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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
Journal of Computational and Applied Mathematics 221 (2008) 66–75
www.elsevier.com/locate/cam

Multiplicity of positive radial solutions for an elliptic inclusion system on an annulus

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Received 22 July 2007

Abstract

In this paper, we present the sufficient conditions for existence and multiplicity of positive radial solutions for elliptic inclusion systems. Our results are obtained by utilizing the generalization of Leggett and Williams's fixed point theorem, established in this paper, for the norm-type cone expansion and compression of multivalued operators.
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MSC: 46N10

Keywords: Multivalued operator; Cone expansion and compression fixed point theorem; Elliptic differential inclusion system

1. Introduction

We shall establish the new result of existence of positive radial solutions for elliptic systems of the form

$$\begin{cases} \triangle u \in \lambda k_1(|x|)F(u,v) & \text{in } \Omega, \\ \triangle v \in \mu k_2(|x|)G(u,v) & \text{in } \Omega, \\ a_1 u + b_1 \dfrac{\partial u}{\partial n} = 0, \qquad a_2 v + b_2 \dfrac{\partial v}{\partial n} = 0 & \text{on } |x| = R_1, \\ c_1 u + d_1 \dfrac{\partial u}{\partial n} = 0, \qquad c_2 v + d_2 \dfrac{\partial v}{\partial n} = 0 & \text{on } |x| = R_2, \end{cases} \tag{1}$$

where $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ with $\Omega = \{x \in \mathbb{R}^n : R_1 < |x| < R_2, R_2 > R_1 > 0\}$ an annulus with boundary $\partial\Omega$, $a_i, b_i, c_i, d_i \geq 0$, and $\rho_i \equiv c_i b_i + a_i c_i + a_i d_i > 0$ for $i = 1, 2$.

Under the hypotheses that F, G are single-valued functions, the existence and multiplicity of positive solutions for elliptic systems (1) subjected to different boundary conditions on a ball has been studied in [1–3]. For instance, subject to the following boundary condition

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0,$$

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doi:10.1016/j.cam.2007.10.010

Peletier and van der Vorst studied the existence of positive solutions for elliptic systems on a ball in [3], Dunninger and Wang used a fixed point theorem of cone expansion/compression type which allows to establish not only existence, but also multiplicity in [2]. If parameters $\lambda = \mu$, moreover, Dunninger and Wang proved the existence, nonexistence and multiplicity of solutions to (1) based on the difference of the value of parameter λ in [1]. They employed upper and lower solution methods together with fixed point index theorems. Our purpose here is to deal with more general F and G, i.e., to allow that they are the multivalued functions. Hence we extend the results of [1]. In fact, we apply the generalization of Leggett and Williams's fixed point theorem, established in this paper, for the norm-type cone expansion and compression of multivalued operators to establish the multiplicity and existence of solutions to (1) for all positive parameters λ, μ without the assumptions of the nonlinear terms F and G being monotone.

Let $(E, |\cdot|)$ be a Banach space. A nonempty convex closed set $P \subset E$ is called a cone of E if the following conditions hold

$$x \in P, \lambda \geq 0 \text{ implies } \lambda x \in P; \quad \text{and} \quad x \in P \text{ and } -x \in P \text{ implies } x = \theta,$$

where θ stands for a zero element of E.

Let us introduce a partially ordered "$\leq$" in E, i.e., $x \leq y$ if and only if $y - x \in P$ for any $x, y \in E$. $x < y$ if and only if $x \leq y$ and $x \neq y$.

The cone P is said to be normal if there exists a positive constant N, which is called the normal constant of P, such that $\theta \leq x \leq y$ $(x, y \in E)$ implies that $|x| \leq N|y|$.

Given a cone P of E, denote $P^+ = P \setminus \{\theta\}$. For $u_0 \in P^+$, denote

$$P(u_0) = \{x \in P : \lambda u_0 \leq x \text{ for some } \lambda > 0\}.$$

For notational purposes for $\eta > 0$ let

$$\Omega_\eta = \{y \in E : |y| < \eta\}, \qquad \partial\Omega_\eta = \{y \in E : |y| = \eta\}, \quad \text{and}$$
$$\overline{\Omega}_\eta = \{y \in E : |y| \leq \eta\}.$$

