aligned}&\|u\|_{L^\infty(t,t+1;H)} + \|\partial_t u\|_{L^{4/3}(t,t+1;V')}\\ &\quad\leqslant \liminf_{n\to\infty} \|u_n\|_{L^\infty(t,t+1;H)} + \liminf_{n\to\infty} \|\partial_t u_n\|_{L^{4/3}(t,t+1;V')} \leqslant 3R_0, \qquad \forall t \geqslant 0.\end{aligned}$$

Therefore $u \in \mathcal{P}$ and $\Pi_T u \in \mathcal{C}([0, T]; H^{-\eta})$. The proof is complete. □

4. Regular pullback trajectory attractor and pullback trajectory asymptotic smoothing effect for the incompressible non-Newtonian fluid

In this section, we first prove the existence of regular pullback trajectory attractor $\mathscr{A}^{\mathrm{tr}} = \{\mathscr{A}_g^{\mathrm{tr}}\}_{g\in\mathcal{H}(g_0)} \subset L^\infty(\mathbb{R}_+; V) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$ for the incompressible non-Newtonian fluid. Then we establish that $\mathcal{A}_g^{\mathrm{tr}} = \mathscr{A}_g^{\mathrm{tr}}$ for each $g \in \mathcal{H}(g_0)$.

4.1. Regular pullback trajectory attractor

We first specify the definition of a regular weak solution to problem (3.12). A function $u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$ is called a regular weak solution of problem (3.12) on the interval $[0,T]$ if u, together with its derivative $\dfrac{\partial u}{\partial t}$, satisfies (3.12) in the sense of distributions in $\mathcal{D}'(0,T;H)$. We can prove by the Galerkin method that (3.12) possesses at least one regular weak solution u defined on the interval $[0,T]$ $(\forall T > 0)$ and satisfying the following energy inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u(t), Au(t)) + 2\mu_1\|Au(t)\|^2 + \langle Bu(t), Au(t)\rangle + \langle N(u(t)), Au(t)\rangle \\ \leqslant \langle g, Au(t)\rangle, \qquad \forall t \in [0,T], \tag{4.1}$$

in the following sense:

$$-\frac{1}{2}\int_0^T (u(t), Au(t))\psi'(t)\,\mathrm{d}t + 2\mu_1\int_0^T \|Au(t)\|^2\psi(t)\,\mathrm{d}t \\ + \int_0^T \langle B(u(t)), Au(t)\rangle\psi(t)\,\mathrm{d}t + \int_0^T \langle N(u(t)), u(t)\rangle\psi(t)\,\mathrm{d}t \\ \leqslant \int_0^T \langle g, Au(t)\rangle\psi(t)\,\mathrm{d}t, \qquad \forall\,\psi(t) \in \mathcal{C}_0^\infty([0,T]), \qquad \psi(t) \geqslant 0, \forall T > 0. \tag{4.2}$$

Definition 4.1. *For each $g \in \mathcal{H}(g_0)$, the regular trajectory space $\mathcal{U}_g^{\mathrm{tr}}$ of equation (3.12) with symbol g consists of functions $u \in L^\infty(\mathbb{R}_+; V) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; D(A))$ such that for any $T > 0$ the function $\Pi_T u(t)$ is a regular weak solution of (3.12) on $[0,T]$ and $\Pi_T u(t)$ satisfies (4.1) in the sense of (4.2). $\mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}} = \bigcup_{g\in\mathcal{H}(g_0)} \mathcal{U}_g^{\mathrm{tr}}$ is the regular united trajectory space of equation (3.12).*

Similar to (3.18), the θ-cocycle ϕ can be defined on $\mathbb{R}_+ \times \mathcal{H}(g_0) \times \mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ as

$$\phi(t, g, u(\cdot)) = u_{\theta_t(g)}(t+\cdot), \qquad \forall\,(t, g, u(\cdot)) \in \mathbb{R}_+ \times \mathcal{H}(g_0) \times \mathcal{U}_g^{\mathrm{tr}}. \tag{4.3}$$

Lemma 4.1. *If $u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$, then Au, $B(u)$ and $N(u)$ all belong to $L^{4/3}(0,T;H)$.*

Proof. Since $u \in L^2(0,T;D(A))$, $Au \in L^2(0,T;H)$ is obvious. Note that $(0,T)$ is a finite interval, thus $Au \in L^2(0,T;H) \subset L^{4/3}(0,T;H)$. In the following, $C(\cdot,\cdot,\cdot)$ denotes the constant depending on the numbers appearing in the bracket. Now for any $\psi(t) \in L^4(0,T;H)$, we have, using Gagliardo–Nirenberg inequality,

$$\left|\int_0^T (B(u(t)), \psi(t))\,\mathrm{d}t\right| \leqslant \int_0^T \|B(u(t))\|\,\|\psi(t)\|\,\mathrm{d}t \\ \leqslant \lambda\int_0^T \|u(t)\|^{5/8}\|\Delta u(t)\|^{3/8}\|u(t)\|^{1/8}\|\Delta u(t)\|^{7/8}\|\psi(t)\|\,\mathrm{d}t \\ \leqslant C(\lambda, T)\|u(t)\|^2_{L^\infty(0,T;V)}\|\psi(t)\|_{L^4(0,T;H)},$$

which implies that $B(u(t)) \in L^{4/3}(0, T; H)$. Similar to the derivation of (3.11) in [53], we have

$$\left|\int_0^T (N(u(t)), \psi(t))\,\mathrm{d}t\right| \leqslant C(\mu_0, \epsilon, \alpha)\int_0^T (\|\nabla u(t)\| + \|\Delta u(t)\|)\|\psi(t)\|\,\mathrm{d}t$$
$$\leqslant C(\mu_0, \epsilon, \alpha, T)\|u(t)\|_{L^\infty(0,T;V)}\|\psi(t)\|_{L^4(0,T;H)},$$

which also implies that $N(u(t)) \in L^{4/3}(0, T; H)$. The proof is complete. □

Lemma 4.2. *(i) For each $g \in \mathcal{H}(g_0)$ and any $u_0 \in V$, there exists at least one regular trajectory $u(t) \in \mathcal{U}_g^{\mathrm{tr}} \subset \mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ such that $u(0) = u_0$.*
(ii) For any $t \in \mathbb{R}_+$, $g \in \mathcal{H}(g_0)$ and $u \in \mathcal{U}_g^{\mathrm{tr}}$, there holds $\phi(t, g, u) \in \mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$.

