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Dynamics of a nonstandard finite-difference scheme for Mackey–Glass system ☆

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ABSTRACT

In this article, we study the dynamics of Mackey–Glass system applied a nonstandard finite-difference scheme. For the discrete system we show that a sequence of Hopf bifurcations occur at the positive fixed point as the delay increasing, analyze the stability of the fixed point and calculate the direction of the Hopf bifurcations. At last, by giving some numerical experiments, we illustrate the relation between the time of producing blood cells and symptom.

1. Introduction

Recently, it has been recognized that differential equations with delay describe various models of mathematical biology more adequately than equations without delay. For example, delay differential equation

$$\dot{p}(t) = -\gamma p(t) + \frac{\beta\theta^n p(t-\tau)}{\theta^n + p^n(t-\tau)}, \quad t \geqslant 0, \tag{1.1}$$

characterize the regulation of hematopoiesis. Here, $p(t)$ denotes the density of mature cells in blood circulation, τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstreams, β, θ, n and γ are all positive constants and

$$\frac{\beta}{\gamma} > 1. \tag{1.2}$$

In [1], Michael C. Mackey and Leon Glass associated the onset of disease with bifurcations in Eq. (1.1). The fluctuations in peripheral while blood cell counts in chronic granulocytic leukemia (CGL). So studying the dynamics of Eq. (1.1) is significant for medical research.

But considering the need of scientific computation and real-time simulation, our interest is focused on the behaviors of discrete dynamics system corresponding to (1.1). Most of this time, it is desired that a difference equation derived from a differential equation, preserves the dynamical features of the corresponding continuous-time model. That is, the discrete-time model is 'dynamically consistent' with the continuous-time model.

In [2], Volker Wulf and Neville J. Ford show that, if applying Euler forward method to solve the delay differential equation, then the discrete scheme is 'dynamically consistent.' It means that for all sufficiently small step sizes the discrete

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model undergoes a Hopf bifurcation of the same type with the original model, and the bifurcation point λ_h of the discrete model is $O(h)$ close to the bifurcation point λ_*, which corresponds to the continuous-time model.

In this paper, we use the nonstandard finite-difference [6,7] to make the discretization for system (1.1). Firstly, we consider the autonomous delay differential equations

$$\dot{u} = f\big(u(t), u(t-1)\big), \quad t \geqslant 0; \qquad u(t) = \phi(t), \quad -1 \leqslant t \leqslant 0.$$

The first-order derivative is approximated by modified forward Euler expression

$$\frac{du(t)}{dt} \to \frac{u_{k+1} - u_k}{\phi}$$

with the 'denominator function' ϕ such that

$$\phi(h) = h + O\big(h^2\big),$$

where $h = \frac{1}{m}$ ($m \in \mathbb{N}^+$) stands for stepsize, and u_k denotes the approximate value to $u(kh)$. So we get the method as following:

$$u_{k+1} - u_k = \phi(h) f(u_k, u_{k-m}). \tag{1.3}$$

This method can be seem as a modified forward Euler method.

With a similar analysis in [2], it is known that the discrete model is 'dynamically consistent,' and the bifurcation point λ_h of the discrete model also is $O(h)$ close to the bifurcation point λ_* corresponding continuous-time model. On the other hand, many authors [7] have shown that the nonstandard finite-difference scheme, applied to the structurally unstable dynamical system, can obtain numerical solutions having the correct behavior.

This paper is organized as follows. In Section 2, using the Hopf bifurcation theory of discrete system [3–5], the stability of positive fixed point and the existence of the local Hopf bifurcations at the fixed point are investigated. In Section 3, direction and stability of the Hopf bifurcation are established. In Section 4, some numerical simulations are provided to illustrate the results found. At last, roughly, we apply our results to explain the symptom of CGL.

2. Stability of the positive equilibrium and local Hopf bifurcations

In this section, we shall see that, when nonstandard finite-difference scheme is applied to system (1.1), it gives rise a discrete dynamics system, and we shall study the stability of the positive fixed point and the existence of local Hopf bifurcations of the discrete dynamical system, which inherits certain dynamics of system (1.1).

Under transformation $p(t) = \theta x(t)$, Eq. (1.1) becomes

$$\dot{x}(t) = -\gamma x(t) + \frac{\beta x(t-\tau)}{1 + x^n(t-\tau)}. \tag{2.1}$$

Let $u(t) = x(\tau t)$, Eq. (2.1) can be rewritten as

$$\dot{u}(t) = -\gamma\tau u(t) + \frac{\beta\tau u(t-1)}{1 + u^n(t-1)}. \tag{2.2}$$

In view of the application of system (1.1) in practice, we only take an interest in the positive equilibrium point of (1.1). Assume that u_* is the positive equilibrium point of Eq. (2.2), if and only if

$$u_* = \sqrt[n]{\frac{\beta}{\gamma} - 1}.$$

If employ the nonstandard finite-difference scheme (1.3) to Eq. (2.2) and choose the 'denominator function' ϕ as

$$\phi(h) = \frac{1 - e^{-\gamma\tau h}}{\gamma\tau}, \tag{2.3}$$

it yields difference equation

$$u_{k+1} = e^{-\gamma\tau h} u_k + \frac{(1 - e^{-\gamma\tau h})\beta u_{k-m}}{\gamma(1 + u_{k-m}^n)}. \tag{2.4}$$