Since Krasnoselskii gave the original result of cone expansion and compression in [4], multifarious fixed point theorems of the expansion and compression have been obtained (see, for example, [4–9]). For instance, in [5] Guo and Lakshmikantham gave the result of the norm type and in [6] Anderson and Avery obtained a generalization of Krasnoselskiis fixed point theorem of the norm type by applying conditions formulated in terms of two functionals replacing the norm-type assumptions. It should be noted that the use of fixed point theorems of cone expansion/compression type in the study of the existence and multiplicity of positive solutions for differential and integral equations has recently been quite extensive (see [2,7,9–12]). However, in general, the expansion may be easily verified for a large class of nonlinear integral operators, compression is a rather stringent condition and is usually not easily verified. In [7] Leggett and Williams improved the compression by replacing it with the weaker condition in which the operator is a compression on $P(u_0) \cap \partial\Omega_\eta$. In [8] one can find some refinements of [7]. In [9] Zima proved the following result via replacing Leggett and Williams type ordering conditions by the conditions of the norm type (see [9, Theorem 1]):

Proposition 1. *Let P be a normal cone in E with the normal constant N. Assume that Ω_1 and Ω_2 are bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $F : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator and $u_0 \in P^+$. If either*

(1) *$N|x| \leq |Fx|$ for $x \in P(u_0) \cap \partial\Omega_1$ and $|Fx| \leq |x|$ for $x \in P \cap \partial\Omega_2$, or*
(2) *$|Fx| \leq |x|$ for $x \in P \cap \partial\Omega_1$ and $N|x| \leq |Fx|$ for $x \in P(u_0) \cap \partial\Omega_2$*

is satisfied, where $\partial\Omega$ denotes the boundary of Ω, then F has a fixed point in the set $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In [13] Agarwal and O'Regan extended Krasnoselskii's fixed point theorem of norm type to multivalued operator problems and obtained the following fixed point theorem for k-set contractive multivalued operators (see Theorem 2.4 and Theorem 2.8 in [13]):

Proposition 2. *Let $E = (E, |\cdot|)$ be a Banach space, $P \subset E$ a cone and let $|\cdot|$ be increasing with respect to P. Also r, R are constants with $0 < r < R$. Suppose $A : \overline{\Omega}_R \cap P \to CK(P)$ be a u.s.c., k-set contractive (here $0 \leq k < 1$) map and assume one of the following conditions holds:*

(C1) (i) $|y| \le |x|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_R \cap P$ *and*
(ii) $|y| > |x|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_r \cap P$.
(C2) (i) $|y| > |x|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_R \cap P$ *and*
(ii) $|y| \le |x|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_r \cap P$.

Then A *has a fixed point in* $P \cap \{x \in r \le |x| \le R\}$.

Impelled by the advantage of usually being easier to apply, in Section 2 of this paper we will extend Leggett and Williams fixed point theorem to k-set contractive multivalued operator problems. This is also a generalization of Proposition 1 and an improvement of Proposition 2, moreover, is a fundamental tool to prove our main results. In Section 3 we prove the multiplicity and existence of solutions to (1) for all positive parameters λ, μ.

2. Existence of fixed points

We begin this section with gathering together some definitions and known facts. For two subsets C, D of E, we write $C \le D$ (or $D \ge C$) if

$$\forall p \in D, \quad \exists q \in C \text{ such that } q \le p.$$

A multivalued operator A is called upper semi-continuous (u.s.c.) on E if for each $x \in E$ the set $A(x)$ is a nonempty closed subset of E, and if for each open set B of E containing $A(x)$, there exists an open neighborhood V of x such that $A(V) \subseteq B$.

A is called a k-set contraction if $\gamma(A(D)) \le k\gamma(D)$ for all bounded sets D of E and $A(D)$ is bounded, where γ denotes the Kuratowskii measure of noncompactness.

Throughout this paper, we denote by $CK(C)$ the family of nonempty compact, convex subsets of set C.

The following nonzero fixed point theorems for multivalued operator will be applied in this section.

Lemma 1 ([13]). *Let* E *be an ordered Banach space and* P *a cone in* E *and let* r, R *be constants with* $R > r > 0$. *Assume that* $A : \overline{\Omega}_R \multimap CK(P)$ *is a u.s.c., k-set contractive (here* $0 \le k < 1$*) map and assume that one of the following conditions hold:*

$$x \notin \lambda Ax \quad \text{for all } \lambda \in [0,1) \quad \text{and} \quad x \in \partial\Omega_R \cap P, \quad \text{and} \tag{2}$$

$$\text{there exists a } v \in P^+ \text{ with } x \notin Ax + \delta v \text{ for } x \in \partial\Omega_r \cap P \text{ and } \delta \ge 0. \tag{3}$$

or

$$x \notin \lambda Ax \quad \text{for all } \lambda \in [0,1) \quad \text{and} \quad x \in \partial\Omega_r \cap P, \quad \text{and} \tag{4}$$

$$\text{there exists a } v \in P^+ \text{ with } x \notin Ax + \delta v \text{ for } x \in \partial\Omega_R \cap P \text{ and } \delta \ge 0. \tag{5}$$

Then A *has at least one fixed point* $y \in P$ *with* $r \le y \le R$.