The proof of lemma 4.2 is similar to that of lemma 3.4 and we omit it here. Evidently, we have $\phi(t, g, \mathcal{U}_g^{\mathrm{tr}}) \subseteq \mathcal{U}_{\theta_t(g)}^{\mathrm{tr}} \subseteq \mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ for any $t \in \mathbb{R}_+$ and $g \in \mathcal{H}(g_0)$.

Lemma 4.3. $\mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}} \subseteq \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta}) \cap L^\infty(\mathbb{R}_+; V)$.

Proof. We only need to prove that for each $g \in \mathcal{H}(g_0)$, $\mathcal{U}_g^{\mathrm{tr}} \subseteq \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta}) \cap L^\infty(\mathbb{R}_+; V)$. By the definition of regular trajectory space, for any $u(\cdot) \in \mathcal{U}_g^{\mathrm{tr}}$, $u(\cdot) \in L^\infty(\mathbb{R}_+; V)$ is obvious. Meanwhile, since $u(\cdot) \in L^\infty(\mathbb{R}_+; V) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; D(A))$ and $g \in \mathcal{H}(g_0) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+; H)$, we combine lemma 4.1 and equation (3.12) to obtain that $\partial_t u \in L^{3/4}_{\mathrm{loc}}(\mathbb{R}_+; H)$. Since $V \hookrightarrow H^{2-\eta} \subseteq H$ and the embedding $V \hookrightarrow H^{2-\eta}$ is compact, we then get from lemma 3.3 that $u(\cdot) \in \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$. The proof is complete. □

Definition 4.2. *A family of compact sets $\{\mathscr{A}_g^{\mathrm{tr}}\}_{g\in\mathcal{H}(g_0)} \subseteq \mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ is called the pullback trajectory attractor of equation (3.12) with respect to the topology $\mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$ if $\mathscr{A}_g^{\mathrm{tr}} \subset \mathcal{U}_g^{\mathrm{tr}}$ for each $g \in \mathcal{H}(g_0)$ and*

(1) compactness: $\mathscr{A}_g^{\mathrm{tr}}$ is compact in $\mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$ and bounded in $L^\infty(\mathbb{R}_+; V)$;
(2) ϕ-invariance: $\phi(t, g, \mathscr{A}_g^{\mathrm{tr}}) = \mathscr{A}_{\theta_t(g)}^{\mathrm{tr}}$, $\forall t \geqslant 0$;
(3) pullback attracting property: for any family of bounded (in $L^\infty(\mathbb{R}_+; V)$ norm) sets $\{\mathscr{B}_g\}_{g\in\mathcal{H}(g_0)} \subset \mathscr{A}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ with $\mathscr{B}_g \subset \mathcal{U}_g^{\mathrm{tr}}$ there holds

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{C}([0,T];H^{2-\eta})}\left(\Pi_T\phi(t, \theta_{-t}(g), \mathscr{B}_{\theta_{-t}(g)}), \Pi_T\mathscr{A}_g^{\mathrm{tr}}\right) = 0, \qquad \forall T > 0; \tag{4.4}$$

(4) minimality: if $\{\mathscr{E}_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$ is a family of non-empty compact sets that satisfy (3), then $\mathscr{A}_g \subseteq \mathscr{E}_g$ for each $g \in \mathcal{H}(g_0)$.

Lemma 4.4. *For any $g \in \mathcal{H}(g_0)$ and any trajectory $u \in \mathcal{U}_{\theta_{-t}(g)}^{\mathrm{tr}} \subseteq \mathcal{U}_{\mathcal{H}(g_0)}^{\mathrm{tr}}$, there exists a positive constant δ which is independent of g and u such that*

$$\|\phi(t, \theta_{-t}(g), u)\|_{L^\infty(\mathbb{R}_+;V)} + \|\phi(t, \theta_{-t}(g), u)\|_{L^2(0,1;D(A))} + \|\partial_t\phi(t, \theta_{-t}(g), u)\|_{L^{4/3}(0,1;H)}$$
$$= \operatorname*{esssup}_{s\geqslant 0}\|u(s+t)\|_V + \left(\int_0^1 \|Au(s+t)\|^2\,\mathrm{d}s\right)^{1/2} + \left(\int_0^1 \|\partial_t u(t+s)\|^{4/3}\,\mathrm{d}s\right)^{3/4}$$
$$= \operatorname*{esssup}_{\rho\geqslant t}\|u(\rho)\|_V + \left(\int_t^{t+1} \|Au(\rho)\|^2\,\mathrm{d}\rho\right)^{1/2} + \left(\int_t^{t+1} \|\partial_t u(\rho)\|^{4/3}\,\mathrm{d}\rho\right)^{3/4}$$
$$\leqslant \mathcal{F}_3(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}) + \left(1 + \frac{1}{\mu_1}\right)\|g_0\|_{L_b^2}, \qquad \forall t \geqslant 0, \tag{4.5}$$

where $\mathcal{F}_3(\cdot)$ is a continuous monotone function of $\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}$.