Note that the u_* also is the unique positive fixed point of (2.4). Set $y_k = u_k - u_*$. Then there is

$$y_{k+1} = e^{-\gamma\tau h} y_k + \frac{(1 - e^{-\gamma\tau h})\beta}{\gamma}\left[\frac{y_{k-m} + u_*}{1 + (y_{k-m} + u_*)^n} - \frac{u_*}{1 + u_*^n}\right]. \tag{2.5}$$

By introducing a new variable $Y_k = (y_k, y_{k-1}, \ldots, y_{k-m})^T$, we have the map

$$Y_{k+1} = F(Y_k, \tau), \tag{2.6}$$

where $F = (F_0, F_1, \dots, F_m)^T$, and

$$F_i = \begin{cases} e^{-\gamma\tau h} y_k + \frac{(1-e^{-\gamma\tau h})\beta}{\gamma}\left(\frac{y_{k-m}+u_*}{1+(y_{k-m}+u_*)^n} - \frac{u_*}{1+u_*^n}\right), & i = 0, \\ y_{k-i+1}, & 1 \leqslant i \leqslant m. \end{cases} \tag{2.7}$$

Clearly, the origin is a fixed point of map (2.6), and the linear part of map (2.6) is

$$Y_{k+1} = AY_k.$$

Here

$$A = \begin{pmatrix} a_m & 0 & \cdots & 0 & 0 & a_0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

in which $a_m = e^{-\gamma\tau h}$ and $a_0 = (1 - e^{-\gamma\tau h})[n(\frac{\gamma}{\beta} - 1) + 1]$. The characteristic equation of A is

$$\lambda^{m+1} - e^{-\gamma\tau h}\lambda^m - \left(1 - e^{-\gamma\tau h}\right)\left[n\left(\frac{\gamma}{\beta} - 1\right) + 1\right] = 0. \tag{2.8}$$

In order to proof the existence of the local Hopf bifurcation at fixed point, we need some lemmata as follows.

Lemma 2.1. *All roots of Eq.* (2.8) *have modulus less than one for sufficiently small positive* $\tau > 0$.

Proof. When $\tau = 0$, Eq. (2.8) is equal to

$$\lambda^{m+1} - \lambda^m = 0.$$

The equations have an m-fold root and a simple root $\lambda = 1$.

Consider the root $\lambda(\tau)$ of Eq. (2.8) such that $\lambda(0) = 1$. This root depends continuously on τ and Eq. (2.8) is differentiable about τ, from which we have

$$\frac{d\lambda}{d\tau} = \frac{\gamma h e^{-\gamma\tau h}[\lambda^m - n(\frac{\gamma}{\beta} - 1) - 1]}{m e^{-\gamma\tau h}\lambda^{m-1} - (m+1)\lambda^m} \tag{2.9}$$

and

$$\frac{d\bar{\lambda}}{d\tau} = \frac{\gamma h e^{-\gamma\tau h}[\bar{\lambda}^m - n(\frac{\gamma}{\beta} - 1) - 1]}{m e^{-\gamma\tau h}\bar{\lambda}^{m-1} - (m+1)\bar{\lambda}^m}. \tag{2.10}$$

Since $\frac{d|\lambda|^2}{d\tau} = \lambda\frac{d\bar{\lambda}}{d\tau} + \bar{\lambda}\frac{d\lambda}{d\tau}$,

$$\left.\frac{d|\lambda|^2}{d\tau}\right|_{\tau=0,\lambda=1} = 2hn\gamma\left(\frac{\gamma}{\beta} - 1\right) < 0.$$

So with the increasing of $\tau > 0$, λ cannot cross $\lambda = 1$. Consequently, all roots of Eq. (2.8) lie in the unit circle for sufficiently small positive $\tau > 0$. □

A Hopf bifurcation occurs when a complex conjugate pair of eigenvalues of A cross the unit circle as τ varies. We have to find values of τ such that there are roots on the unit circle. Denote the roots on the unit circle by $e^{i\omega}$, $\omega \in (-\pi, \pi]$. Since we are dealing with complex roots of a real polynomial, we only need to look for $\omega \in (0, \pi]$. For $\omega \in (0, \pi]$, $e^{i\omega}$ is a root of (2.8) if and only if

$$e^{i\omega} - e^{-\gamma\tau h} - \left(1 - e^{-\gamma\tau h}\right)\left[n\left(\frac{\gamma}{\beta} - 1\right) + 1\right]e^{-im\omega} = 0. \tag{2.11}$$

Separating the real part and imaginary part from Eq. (2.11), there are

$$\cos\omega - \left(1 - e^{-\gamma\tau h}\right)\left[n\left(\frac{\gamma}{\beta} - 1\right) + 1\right]\cos m\omega = e^{-\gamma\tau h} \tag{2.12}$$

and

$$\sin\omega + \left(1 - e^{-\gamma\tau h}\right)\left[n\left(\frac{\gamma}{\beta} - 1\right) + 1\right]\sin m\omega = 0. \tag{2.13}$$

So

$$\cos\omega = 1 - \frac{n(\frac{\gamma}{\beta}-1)[n(\frac{\gamma}{\beta}-1)+2](1-e^{-\gamma\tau h})^2}{2e^{-\gamma\tau h}}.$$

Let $1 < \frac{\beta}{\gamma} < \frac{n}{n-2}$, then $\cos\omega > 1$, which yields a contradiction. So we have the following result.