Lemma 2 ([14]). *Let* E *be a Banach space,* C *a closed convex subset of* E, *and* U *an open subset of* C *with* $\theta \in U$. *Suppose that* $A : \overline{U} \multimap CK(C)$ *is u.s.c, k-set contractive (here* $0 \le k < 1$*). Then either*

(h1) *there exists* $x \in \overline{U}$ *with* $x \in Ax$; *or*
(h2) *there exists* $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u \in \lambda Ax$.

Theorem 1. *Let* P *be a normal cone in* E *with the normal constant* N. *Assume that* Ω_1 *and* Ω_2 *are bounded open sets in* E *such that* $\theta \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Let* $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to CK(P)$ *be a u.s.c, k-set contractive (here* $0 \le k < 1$*) operator and* $u_0 \in P^+$. *If there exist* $0 < r < R$ *with* $\Omega_r \subset \Omega_1$, $\Omega_R \subset \Omega_2$ *such that either*

(H1) (i) $N|x| < |y|$ *for all* $y \in Ax$ *and* $x \in P(u_0) \cap \partial\Omega_r$ *and*
(ii) $|y| \le |x|$ *for all* $y \in Ax$ *and* $x \in P \cap \partial\Omega_R$, *or*
(H2) (i) $|y| \le |x|$ *for all* $y \in Ax$ *and* $x \in P \cap \partial\Omega_r$ *and*
(ii) $N|x| < |y|$ *for all* $y \in Ax$ *and* $x \in P(u_0) \cap \partial\Omega_R$

is satisfied, then A *has a positive fixed point in* $P \cap (\overline{\Omega}_R \setminus \Omega_r)$.

Proof. From the hypotheses of Ω_1 and Ω_2 it follows that there exist positive numbers r, R with $0 < r < R$ such that $\Omega_r \subset \Omega_1$, $\overline{\Omega}_r \subset \Omega_R \subset \Omega_2$, and either (H1) or (H2) holds on $\partial\Omega_r$ and $\partial\Omega_R$. We seek to apply Lemma 1. It is sufficient to check that A satisfies the conditions (2) and (3) in Ω_r and in Ω_R, respectively, provided that the condition (H1) holds. First, (H1)(ii) with $x \in P \cap \partial\Omega_R$ instead of $x \in P \cap \Omega_2$ implies that (2) is true. To see this suppose there exists $x \in P \cap \partial\Omega_R$ and $\lambda \in [0, 1)$ with $x \in \lambda Ax$. Then there exists $y \in Ax$ with $x = \lambda y$ and so

$$R = |x| = \lambda|y| < |y| \le |x| = R,$$

a contradiction. Next, we will prove that for any $x \in P \cap \partial\Omega_r$ and any $\delta \ge 0$,

$$x \notin Ax + \delta u_0. \tag{6}$$

Suppose, on the contrary, that there exist $x_0 \in P \cap \partial\Omega_r$ and $t \ge 0$ such that $x_0 \in Ax_0 + tu_0$, that is,

$$x_0 - tu_0 \in Ax_0. \tag{7}$$

Clearly, $t \neq 0$ (otherwise, this proof is completed). Note that $Ax_0 \subset P$, we conclude that

$$tu_0 \le tu_0 + y$$

for all $y \in Ax_0$. This, combining (7), we get $x_0 \in P(u_0)$. Also, (7) shows that there exists $y_0 \in Ax_0$ such that $x_0 - tu_0 = y_0$. This implies

$$\theta \le y_0 \le x_0.$$

In virtue of the normality of P we have

$$|y_0| \le N|x_0|. \tag{8}$$

Since $x_0 \in P \cap \partial\Omega_r$, (8) contradicts (H2)(i). Hence, (6) is true. This implies that (3) is true. The result of Theorem 1 now follows from Lemma 1.

Similarly, we can prove that the result of Theorem 1 follows if (H2) holds. This proof is completed. □

Remark 1. Under the conditions of Theorem 1 if $\Omega_1 = \Omega_r$ and $\Omega_2 = \Omega_R$ with $0 < r < R$, then A has a positive fixed point in the set $P \cap (\overline{\Omega}_R \setminus \Omega_r)$.