Proof. Let $u(t) \in \mathcal{U}^{\mathrm{tr}}_{\theta_{-t}(g)} \subseteq \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$, then $u_t \in L^{4/3}(0, T; H)$. Multiplying (3.12) by u_t and integrating over Ω, we obtain

$$\|u_t\|^2 + 2\mu_1 a(u(t), u(t)) + b(u(t), u(t), u_t) + \langle N(u(t)), u_t\rangle = (g(t), u_t). \tag{4.6}$$

Set

$$\Gamma(|e(u)|) = \int_0^{|e(u)|^2} \mu_0(\epsilon + s)^{-\alpha/2}\,\mathrm{d}s \quad \text{such that} \quad \frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \sum_{i,j=1}^{3} \mu(u)e_{ij}(u)\frac{\partial e_{ij}(u)}{\partial t}.$$

Then,

$$\langle N(u), u_t\rangle = \sum_{i,j=1}^{3} \int_\Omega \mu(u)e_{ij}(u)e_{ij}(u_t)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \Gamma(|e(u)|)\,\mathrm{d}x\right). \tag{4.7}$$

Substituting (4.7) into (4.6), we have

$$\begin{aligned}
\|u_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}&\left(2\mu_1 a(u(t), u(t)) + \int_\Omega \Gamma(|e(u)|)\,\mathrm{d}x\right)\\
&= -b(u, u, u_t) + (g(t), u_t)\\
&\leqslant \left|\sum_{i,j=1}^{3}\int_\Omega u_j \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial t}\,\mathrm{d}x\right| + 4\|g(t)\|^2 + \frac{1}{4}\|u_t\|^2\\
&\leqslant \|u\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|u_t\| + 4\|g(t)\|^2 + \frac{1}{4}\|u_t\|^2.
\end{aligned} \tag{4.8}$$

By the Gagliardo–Nirenberg and Cauchy inequalities,

$$\|u\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|u_t\| \leqslant \lambda\|\Delta u\|^2\|u_t\| \leqslant \lambda^2\|u\|_V^4 + \tfrac{1}{4}\|u_t\|^2.$$

Thus, from (4.8) and lemma 3.1, we obtain

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} \leqslant f(t)y(t) + h(t), \tag{4.9}$$

where

$$y(t) = 2\mu_1 a(u(t), u(t)) + \int_\Omega \Gamma(|e(u)|)\,\mathrm{d}x,$$

$$f(t) = \frac{\lambda^2}{2\mu_1 c_1}\|u\|_V^2 \qquad \text{and} \qquad h(t) = 4\|g(t)\|^2.$$

By (3.28),

$$\int_t^{t+1} f(\tau)\,\mathrm{d}\tau = \frac{\lambda^2}{2\mu_1 c_1}\int_t^{t+1} \|u(\tau)\|_V^2\,\mathrm{d}\tau \leqslant \frac{\lambda^2}{2\mu_1 c_1}\mathcal{F}_1(\|u\|_{L^\infty(0,T;V)}\mathrm{e}^{-\delta t/2}), \tag{4.10}$$

where

$$\mathcal{F}_1(\|u\|_{L^\infty(0,T;V)}\mathrm{e}^{-\delta t/2}) = \left(c_3\|u\|_{L^\infty(0,T;V)}\mathrm{e}^{-\delta t/2} + R_2\right)^2.$$

Obviously,

$$\int_t^{t+1} h(\tau)\,\mathrm{d}\tau = 4\int_t^{t+1} \|g(\tau)\|^2\,\mathrm{d}\tau \leqslant 4\|g_0\|^2_{L^2_b}. \tag{4.11}$$

Now from (3.6) and (3.28) we get

$$\begin{aligned}
\int_t^{t+1}\int_\Omega 2\mu_1 a(u(t), u(t))\,\mathrm{d}x\,\mathrm{d}t &\leqslant 2c_2\mu_1\int_t^{t+1}\|u(\tau)\|_V^2\,\mathrm{d}\tau\\
&\leqslant 2c_2\mu_1\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}).
\end{aligned} \tag{4.12}$$

Since $0 < (\epsilon + s)^{-\alpha/2} \leqslant \epsilon^{-\alpha/2}$ when $s \geqslant 0$ and $\alpha > 0$, thus

$$\Gamma(|e(u)|) = \int_0^{|e(u)|^2} \mu_0(\epsilon + s)^{-\alpha/2}\,\mathrm{d}s \leqslant \mu_0\epsilon^{-\alpha/2}|e(u)|^2,$$

and

$$\begin{aligned}\int_t^{t+1}\int_\Omega \Gamma(|e(u)|)\,\mathrm{d}x\,\mathrm{d}s &\leqslant \mu_0\epsilon^{-\alpha/2}\int_t^{t+1}\int_\Omega |e(u)|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant 9\mu_0\epsilon^{-\alpha/2}\int_t^{t+1}\|u(s)\|_V^2\,\mathrm{d}s\\ &\leqslant 9\mu_0\epsilon^{-\alpha/2}\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}),\end{aligned} \tag{4.13}$$

where the fact that $\int_\Omega |e(u)|^2\,\mathrm{d}x \leqslant 9\|u\|_V^2$ has been used. From (4.12) and (4.13), we get

$$\int_t^{t+1} y(\rho)\,\mathrm{d}\rho \leqslant 9(c_2\mu_1 + \mu_0\epsilon^{-\alpha/2})\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}). \tag{4.14}$$

Now let $t \leqslant \rho \leqslant t+1$. Multiplying (4.9) by $\exp(-\int_t^\rho f(\tau)\,\mathrm{d}\tau)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(y(\rho)\exp\left(-\int_t^\rho f(\tau)\,\mathrm{d}\tau\right)\right) \leqslant h(\rho)\exp\left(-\int_t^\rho f(\tau)\,\mathrm{d}\tau\right) \leqslant h(\rho). \tag{4.15}$$