Lemma 2.2. *Assume that $1 < \frac{\beta}{\gamma} < \frac{n}{n-2}$. Then Eq. (2.8) has no root with modulus one for all $\tau > 0$.*

Since, from Eq. (2.13),

$$-\left(1-e^{-\gamma\tau h}\right)\left[n\left(\frac{\gamma}{\beta}-1\right)+1\right] = \frac{\sin\omega}{\sin m\omega} \tag{2.14}$$

is positive, $\sin\omega$ ($\omega \in [0,\pi]$) has the same sign as $\sin m\omega$, so that if $\frac{\beta}{\gamma} > \frac{n}{n-2}$, according to Lemma 2.2, there exists a sequence of

$$\omega_k \in \left(\frac{2k\pi}{m}, \frac{(2k+1)\pi}{m}\right), \quad k = 0, 1, 2, \ldots, \left[\frac{m-1}{2}\right], \tag{2.15}$$

where $[\cdot]$ denotes the greatest integer function. It is clear that there exists a sequence of the time delay parameters τ_k satisfying Eqs. (2.12), (2.13), according to $\omega = \omega_k$.

Lemma 2.3. *Let $\lambda_k(\tau) = r_k(\tau)e^{i\omega_k(\tau)}$ be a root of (2.8) near $\tau = \tau_k$ satisfying $r_k(\tau_k) = 1$ and $\omega_k(\tau_k) = \omega_k$. Then*

$$\left.\frac{dr_k^2(\tau)}{d\tau}\right|_{\tau=\tau_k,\,\omega=\omega_k} > 0.$$

Proof. From (2.9) and (2.10), we have

$$\frac{dr_k^2(\tau)}{d\tau} = \lambda\frac{d\bar{\lambda}}{d\tau} + \bar{\lambda}\frac{d\lambda}{d\tau} = \frac{2[m+1+me^{-\gamma\tau h}](1-\cos\omega)\gamma h e^{-\gamma\tau h}}{(1-e^{-\gamma\tau h})|(m+1)e^{iw_k} - me^{-\gamma\tau h}|^2} > 0.$$

Thus, the proof is complete. □

Lemma 2.4.

(i) *If $1 < \frac{\beta}{\gamma} < \frac{n}{n-2}$, then all roots of the characteristic equation (2.8) have modulus less than one.*
(ii) *If $\frac{\beta}{\gamma} > \frac{n}{n-2}$, then Eq. (2.8) has a pair of simple roots $e^{\pm i\omega_k}$ on the unit circle when $\tau = \tau_k$, $k = 0, 1, 2, \ldots, [\frac{m-1}{2}]$. Furthermore, if $\tau \in [0, \tau_0)$, then all the roots of Eq. (2.8) have modulus less than one; If $\tau = \tau_0$, then all roots of (2.8) except $e^{\pm i\omega_0}$ have modulus less than one. But if $\tau \in (\tau_k, \tau_{k+1}]$, for $k = 0, 1, 2, \ldots, [\frac{m-1}{2}]$, Eq. (2.8) has $2(k+1)$ roots have modulus more than one.*

Proof. By Lemmas 2.1 and 2.2, and applying a similar result of Ruan and Wei [11, Corollary 2.4], we arrive at the conclusion (i).

If $\frac{\beta}{\gamma} > \frac{n}{n-2}$, let τ_k be as in (2.16). From (2.14), we have that Eq. (2.8) has roots $e^{\pm i\omega_k}$ if and only if $\tau = \tau_k$ and $\omega = \omega_k$ given in (2.15) and (2.16).

By Lemmas 2.1 and 2.2 we know that if $\tau \in [0, \tau_0)$, then all the roots of Eq. (2.8) have modulus less than one; if $\tau = \tau_0$, then all roots of (2.8) except $e^{\pm i\omega_0}$ have modulus less than one; furthermore, by the Rouché's theorem (Dieudonne [8, Theorem 9.17.4]), the statement on the number of eigenvalues with modulus more than one as follows. □

Special properties in Lemma 2.4 immediately lead to stability of the zero solution of Eq. (2.5), namely, of the positive fixed point $u = u_*$ of Eq. (2.4). So we have the theorem about Eq. (2.4).

Theorem 2.5.

(i) *If $1 < \frac{\beta}{\gamma} < \frac{n}{n-2}$, then $u = u_*$ is asymptotically stable for any $\tau \geqslant 0$.*
(ii) *If $\frac{\beta}{\gamma} > \frac{n}{n-2}$, then $u = u_*$ is asymptotically stable for $\tau \in [0, \tau_0)$, and unstable for $\tau > \tau_0$.*
(iii) *For $\frac{\beta}{\gamma} > \frac{n}{n-2}$, Eq. (2.4) undergoes a Hopf bifurcation at u_* when $\tau = \tau_k$, for $k = 0, 1, 2, \ldots, [\frac{m-1}{2}]$.*

3. Direction and stability of the Hopf bifurcation in discrete model

In the previous section, we obtained the conditions of Hopf bifurcation occurring when $\tau = \tau_k$, $k = 0, 1, 2, \ldots, [\frac{m-1}{2}]$. In this section, we study the direction of the Hopf bifurcation and the stability of the bifurcation periodic solutions when $\tau = \tau_0$, using the similar techniques from normal form and center manifold theory (see e.g. Yuri [5]). Proving the main result, we need some preliminary lemmata.