In what follows, we combine Lemma 2 and Theorem 1 to establish existence of multiple fixed points.

Theorem 2. *Assume that the conditions of Theorem 1 hold and*

$$x \notin Ax \quad \textit{for all } x \in \partial\Omega_r \cap P, \tag{9}$$

where $\Omega_r \subset \Omega_1$. Then, there exists a constant R with $R > r$ such that A has at least two fixed points x_1 and x_2 with $x_1 \in \Omega_r \cap P$ and $x_2 \in P \cap (\overline{\Omega}_R \setminus \overline{\Omega}_r)$.

Proof. Theorem 1 implies that A has at least one fixed point x_2 with $x_2 \in P \cap (\overline{\Omega}_R \setminus \Omega_r)$, where $R > r$ and $\Omega_R \subset \Omega_2$ (see the proof of Theorem 1). In addition, we obtain in the proof of Theorem 1 that $x \notin \lambda Ax$ for all $\lambda \in [0, 1)$ and $x \in \partial\Omega_r \cap P$. Hence, we combine (9) and Lemma 2 to conclude that A has a fixed point $x_1 \in \Omega_r \cap P$. This completes the proof of Theorem 2. □

For constants L, r, R with $0 < r < L < R$, let us suppose

(H3) $x \notin Ax$ for all $x \in \partial\Omega_L \cap P$.
(H4) (i) $N|x| < |y|$ for all $y \in Ax$ and $x \in P(u_0) \cap \partial\Omega_r$,
(ii) $|y| \le |x|$ for all $y \in Ax$ and $x \in P \cap \partial\Omega_L$,
(iii) $N|x| < |y|$ for all $y \in Ax$ and $x \in P(u_0) \cap \partial\Omega_R$.
(H5) (i) $|y| \le |x|$ for all $y \in Ax$ and $x \in P \cap \partial\Omega_r$,
(ii) $N|x| < |y|$ for all $y \in Ax$ and $x \in P(u_0) \cap \partial\Omega_L$,
(iii) $|y| \le |x|$ for all $y \in Ax$ and $x \in P \cap \partial\Omega_R$.

Theorem 3. *Let P be a normal cone in E with the normal constant N, $A : \overline{\Omega}_R \cap P \to CK(P)$ be a u.s.c., k-set contractive (here $0 \le k < 1$) operator. If either the conditions* (H3) *and* (H4) *or the conditions* (H3) *and* (H5) *hold, then A has at least two fixed points x_1 and x_2 with $x_1 \in P \cap (\Omega_L \setminus \Omega_r)$ and $x_2 \in P \cap (\overline{\Omega}_R \setminus \overline{\Omega}_L)$.*

Proof. Remark 1 implies that A has a fixed point $x_1 \in P \cap (\overline{\Omega}_L \setminus \Omega_r)$. (H3) shows that $x_1 \notin \partial\Omega_L$. Hence, $x_1 \in P \cap (\Omega_L \setminus \Omega_r)$. Again, Remark 1 guarantees the existence of x_2. This proof is completed. □

3. Positive radial solutions for elliptic systems

In this section, we prove the multiplicity and existence of solutions to (1) for all positive parameters λ, μ by applying the result involving Section 2.

Let $C[0, 1]$ be a set consisting of all continuous functions from $[0, 1]$ into $\mathbb{R}$ and endowed the norm with $\|x\| = \sup_{t\in[0,1]} |x(t)|$. We define, for $u, v \in C[0, 1]$, $u \le v$ if and only if $u(t) \le v(t)$ for every $t \in [0, 1]$. By a positive solution of (1) we understand that a component (u, v) with $u(t) \ge 0$, $v(t) \ge 0$ for all $t \in [0, 1]$ satisfies (1) and either $u \not\equiv 0$ or $v \not\equiv 0$. By the maximum principle, each nontrivial component of (u, v) is thus positive in Ω. Throughout this section we assume that the following conditions hold:

$$F, G : [0, \infty) \times [0, \infty) \to CK([0, \infty)) \text{ and } (u, v) \to F(u, v) \text{ and } (u, v) \to G(u, v) \text{ are u.s.c.}$$