Letting $t_1 \in [t, t+1]$ and integrating (4.15) over $[t_1, t+1]$, we obtain

$$y(t+1)\exp\left(-\int_t^{t+1} f(\tau)\,\mathrm{d}\tau\right) \leqslant y(t_1)\exp\left(-\int_t^{t+1} h(\tau)\,\mathrm{d}\tau\right) + \int_t^{t+1} h(\tau)\,\mathrm{d}\tau.$$

Therefore, we have for any $t \in \mathbb{R}_+$ and $t_1 \in [t, t+1]$ that

$$\begin{aligned}y(t+1) &\leqslant y(t_1) + \int_t^{t+1} h(\tau)\,\mathrm{d}\tau\exp\left(\int_t^{t+1} f(\tau)\,\mathrm{d}\tau\right)\\ &\leqslant y(t_1) + 4\|g_0\|_{L_b^2}^2\exp\left\{\frac{\lambda^2}{2\mu_1 c_1}\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2})\right\}.\end{aligned} \tag{4.16}$$

Integrating (4.16) (with respect to t_1) over $[t, t+1]$, we get

$$\begin{aligned}y(t+1) &\leqslant \int_t^{t+1} y(t_1)\,\mathrm{d}t_1 + 4\|g_0\|_{L_b^2}^2\exp\left\{\frac{\lambda^2}{2\mu_1 c_1}(c_3\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2} + R_2)^2\right\}\\ &\leqslant 9(c_2\mu_1 + \mu_0\epsilon^{-\alpha/2})\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2})\\ &\quad + 4\|g_0\|_{L_b^2}^2\exp\left\{\frac{\lambda^2}{2\mu_1 c_1}\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2})\right\}, \qquad \forall t \in \mathbb{R}_+.\end{aligned} \tag{4.17}$$

It then follows from (4.17) and lemma 3.1 that

$$\begin{aligned}\|u(t+1)\|_V^2 &\leqslant \frac{1}{c_1}a(u(t+1), u(t+1))\\ &\leqslant \frac{1}{2c_1\mu_1}\left\{(9(c_2\mu_1 + \mu_0\epsilon^{-\alpha/2})\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2})\right.\\ &\quad \left. +4\|g_0\|_{L_b^2}^2\exp\left(\frac{\lambda^2}{2\mu_1 c_1}\mathcal{F}_1(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2})\right)\right\}\\ &\doteq \mathcal{F}_2(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}), \qquad \forall t \in \mathbb{R}_+,\end{aligned} \tag{4.18}$$

and thus for any $t \in [1, +\infty)$ there holds

$$\|\phi(t, \theta_{-t}(g), u)\|_{L^\infty(\mathbb{R}_+;V)} \leqslant \mathcal{F}_2^{1/2}(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}). \tag{4.19}$$

Now multiplying (3.12) by Au, we get

$$(u_t, Au) + 2\mu_1\|Au\|^2 + \langle B(u), Au\rangle + \langle N(u), Au\rangle = (g(t), Au). \tag{4.20}$$

We next estimate the terms in (4.20).

$$(u_t, Au(t)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u(t), Au(t)), \tag{4.21}$$

$$\langle B(u(t)), Au(t)\rangle = (B(u(t)), Au(t)) \leqslant \frac{\mu_1}{2}\|Au(t)\|^2 + \frac{1}{2\mu_1}\|B(u(t))\|^2, \tag{4.22}$$

$$\langle Bu(t), Au(t)\rangle = (N(u(t)), Au(t)) \leqslant \frac{\mu_1}{2}\|Au(t)\|^2 + \frac{1}{2\mu_1}\|N(u(t))\|^2, \tag{4.23}$$

$$(g(t), Au(t)) = (g(t), Au(t)) \leqslant \frac{\mu_1}{2}\|Au(t)\|^2 + \frac{1}{2\mu_1}\|g\|^2. \tag{4.24}$$

Combining (4.20)–(4.24), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t), Au(t)) + \mu_1\|Au\|^2 \leqslant \frac{1}{\mu_1}(\|B(u(t))\|^2 + \|N(u(t))\|^2 + \|g(t)\|^2). \tag{4.25}$$

Integrating (4.25) over $[t, t+1]$, we get

$$\begin{aligned}(u(t+1), Au(t+1)) &+ \mu_1\int_t^{t+1}\|Au(\tau)\|^2\,\mathrm{d}\tau\\ &\leqslant \frac{1}{\mu_1}\int_t^{t+1}\|B(u(\tau))\|^2\,\mathrm{d}\tau + \frac{1}{\mu_1}\int_t^{t+1}\|N(u(\tau))\|^2\,\mathrm{d}\tau\\ &\quad+\frac{1}{\mu_1}\int_t^{t+1}\|g(\tau)\|^2\,\mathrm{d}\tau + (u(t), Au(t)).\end{aligned} \tag{4.26}$$

Now from the proof of lemma 4.1 we can see that

$$\int_t^{t+1}\|B(u(\tau))\|^2\,\mathrm{d}\tau \leqslant \lambda\|u\|^4_{L^\infty(\mathbb{R}_+;V)}, \tag{4.27}$$

$$\int_t^{t+1}\|N(u(\tau))\|^2\,\mathrm{d}\tau \leqslant C(\mu_0, \epsilon, \alpha)\|u\|^2_{L^\infty(\mathbb{R}_+;V)}, \tag{4.28}$$

and thus

$$\begin{aligned}\int_t^{t+1}\|Au(\tau)\|^2\,\mathrm{d}\tau \leqslant{}& \frac{c_2 + C(\mu_0, \epsilon, \alpha)}{\mu_1^2}\mathcal{F}_2(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2})\\ &+ \frac{\lambda}{\mu_1^2}\mathcal{F}_2^2(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}) + \frac{1}{\mu_1^2}\|g_0\|^2_{L^2_b}.\end{aligned} \tag{4.29}$$