Set $\tau = \tau_0 + \mu$, $\mu \in R$. Then $\mu = 0$ is a Hopf bifurcation value for Eq. (2.5). Rewrite Eq. (2.5) as

$$y_{k+1} = e^{-\gamma\tau h} y_k + \left(1 - e^{-\gamma\tau h}\right)\left[\frac{\beta - n(\beta-\gamma)}{\beta} y_{k-m} + \frac{n(\beta-\gamma)((n-1)\beta - 2n\gamma)}{2u_*\beta^2} y_{k-m}^2 - \frac{n(\beta-\gamma)((n^2-1)\beta^2 - 6n^2\beta\gamma + 6n^2\gamma^2)}{6u_*^2\beta^3} y_{k-m}^3\right] + O\left(|y_{k-m}|^4\right).$$

So system (2.6) is turned into

$$Y_{k+1} = AY_k + \frac{1}{2}B(Y_k, Y_k) + \frac{1}{6}C(Y_k, Y_k, Y_k) + O\left(\|Y_k\|^4\right),$$

where

$$B(Y_k, Y_k) = \left(b_0(Y_k, Y_k), 0, \ldots, 0\right), \qquad C(Y_k, Y_k, Y_k) = \left(c_0(Y_k, Y_k, Y_k), 0 \ldots, 0\right),$$

in which

$$\begin{cases} b_0(\phi, \psi) = \tilde{b}\phi_m\psi_m, \\ c_0(\phi, \psi, \eta) = \tilde{c}\phi_m\psi_m\eta_m, \end{cases} \tag{3.1}$$

$$\tilde{b} = \left(1 - e^{-\gamma\tau h}\right)\frac{n(\beta-\gamma)((n-1)\beta - 2n\gamma)}{2u_*\beta^2},$$

$$\tilde{c} = -\left(1 - e^{-\gamma\tau h}\right)\frac{\gamma(\beta-\gamma)((n^2-1)\beta^2 - 6n^2\beta\gamma + 6n^2\gamma^2)}{6u_*^2\beta^4}.$$

Let $q = q(\tau_0) \in \mathbb{C}^{m+1}$ be an eigenvector of A corresponding to $e^{i\omega_0}$, then

$$Aq = e^{i\omega_0}q, \qquad A\bar{q} = e^{-i\omega_0}\bar{q}.$$

We also introduce an adjoint eigenvector $q^* = q^*(\tau) \in \mathbb{C}^{m+1}$ having the properties

$$A^Tq^* = e^{-i\omega_0}q^*, \qquad A^T\bar{q}^* = e^{i\omega_0}\bar{q}^*,$$

and satisfying the normalization $\langle q^*, q\rangle = 1$, where $\langle q^*, q\rangle = \sum_{i=0}^m \bar{q}_i^* q_i$.

Lemma 3.1. *(See [9].) Define a vector valued function $q : \mathbb{C} \to \mathbb{C}^{m+1}$ by*

$$p(\xi) = \left(\xi^m, \xi^{m-1}, \ldots, 1\right)^T.$$

If ξ is an eigenvalue of A, then $Ap(\xi) = \xi p(\xi)$.

In view of Lemma 3.1, we have

$$q = p\left(e^{i\omega_0}\right) = \left(e^{im\omega_0}, e^{i(m-1)\omega_0}, \ldots, e^{i\omega_0}, 1\right)^T. \tag{3.2}$$

Lemma 3.2. *Suppose $q^* = (q_0^*, q_1^*, \ldots, q_m^*)^T$ is the eigenvector of A^T corresponding to eigenvalue $e^{-i\omega_0}$, and $\langle q^*, q\rangle = 1$. Then*

$$q^* = \bar{K}\left(1, a_0e^{im\omega_0}, a_0e^{i(m-1)\omega_0}, \ldots, a_0e^{i2\omega_0}, a_0e^{i\omega_0}\right)^T \tag{3.3}$$

and

$$K = \frac{1}{e^{im\omega_0} + ma_0e^{-i\omega_0}}. \tag{3.4}$$

Proof. Assign q^* satisfies $A^Tq^* = \bar{z}q^*$ with $\bar{z} = e^{-i\omega_0}$, then the following identities hold

$$\begin{cases} a_mq_0^* + q_1^* = e^{-i\omega_0}q_0^*, \\ q_k^* = e^{-i\omega_0}q_{k-1}^*, \quad k = 2, \ldots, m, \\ a_0q_0^* = e^{-i\omega_0}q_m^*. \end{cases} \tag{3.5}$$

Let $q_m^* = a_0 e^{i\omega_0} \bar{K}$, then

$$q^* = \bar{K}\big(1, a_0 e^{im\omega_0}, a_0 e^{i(m-1)\omega_0}, \ldots, a_0 e^{i2\omega_0}, a_0 e^{i\omega_0}\big)^T.$$

From normalization $\langle q^*, q\rangle = 1$ and computation, Eq. (3.4) is hold. $\square$