The following characterization is due to [1]. We seek the criteria for the existence of positive radial solutions $u = u(r)$, $v = v(r)$ of (1) which then satisfy

$$\begin{aligned}
&u''(r) \in \frac{n-1}{r}u'(r) + \lambda k_1(r)F(u(r), v(r)), \quad (R_1 < r < R_2),\\
&v''(r) \in \frac{n-1}{r}v'(r) + \mu k_2(r)G(u(r), v(r)), \quad (R_1 < r < R_2),\\
&a_1u(R_1) - b_1u'(R_1) = 0, \qquad a_2v(R_1) - b_2v'(R_1) = 0,\\
&c_1u(R_2) + d_1u'(R_2) = 0, \qquad c_2v(R_2) + d_2v'(R_2) = 0.
\end{aligned} \tag{10}$$

By applying the change of variables $s = -\int_r^{R_2}(1/t^{n-1})\mathrm{d}t$, followed by the change of variables $t = (m - s)/m$, with $m = -\int_{R_1}^{R_2}(1/t^{n-1})\mathrm{d}t$, (10) can be brought into the form

$$\begin{aligned}
&u''(t) \in \lambda h_1(t)F(u(t), v(t)), \quad (0 < t < 1),\\
&v''(t) \in \mu h_2(t)G(u(t), v(t)), \quad (0 < t < 1),\\
&a_1u(0) - b_1u'(0) = 0, \qquad a_2v(0) - b_2v'(0) = 0,\\
&c_1u(1) + d_1u'(1) = 0, \qquad c_2v(1) + d_2v'(1) = 0,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
h_1(t) &= m^2r^{2(n-1)}(m(1-t))k_1(r(m(1-t))),\\
h_2(t) &= m^2r^{2(n-1)}(m(1-t))k_2(r(m(1-t))),
\end{aligned}$$

and b_i, d_i are relabels of $-b_iR_1^{1-n}/m$, $-d_iR_2^{1-n}/m$, respectively. The system (11), in turn, is equivalent to the system of integral inclusions

$$\begin{cases}
u(t) \in \lambda \displaystyle\int_0^1 K_1(t, s)h_1(s)F(u(s), v(s))\mathrm{d}s,\\
v(t) \in \mu \displaystyle\int_0^1 K_2(t, s)h_2(s)G(u(s), v(s))\mathrm{d}s
\end{cases} \tag{12}$$

with $K_i(t, s)$, $i = 1, 2$, being the Green's function

$$K_i(t, s) = \frac{1}{\rho_i}\begin{cases}(c_i + d_i - c_it)(b_i + a_is), & s \le t,\\ (b_i + a_it)(c_i + d_i - c_is), & t \le s.\end{cases}$$

It is easy to see that for $i = 1, 2$

$$K_i(t,s) \geq 0, \quad (t,s) \in [0,1] \times [0,1]$$

and there exists a constant $0 < \rho < 1$ such that

$$K_i(t,s) \geq \rho(c_i + d_i - c_i t)(b_i + a_i t) K_i(s,s), \quad t, s \in [0,1].$$

Let $\sigma_i(t) = \rho(c_i + d_i - c_i t)(b_i + a_i t)$. Clearly, $\sigma_i(t) \geq 0$. Denote

$$M_1 = \int_0^1 K_1(s,s) h_1(s) \sigma(s) \mathrm{d}s, \qquad M_2 = \int_0^1 K_2(s,s) h_2(s) \sigma(s) \mathrm{d}s$$

with $\sigma(t) = \min\{\sigma_1(t), \sigma_2(t)\}$. Then it is clear that $M_1, M_2 > 0$. Define multivalued operators as follows

$$A_\lambda(u,v)(t) = \lambda \int_0^1 K_1(t,s) h_1(s) F(u(s), v(s)) \mathrm{d}s,$$
$$B_\mu(u,v)(t) = \mu \int_0^1 K_2(t,s) h_2(s) G(u(s), v(s)) \mathrm{d}s,$$
$$T_{\lambda\mu}(u,v)(t) = (A_\lambda(u,v)(t), B_\mu(u,v)(t)).$$

Then (12) is equivalent to fixed point problem

$$(u,v) \in T_{\lambda\mu}(u,v)$$

in usual Banach space $E =: C[0,1] \times C[0,1]$ with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$. Let E be introduced as partially ordered by $(u_1, v_1) \leq (u_2, v_2)$ if and only if $u_1 \leq u_2$ and $v_1 \leq v_2$ and a set $P \subset E$ be defined by

$$P = \{(u,v) \in E : u(t) \geq 0, v(t) \geq 0, t \in [0,1]\}.$$

Clearly, P is a normal cone of E with the normal constant $N = 1$. Fix $U_0(t) = (u_0(t), v_0(t)) \equiv (1,1)$, define

$$P(U_0) = \{(u,v) \in P : u(t) > \sigma_1(t)\|(u,v)\|, v(t) > \sigma_2(t)\|(u,v)\|, t \in [0,1]\}.$$

Theorem 4. *Assume that*

(S1) *There exist functions* $\xi_1, \xi_2 \in L^2[0,1]$, *satisfying*

$$f(u(t), v(t)) \leq \xi_1(t), \qquad g(u(t), v(t)) \leq \xi_2(t)$$

for all $f \in F(u,v)$ *and* $g \in G(u,v)$ *with* $(u,v) \in P$.