Hence

$$\begin{aligned}\|\partial_t u(\tau)\|_{L^{4/3}(t,t+1;H)} &\leqslant \|Au(\tau)\|_{L^{4/3}(t,t+1;H)} + \|B(u(\tau))\|_{L^{4/3}(t,t+1;H)}\\ &\quad+\|N(u(\tau))\|_{L^{4/3}(t,t+1;H)} + \|g(t)\|_{L^{4/3}(t,t+1;H)}\\ &\leqslant \left(\frac{\sqrt{c_2 + C(\mu_0, \epsilon, \alpha)}}{\mu_1} + C(\mu_0, \epsilon, \alpha)\right)\mathcal{F}_2^{1/2}(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2})\\ &\quad+\frac{\sqrt{\lambda}}{\mu_1}\mathcal{F}_2(\|u\|_{L^\infty(0,1;V)}\mathrm{e}^{-\delta t/2}) + \left(1 + \frac{1}{\mu_1}\right)\|g_0\|_{L^2_b}.\end{aligned} \tag{4.30}$$

The proof of this lemma is complete. □

Lemma 4.5. *Let $g_0 \in L_b^2(\mathbb{R}; H)$, then the θ-cocycle $\phi(t, g, \cdot)$ (defined by (4.3)) possesses a pullback trajectory absorbing set $\{\Lambda_g\}_{g\in\mathcal{H}(g_0)} \subset \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ with $\Lambda_g \subset \mathcal{U}^{\mathrm{tr}}_g$ for each $g \in \mathcal{H}(g_0)$. Moreover, there is a set $\Lambda \subset \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ such that $\bigcup_{g\in\mathcal{H}(g_0)} \Lambda_g \subset \Lambda$ and Λ is bounded in $L^\infty(\mathbb{R}_+; V)$.*

Proof. Set

$$\Lambda_g = \left\{ u \in \mathcal{U}^{\mathrm{tr}}_g : \sup_{t\in\mathbb{R}_+}\{\|u\|_{L^\infty(t,t+1;V)} + \|\partial_t u\|_{L^{4/3}(t,t+1;H)}\} \leqslant \mathcal{F}_3(\hat{C}) + \left(1+\frac{1}{\mu_0}\right)\|g_0\|_{L_b^2} \right\},$$

$$\Lambda = \left\{ u \in \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)} : \sup_{t\in\mathbb{R}_+}\{\|u\|_{L^\infty(t,t+1;V)} + \|\partial_t u\|_{L^{4/3}(t,t+1;H)}\} \leqslant \mathcal{F}_3(\hat{C}) + \left(1+\frac{1}{\mu_0}\right)\|g_0\|_{L_b^2} \right\}, \tag{4.31}$$

where $\mathcal{F}_3$ is the continuous monotone function from lemma 4.4 and $\hat{C}$ is a positive constant depending on $\mu_0, \mu_1, \epsilon, \alpha, c_1, c_2, c_3, \delta, \|g_0\|_{L_b^2}$ and R_2, but not on u. We next prove that Λ is the bounded pullback trajectory absorbing set for the translation cocycle $\phi(t, g, \cdot)$. Indeed, for any family of bounded (in $L^\infty(\mathbb{R}_+; V)$ norm) sets $\{\mathscr{B}_g\}_{g\in\mathcal{H}(g_0)}$ of $\mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ with $\mathscr{B}_g \subset \mathcal{U}^{\mathrm{tr}}_g$, we see from lemma 4.4 that for $\forall\, u \in \mathscr{B}_{\theta_{-t}(g)} \subset \mathcal{U}^{\mathrm{tr}}_{\theta_{-t}(g)}$, there exists a $t^* > 1$ such that

$$\|u\|_{L^\infty(t,t+1;V)} + \|\partial_t u\|_{L^{4/3}(t,t+1;H)} \leqslant \mathcal{F}_3(\hat{C}) + \left(1+\frac{1}{\mu_0}\right)\|g_0\|_{L_b^2}, \qquad \forall\, t \geqslant t^*, \tag{4.32}$$

which implies that $\phi(t, \theta_{-t}(g), \mathscr{B}_{\theta_{-t}(g)}) \subseteq \Lambda_g$ for $\forall\, t \geqslant t^*$ and thus $\{\Lambda_g\}_{g\in\mathcal{H}(g_0)}$ is a pullback trajectory absorbing set for $\phi(t, g, \cdot)$ in $\mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$. Obviously, Λ_g is bounded (in $L^\infty(\mathbb{R}_+; V)$ norm) in $\mathcal{U}^{\mathrm{tr}}_g$ for each $g \in \mathcal{H}(g_0)$. Also, $\bigcup_{g\in\mathcal{H}(g_0)} \Lambda_g \subset \Lambda$ and Λ is bounded in $L^\infty(R_+; V)$. The proof of lemma 4.5 is complete. □

Lemma 4.6. *Let $\{u_n\}$ be a bounded (in the norm of $L^\infty(\mathbb{R}_+; V)$) sequence in $\mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ and there exists a function $u_* \in \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$ such that*

$$u_n \longrightarrow u_* \quad \text{strongly in} \quad \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta}) \quad \text{as} \quad n \to \infty. \tag{4.33}$$

Then $u_* \in \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$.