Let $a(\lambda)$ be characteristic polynomial of A and $\lambda_0 = e^{i\omega_0}$. Following the algorithms in [5] and using a computation process similar to that in [9], we can compute an expression for the critical coefficient $c_1(\tau_0)$,

$$c_1(\tau_0) = \frac{g_{20}g_{11}(1-2\lambda_0)}{2(\lambda_0^2-\lambda_0)} + \frac{|g_{11}|^2}{1-\bar{\lambda}_0} + \frac{|g_{02}|^2}{2(\lambda_0^2-\bar{\lambda}_0)} + \frac{g_{21}}{2}, \tag{3.6}$$

where

$$g_{20} = \big\langle q^*, B(q,q)\big\rangle, \qquad g_{11} = \big\langle q^*, B(q,\bar{q})\big\rangle, \qquad g_{02} = \big\langle q^*, B(\bar{q},\bar{q})\big\rangle,$$
$$g_{21} = \big\langle q^*, B(\bar{q},\omega_{20})\big\rangle + 2\big\langle q^*, B(q,\omega_{11})\big\rangle + \big\langle q^*, C(q,q,\bar{q})\big\rangle,$$

and

$$\omega_{20} = \frac{b_0(q,q)}{a(\lambda_0^2)} p\big(\lambda_0^2\big) - \frac{\langle q^*, B(q,q)\rangle}{\lambda_0^2-\lambda_0} q - \frac{\langle \bar{q}^*, B(q,q)\rangle}{\lambda_0^2-\bar{\lambda}_0}\bar{q},$$
$$\omega_{11} = \frac{b_0(q,\bar{q})}{a(1)} p(1) - \frac{\langle q^*, B(q,\bar{q})\rangle}{1-\lambda_0} q - \frac{\langle \bar{q}^*, B(q,\bar{q})\rangle}{1-\bar{\lambda}_0}\bar{q}.$$

By (3.1), (3.2) and Lemma 3.2, we get

$$\begin{cases} b_0\big(\bar{q}, p\big(e^{i2\omega_0}\big)\big) = \tilde{b}, \\ b_0(q,q) = \tilde{b}, \\ b_0(q,\bar{q}) = \tilde{b}, \\ c_0(q,q,\bar{q}) = \tilde{c}, \\ a\big(e^{i2\omega_0}\big) = e^{i2(m+1)\omega_0} - a_m e^{i2m\omega_0} - a_0, \\ a(1) = 1 - a_m - a_0, \\ b_0\big(q, p(1)\big) = \tilde{b}. \end{cases} \tag{3.7}$$

Substituting these into (3.6) we have

$$c_1(\tau_0) = \frac{K}{2}\left(\frac{\tilde{b}^2}{a(e^{i2\omega_0})} + \frac{2\tilde{b}^2}{a(1)} + \tilde{c}\right). \tag{3.8}$$

Lemma 3.3. *(See [10].) Given the map* (2.6) *and assume*

(1) $\lambda(\tau) = r(\tau)e^{i\omega(\tau)}$, *where* $r(\tau^*) = 1$, $r'(\tau^*) \neq 0$ *and* $\omega(\tau^*) = \omega^*$;
(2) $e^{ik\omega^*} \neq 1$ *for* $k = 1, 2, 3, 4$;
(3) $\mathrm{Re}[e^{-i\omega^*}c_1(\tau^*)] \neq 0$.

Then an invariant closed curve, topologically equivalent to a circle, for map (2.6) *exists for* τ *in a one side neighborhood of* τ^*. *The radius of the invariant curve grows like* $O(\sqrt{|\tau-\tau^*|})$. *One of the four cases below applies*:

(1) $r'(\tau^*) > 0$, $\mathrm{Re}[e^{-i\omega^*}c_1(\tau^*)] < 0$. *The origin is asymptotically stable for* $\tau < \tau^*$ *and unstable for* $\tau > \tau^*$. *An attracting invariant closed curve exists for* $\tau > \tau^*$.
(2) $r'(\tau^*) > 0$, $\mathrm{Re}[e^{-i\omega^*}c_1(\tau^*)] > 0$. *The origin is asymptotically stable for* $\tau < \tau^*$ *and unstable for* $\tau > \tau^*$. *A repelling invariant closed curve exists for* $\tau < \tau^*$.
(3) $r'(\tau^*) < 0$, $\mathrm{Re}[e^{-i\omega^*}c_1(\tau^*)] < 0$. *The origin is asymptotically stable for* $\tau > \tau^*$ *and unstable for* $\tau < \tau^*$. *An attracting invariant closed curve exists for* $\tau < \tau^*$.
(4) $r'(\tau^*) < 0$, $\mathrm{Re}[e^{-i\omega^*}c_1(\tau^*)] > 0$. *The origin is asymptotically stable for* $\tau > \tau^*$ *and unstable for* $\tau < \tau^*$. *A repelling invariant closed curve exists for* $\tau > \tau^*$.

From the discussion in Section 2, we know that $r'(\tau^*) > 0$, therefore, by Lemma 3.3 we have the following result.

Theorem 3.4. *If* $\frac{\beta}{\gamma} > \frac{n}{n-2}$, *then* $u = u_*$ *is asymptotically stable for* $\tau \in [0, \tau_0)$, *and unstable for* $\tau > \tau_0$. *An attracting* (*repelling*) *invariant closed curve exists for* $\tau > \tau_0$ *if* $\mathrm{Re}[e^{-i\omega_0}c_1(\tau_0)] < 0$ (> 0).

Table 1
The bifurcation points and directions of bifurcation at μ_* for different step sizes

h	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	$\mathrm{Re}[e^{-i\omega_0}c_1(\tau_0)]$
$1/2$	3.64642						−0.0374223
$1/2^2$	4.14285	15.9481					−0.0210292
$1/2^3$	4.41861	18.8566	30.6152	38.235			−0.0110845
$1/2^4$	4.56206	20.0616	34.7746	48.1599	59.7469	···	−0.00567481

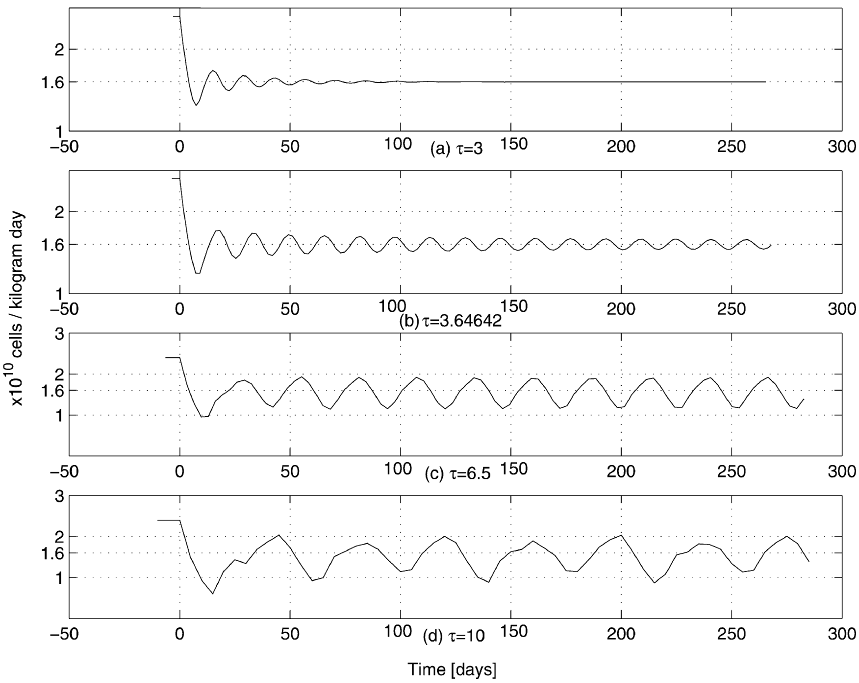

Fig. 1. The numerical solution of Eq. (1.1) with nonstandard finite-difference scheme corresponding to step size $h = 1/2$.

4. Numerical test

One of the purposes of this section is to test the results in Sections 2 and 3 by some examples; the second one is to explain the main theorem more clearly; the third one is to show that nonstandard scheme is better than Euler method in some aspects. Such as for certain step size, the Euler discrete model cannot get the bifurcation solution, but the nonstandard discrete can get it, which is the same as the analysis solution of Eq. (1.1).

From [1] we get some indexes of a person, that is $\beta = 0.2$ per day, $\gamma = 0.1$ per day, $n = 10$, $\theta = 1.6 \times 10^{10}$ cells per kilogram, initial condition $p(t) = 1.5$ $(-\tau < t < 0)$ cells per kilogram per day, and steady-state granulocyte turnover rate of 1.63×10^{10} cells per kilogram per day.

We compute the bifurcation points and direction of bifurcation at fixed point of numerical solutions of Eq. (1.1) with nonstandard finite-difference scheme Eq. (2.4) for some step sizes. The figures refer to Table 1 and Figs. 1–4.

From Theorem 3.4 and Table 1, we conclude that all these numerical solution should be asymptotically stable for $\tau \in [0, \tau_0)$, unstable for $\tau > \tau_0$ and an attracting invariant closed curve exists for $\tau > \tau_0$. This is just what can see from Figs. 1–4.

Secondly, using Euler method we draw the numerical solutions of Eq. (1.1) with the same coefficients. From Fig. 5 we can see that for $h = \frac{1}{2}$, when $\tau \leqslant \tau_0$ the fixed point is not asymptotically stable and with the increasing of τ, it is unstable; Fig. 6 ($h = \frac{1}{8}$) is similar with Fig. 2 ($h = \frac{1}{4}$). And then, by comparing with the figures in [1], Figs. 1–4 correctly describe the condition of circulating white blood cells. So we could argue that nonstandard finite-difference scheme is superior to Euler method under the means of describing approximately the dynamics of the original system with the same step size.

5. Conclusions

In this section, we apply the results to analyze the symptom of CGL. In normal healthy adults, circulating levels of granulocytes are either constant or show a mild oscillation with a period of 14 to 24 days. Cyclical neutropenia is a disease