Proof. We need to prove that $u_* \in L^\infty(\mathbb{R}_+; V) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; D(A))$ and for any $T > 0$, $\Pi_T u_*$ is a regular weak solution of (3.12) on the interval $[0, T]$ satisfying the energy inequality (4.1). Indeed, since $\{u_n\} \subset \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ and is bounded in $L^\infty(\mathbb{R}_+; V)$, by (4.5) we conclude that $\{u_n\}$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}_+; D(A))$ and $\{\frac{\partial u_n}{\partial t}\}$ is bounded in $L^{4/3}_{\mathrm{loc}}(\mathbb{R}_+; H)$. By the diagonal procedure, we find that there exists a function $u \in L^\infty(\mathbb{R}_+; V) \cap \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; D(A))$ and a subsequence (still denoted by $\{u_n\}$) of $\{u_n\}$ such that

$$\Pi_T u_n \rightharpoonup \Pi_T u \quad \text{weakly in} \quad L^2(0, T; D(A)) \quad \text{as} \quad n \to \infty, \tag{4.34}$$

$$u_n \rightharpoonup u \quad \text{weakly star in} \quad L^\infty(\mathbb{R}_+; V) \quad \text{as} \quad n \to \infty, \tag{4.35}$$

$$\partial_t \Pi_T u_n \rightharpoonup \partial_t \Pi_T u \quad \text{weakly in} \quad L^{4/3}(0, T; H) \quad \text{as} \quad n \to \infty. \tag{4.36}$$

Obviously, $\partial_t u \in L^{4/3}_{\mathrm{loc}}(\mathbb{R}_+; H)$. By lemma 3.3 we obtain $\Pi_T u \in \mathcal{C}([0, T]; H^{2-\eta})$ since the embedding $V \hookrightarrow H^{2-\eta}$ is compact. From (4.33) and the uniqueness of limit we have $u = u_*$. Next we verify that $\Pi_T u_*$ is a regular weak solution of (3.12) on the interval $[0, T]$ satisfying (4.1). To this end, we prove the following relations.

$$A\Pi_T u_n \rightharpoonup A\Pi_T u_* \quad \text{weakly in} \quad L^{4/3}(0, T; H) \quad \text{as} \quad n \to \infty, \tag{4.37}$$

$$B(\Pi_T u_n) \rightharpoonup B(\Pi_T u_*) \quad \text{weakly in} \quad L^{4/3}(0, T; H) \quad \text{as} \quad n \to \infty, \tag{4.38}$$

$$N(\Pi_T u_n) \rightharpoonup N(\Pi_T u_*) \quad \text{weakly in} \quad L^{4/3}(0, T; H) \quad \text{as} \quad n \to \infty. \tag{4.39}$$

Obviously, (4.37) can be deduced directly from (4.34). Also from (4.34) and the compact embedding $V \hookrightarrow H_0^1(\Omega) \hookrightarrow H$, we obtain

$$\Pi_T u_n \longrightarrow \Pi_T u_* \text{ strongly in } L^{4/3}(0, T; H_0^1(\Omega)) \text{ as } n \to \infty, \tag{4.40}$$

$$\Pi_T u_n \longrightarrow \Pi_T u_* \text{ strongly in } L^{4/3}(0, T; H) \text{ as } n \to \infty. \tag{4.41}$$

Now for any $\psi(t) \in L^4(0, T; H)$, we have

$$\begin{aligned} &\left| \lim_{n\to\infty} \int_0^T (B(\Pi_T u_n) - B(\Pi_T u_*), \psi(t))\, dt \right| \\ &\quad \leqslant \lim_{n\to\infty} \int_0^T |b(\Pi_T(u_n - u_*), \Pi_T u_n, \psi)|\, dt + \lim_{n\to\infty} \int_0^T |b(\Pi_T u_*, \Pi_T(u_n - u_*), \psi)|\, dt \\ &\quad \doteq I_1 + I_2. \end{aligned} \tag{4.42}$$

By Hölder inequality and Gagliardo–Nirenberg inequality, we get

$$\begin{aligned} I_1 &= \lim_{n\to\infty} \int_0^T |b(\Pi_T(u_n - u_*), \Pi_T u_n, \psi)|\, dt \\ &\leqslant \lim_{n\to\infty} \int_0^T \|\Pi_T(u_n - u_*)\|^{1/4} \|\Pi_T(u_n - u_*)\|^{3/4}_{H_0^1(\Omega)} \|\Pi_T u_n\|^{1/8} \|\Pi_T u_n\|_V^{7/8} \|\psi(t)\|\, dt \\ &\leqslant \lim_{n\to\infty} C \int_0^T \|\Pi_T(u_n - u_*)\|_{H_0^1(\Omega)} \|\psi(t)\|\, dt \\ &\leqslant C \lim_{n\to\infty} \|\Pi_T(u_n - u_*)\|_{L^{4/3}(0,T;H_0^1(\Omega))} \|\psi(t)\|_{L^4(0,T;H)} = 0, \end{aligned} \tag{4.43}$$

where C is a constant depending on the $L^\infty(\mathbb{R}_+; V)$ bound of the sequence $\{u_n\}$ and T, but not on n. Analogously,

$$\begin{aligned} I_2 &= \lim_{n\to\infty} \int_0^T |b(\Pi_T u_*, \Pi_T(u_n - u_*), \psi)|\, dt \\ &\leqslant \lim_{n\to\infty} \int_0^T \|\Pi_T u_*\|_\infty \|\nabla \Pi_T(u_n - u_*)\|\, \|\psi(t)\|\, dt \\ &\leqslant \lim_{n\to\infty} \int_0^T \|\Pi_T u_*\|_V\, \|\Pi_T(u_n - u_*)\|_{H_0^1(\Omega)} \|\psi(t)\|\, dt \\ &\leqslant C \lim_{n\to\infty} \|\Pi_T(u_n - u_*)\|_{L^{4/3}(0,T;H_0^1(\Omega))} \|\psi(t)\|_{L^4(0,T;H)} = 0. \end{aligned} \tag{4.44}$$

It then follows from (4.42)–(4.44) that (4.38) holds. Now from (4.34) we see that $\|A\Pi_T(u_n - u_*)\|_{L^2(0,T;H)}$ is bounded. Similar to the derivation of (3.27) in [51], we get from (4.41) that