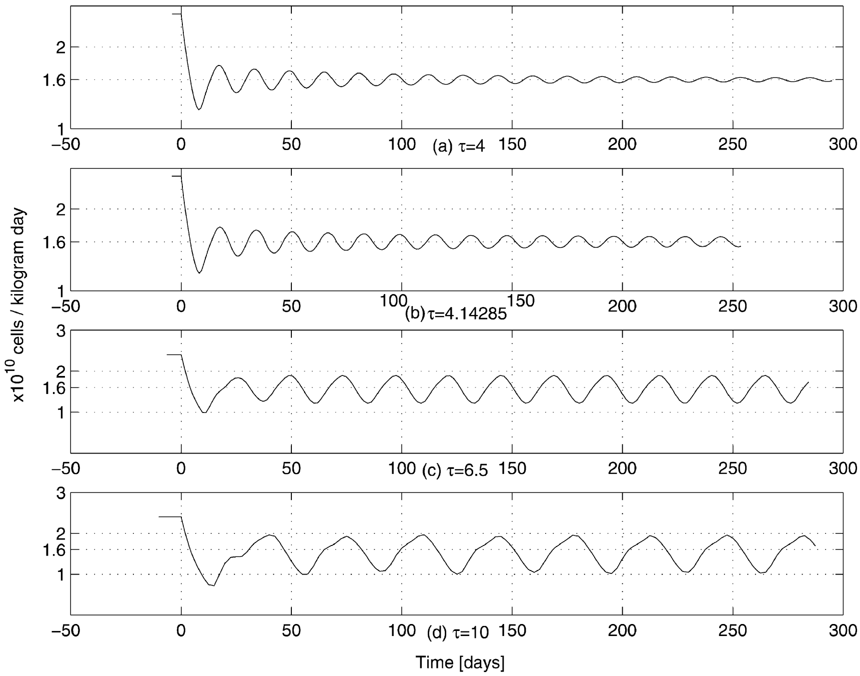

Fig. 2. The numerical solution of Eq. (1.1) with nonstandard finite-difference scheme corresponding to step size $h = 1/4$.

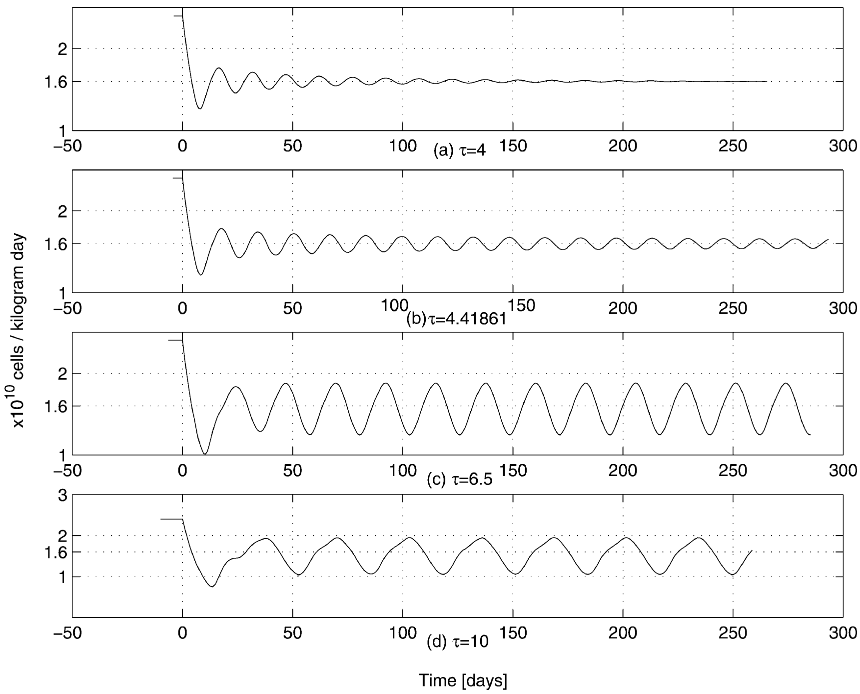

Fig. 3. The numerical solution of Eq. (1.1) with nonstandard finite-difference scheme corresponding to step size $h = 1/8$.

characterized by spontaneous oscillations in granulocyte numbers from normal to subnormal levels with a period of about 21 days. In some patients suffering from CGL circulation granulocyte numbers display large-amplitude oscillations with periodicities ranging from 30 to 70 days, depending on the patient [1].

From Sections 2 and 3, we could see that for an adult when the time delay τ (between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstreams) is less than τ_0, his or her circulating

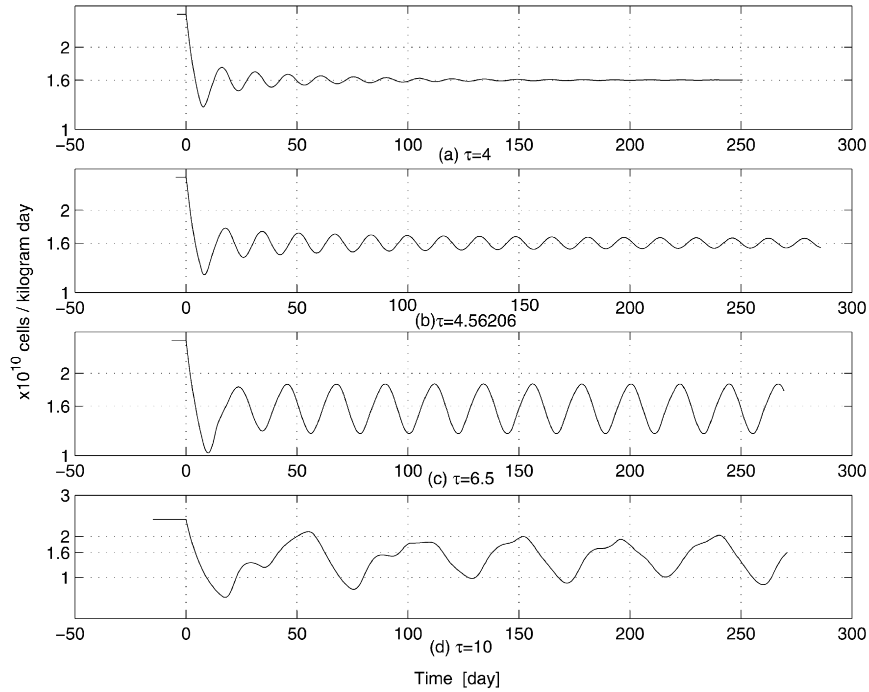

Fig. 4. The numerical solution of Eq. (1.1) with nonstandard finite-difference scheme corresponding to step size $h = 1/16$.

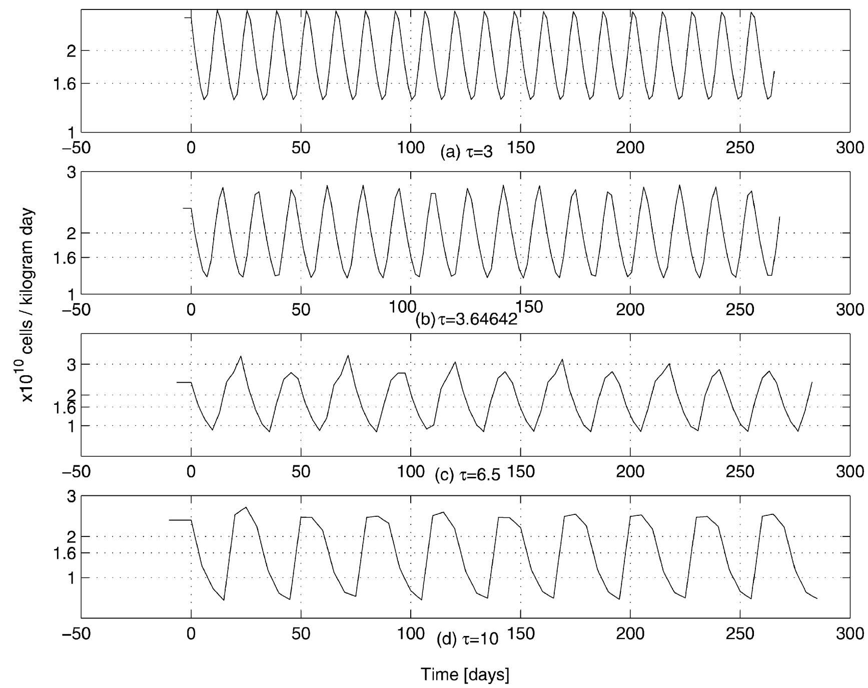

Fig. 5. The numerical solution of Eq. (1.1) with Euler method corresponding to step size $h = 1/2$.

levels are constant which is normal. Once τ crosses τ_0, there will be spontaneous oscillations of circulating levels which is a symptom of CGL. So people should try to control the delay time no more than the critical value. However, this article just provide a method to compute the critical time for different persons.

Using the method, we get Figs. 1–4, so diagnosis that if the level of the circulating of some people is the same as (a) which is constant, or (b) whose periodic is about is 16 days, then the people is healthy. If it is the same as (c), whose period

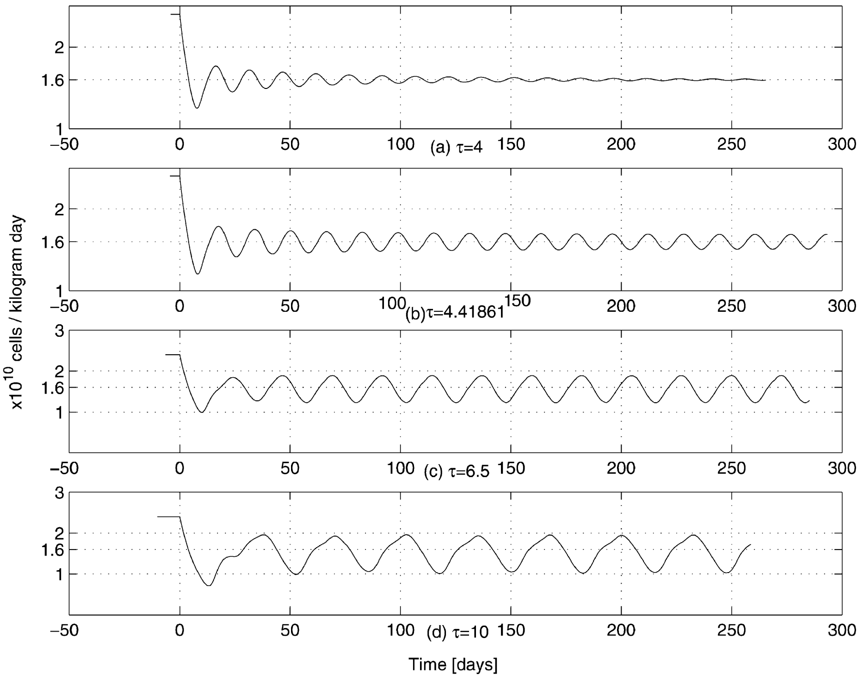

Fig. 6. The numerical solution of Eq. (1.1) with Euler method corresponding to step size $h = 1/8$.

is about 22 days, then he (or she) owns subnormal level. If it likes (d) whose period is 33 days then he (or she) is suffering from CGL.

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