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T |N(\Pi_T u_n) - N(\Pi_T u_*), \psi)|\, dt \\ &\quad = \left| \lim_{n\to\infty} \int_0^T \int_\Omega \{\nabla \cdot [\mu(e(\Pi_T u_n))e(\Pi_T u_n) - \mu(e(\Pi_T u_n))e(\Pi_T u_n)]\} \cdot \psi(t)\, dx\, dt \right| \\ &\quad \leqslant C(\mu_0, \epsilon, \alpha) \lim_{n\to\infty} \int_0^T \|\Pi_T(u_n - u_*)\|^{1/2} \|A\Pi_T(u_n - u_*)\|^{1/2} \|\psi(t)\|\, dt \\ &\quad \leqslant C(\mu_0, \epsilon, \alpha) \lim_{n\to\infty} \|\Pi_T(u_n - u_*)\|^{1/2}_{L^2(0,T;H)} \|A\Pi_T(u_n - u_*)\|^{1/2}_{L^2(0,T;H)} \|\psi(t)\|_{L^2(0,T;H)} \\ &\quad = 0. \end{aligned} \tag{4.45}$$

Equation (4.45) gives (4.39). The proof is complete. □

Theorem 4.1. *Let $g_0 \in L^2_b(\mathbb{R}; H)$. Then equation (3.12) possesses a compact pullback trajectory attractor $\mathscr{A}^{\mathrm{tr}} = \{\mathscr{A}^{\mathrm{tr}}_g\}_{g\in\mathcal{H}(g_0)} = \{\omega_g(\Lambda)\}_{g\in\mathcal{H}(g_0)}$ in $\mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$.*

Proof. According to theorem 2.1 and lemma 4.5, we only need to prove that the set $\Lambda \subseteq \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$ constructed in (4.31) is compact in $\mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{2-\eta})$. Actually, from (4.6) one can see that $\Pi_T\Lambda$ is bounded in $W_{\infty,4/3}(0, T; V, H)$ and thus $\Pi_T\Lambda$ is relatively compact in $\mathcal{C}([0, T]; H^{2-\eta})$ (thanks to lemma 3.3). Hence, it suffices to show that $\Pi_T\Lambda$ is closed in $\mathcal{C}([0, T]; H^{2-\eta})$ for any $T > 0$. Assume $\{u_n\} \subset \Lambda$ and $\Pi_T u_n \longrightarrow \Pi_T u$ strongly in the norm of $\mathcal{C}([0, T]; H^{2-\eta})$ as $n \to \infty$. Since $\{u_n\}$ is bounded in $L^\infty(\mathbb{R}_+; V)$, applying lemma 4.6 we see $u \in \mathcal{U}^{\mathrm{tr}}_{\mathcal{H}(g_0)}$. Moreover, in the proof of lemma 4.6 we know $\partial_t\Pi_T u_n \rightharpoonup \partial_t\Pi_T u$ weakly in $L^{4/3}(0, T; H)$ and $u_n \rightharpoonup u$ weakly star in $L^\infty(\mathbb{R}_+; V)$ as $n \to \infty$. Thus we obtain

$$\begin{aligned}
&\|u\|_{L^\infty(t,t+1;V)} + \|\partial_t u\|_{L^{4/3}(t,t+1;H)}\\
&\qquad \leqslant \liminf_{n\to\infty} \|u_n\|_{L^\infty(t,t+1;V)} + \liminf_{n\to\infty} \|\partial_t u_n\|_{L^{4/3}(t,t+1;H)} \leqslant \mathcal{F}(\hat{C})\\
&\qquad + \left(1 + \frac{1}{\mu_0}\right)\|g_0\|_{L^2_b}, \qquad t \geqslant 0.
\end{aligned}$$

Therefore $u \in \Lambda$ and $\Pi_T u \in \mathcal{C}([0, T]; H^{2-\eta})$. The proof is complete. □

4.2. *Pullback trajectory asymptotic smoothing effect*

In this section, we prove the regularity of the pullback trajectory attractors by showing $\mathscr{A}^{\mathrm{tr}}_g = \mathcal{A}^{\mathrm{tr}}_g$ for each $g \in \mathcal{H}(g_0)$. This regularity implies the pullback trajectory smoothing effect of the incompressible non-Newtonian fluid in the following sense: the trajectories issued from $u_0 \in H$ belong to $\mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+; H^{-\eta}) \cap L^\infty(\mathbb{R}_+; H)$, and (under the pullback acting of the translation cocycle) eventually belong to $\mathcal{C}([0, T]; H^{2-\eta}) \cap L^\infty(\mathbb{R}_+; V)$ after large enough time.

Lemma 4.7. *Let $g_0 \in L^2_b(\mathbb{R}; H)$, $g \in \mathcal{H}(g_0)$ and $\mathcal{B}^H \subset \mathcal{T}^{\mathrm{tr}}_{\theta_{-t}(g)}$ be arbitrary. Let $u(\cdot) = \phi(t, \theta_{-t}(g), u_0(\cdot))$ with $u_0(\cdot) \in \mathcal{B}^H$. Then there exist a time $T_0(\mathcal{B}^H)$ and a positive constant K such that*

$$\|\phi(t, \theta_{-t}(g), u_0(\cdot))\|_{L^\infty(\mathbb{R}_+;V)} \leqslant K, \qquad \forall\, t \geqslant T_0(\mathcal{B}^H). \tag{4.46}$$

Proof. The assertion of this lemma can be deduced from lemma 4.2 of [53] and we omit the detailed proof here. □

Theorem 4.2. *Let $g_0 \in L^2_b(\mathbb{R}; H)$, then*

$$\mathcal{A}^{\mathrm{tr}} = \{\mathcal{A}^{\mathrm{tr}}_g\}_{g\in\mathcal{H}(g_0)} = \{\mathscr{A}^{\mathrm{tr}}_g\}_{g\in\mathcal{H}(g_0)} = \mathscr{A}^{\mathrm{tr}}. \tag{4.47}$$

Proof. We only need to show

$$\mathcal{A}^{\mathrm{tr}}_g = \mathscr{A}^{\mathrm{tr}}_g, \qquad \forall\, g \in \mathcal{H}(g_0). \tag{4.48}$$

On the one hand, $\mathscr{A}^{\mathrm{tr}}_g$ is bounded in $L^\infty(\mathbb{R}_+; V)$ for any $g \in \mathcal{H}(g_0)$. Thus $\mathscr{A}^{\mathrm{tr}}_{\theta_{-t}(g)}$ is bounded in $L^\infty(\mathbb{R}_+; H)$ for any $t \in \mathbb{R}$. By the ϕ-invariance property and pullback attracting property of the pullback trajectory attractor, we have for any $T > 0$ that

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{C}([0,T];H^{-\eta})}(\Pi_T\mathscr{A}^{\mathrm{tr}}_g, \Pi_T\mathcal{A}^{\mathrm{tr}}_g)\\
&\quad = \mathrm{dist}_{\mathcal{C}([0,T];H^{-\eta})}\left(\Pi_T\phi\left(t, \theta_{-t}(g), \mathscr{A}^{\mathrm{tr}}_{\theta_{-t}(g)}\right), \Pi_T\mathcal{A}^{\mathrm{tr}}_g\right) \qquad (\forall\, t \in \mathbb{R}_+)\\
&\quad = \lim_{t\to+\infty} \mathrm{dist}_{\mathcal{C}([0,T];H^{-\eta})}\left(\Pi_T\phi\left(t, \theta_{-t}(g), \mathscr{A}^{\mathrm{tr}}_{\theta_{-t}(g)}\right), \Pi_T\mathcal{A}^{\mathrm{tr}}_g\right)\\
&\quad = 0, \qquad \forall\, g \in \mathcal{H}(g_0),
\end{aligned}$$

which implies

$$\mathscr{A}_g^{\mathrm{tr}} \subseteq \mathcal{A}_g^{\mathrm{tr}}, \qquad \forall\, g \in \mathcal{H}(g_0). \tag{4.49}$$

On the other hand, lemma 4.7 tells us that

$$\mathcal{A}_{\theta_{-t}(g)}^{\mathrm{tr}} = \omega_{\theta_{-t}(g)}(\mathcal{P}) = \bigcap_{s\geqslant 0}\overline{\bigcup_{\tau\geqslant s}\phi(\tau,\theta_{-\tau}\theta_{-t}(g),\mathcal{P}_{\theta_{-\tau}\theta_{-t}(g)})}$$

is bounded in $L^\infty(\mathbb{R}_+; V)$ for any $t \in \mathbb{R}_+$. Also by the ϕ-invariance property and pullback attracting property of the pullback trajectory attractor, we obtain for any $T > 0$ that

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{C}([0,T];H^{-\eta})}(\Pi_T\mathcal{A}_g^{\mathrm{tr}}, \Pi_T\mathscr{A}_g^{\mathrm{tr}})\\
&\quad = \mathrm{dist}_{\mathcal{C}([0,T];H^{-\eta})}\left(\Pi_T\phi\left(t,\theta_{-t}(g),\mathcal{A}_{\theta_{-t}(g)}^{\mathrm{tr}}\right), \Pi_T\mathscr{A}^{\mathrm{tr}}\right) \qquad (\forall\, t \in \mathbb{R}_+)\\
&\quad \leqslant \mathrm{dist}_{\mathcal{C}([0,T];H^{2-\eta})}\left(\Pi_T\phi\left(t,\theta_{-t}(g),\mathcal{A}_{\theta_{-t}(g)}^{\mathrm{tr}}\right), \Pi_T\mathscr{A}^{\mathrm{tr}}\right) \qquad (\forall\, t \in \mathbb{R}_+)\\
&\quad = \lim_{t\to+\infty}\mathrm{dist}_{\mathcal{C}([0,T];H^{2-\eta})}\left(\Pi_T\phi\left(t,\theta_{-t}(g),\mathcal{A}_{\theta_{-t}(g)}^{\mathrm{tr}}\right), \Pi_T\mathscr{A}^{\mathrm{tr}}\right)\\
&\quad = 0, \qquad \forall\, g \in \mathcal{H}(g_0),
\end{aligned}$$

which implies

$$\mathcal{A}_g^{\mathrm{tr}} \subseteq \mathscr{A}_g^{\mathrm{tr}}, \qquad \forall\, g \in \mathcal{H}(g_0). \tag{4.50}$$

We readily get (4.48) from (4.49) and (4.50). The proof is complete. □

5. Conclusions and a remark

The starting point for our interest in this paper is the asymptotic smoothing effect of the weak solutions to 3D non-Newtonian fluid. Our ideas originate from Caraballo *et al* [14, 15, 19] for pullback attractor and Chepyzhov and Vishik [23,24,49] for trajectory attractor. The definition of pullback trajectory attractor is indeed a combination of the definitions of pullback attractor and trajectory attractor. It contains not only the characteristic of pullback attractor but also the characteristic of trajectory attractor. We have constructed the pullback trajectory attractors for the 3D incompressible non-Newtonian fluid and proved its pullback trajectory asymptotic smoothing effect.

Remark. The idea of this paper could be applied to 3D Navier–Stokes equations. One can construct the pullback trajectory attractors in different spaces and then establish the regularity. This regularity also reveals the pullback trajectory asymptotic smoothing effect of the weak solutions to 3D Navier–Stokes equations. This will be the topic of some other papers.

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