(S2) $k_1, k_2 : [R_1, R_2] \to [0, \infty)$ *are continuous and do not vanish identically on any subinterval of* $[R_1, R_2]$.

(S3) *Let* $z_0(u,v) = \min F(u,v)$, $w_0(u,v) = \min G(u,v)$ *and the following limits hold:*

$$\varliminf_{(u,v)\to 0} \frac{z_0(u,v)}{\max\{|u|, |v|\}} = \infty,$$
$$\varliminf_{(u,v)\to 0} \frac{w_0(u,v)}{\max\{|u|.|v|\}} = \infty.$$

Then, for any $\lambda, \mu \in (0, \infty)$, (1) *has at least one positive solution.*

In order to prove our results, we first prove some lemmas.

Lemma 3. *Let us denote*

$$\Phi_1(\tau, z) = \sup_{(u,v)\in\partial\Omega_\tau \cap P} \max_{t\in[0,1]} \int_0^1 K_1(t,s) h_1(s) z(u(s), v(s)) \mathrm{d}s,$$

$$\Phi_2(\tau, w) = \sup_{(u,v)\in\partial\Omega_\tau \cap P} \max_{t\in[0,1]} \int_0^1 K_2(t,s) h_2(s) w(u(s), v(s)) \mathrm{d}s,$$

where $z \in F(u, v)$, $w \in G(u, v)$, then one has

$$\lim_{\tau\to\infty} \frac{\Phi_1(\tau, z)}{\tau} = 0, \qquad \lim_{\tau\to\infty} \frac{\Phi_2(\tau, w)}{\tau} = 0 \tag{13}$$

uniformly for $z \in F(u, v)$ and $w \in G(u, v)$.

Proof. For any $(u, v) \in \partial\Omega_\tau \cap P$, $z(u, v) \in F(u, v)$, and $t \in [0, 1]$, from (S1) we deduce that

$$\begin{aligned}\int_0^1 K_1(t, s)h_1(s)z(u(s), v(s))\mathrm{d}s &\le \left\{\int_0^1 [K_1(t, s)h_1(s)]^2 \mathrm{d}s\right\}^{\frac{1}{2}} \left\{\int_0^1 z^2(u(s), v(s))\mathrm{d}s\right\}^{\frac{1}{2}} \\ &\le \left\{\int_0^1 [K_1(s, s)h_1(s)]^2 \mathrm{d}s\right\}^{\frac{1}{2}} \left\{\int_0^1 \xi_1^2(s)\mathrm{d}s\right\}^{\frac{1}{2}}.\end{aligned}$$

Consequently, $\lim_{\tau\to\infty} \frac{\Phi_1(\tau,z)}{\tau} = 0$ uniformly for $z \in F(u, v)$. Similarly, $\lim_{\tau\to\infty} \frac{\Phi_2(\tau,w)}{\tau} = 0$ uniformly for $w \in G(u, v)$. Hence, (13) holds and this proof is completed. □

Lemma 4. *Let us denote*

$$\Psi_1(\tau, z) = \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \max_{t\in[0,1]} \int_0^1 K_1(t, s)h_1(s)z(u(s), v(s))\mathrm{d}s,$$

$$\Psi_2(\tau, w) = \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \max_{t\in[0,1]} \int_0^1 K_2(t, s)h_2(s)w(u(s), v(s))\mathrm{d}s,$$

where $z \in F(u, v)$, $w \in G(u, v)$, then the following limit is true uniformly for $z \in F(u, v)$, $w \in G(u, v)$.

$$\lim_{\tau\to 0^+} \frac{\Psi_1(\tau, z)}{\tau} = \infty, \qquad \lim_{\tau\to 0^+} \frac{\Psi_2(\tau, w)}{\tau} = \infty. \tag{14}$$

Proof. In virtue of hypothesis (S3), for any natural number m, there exists $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$ when $m \to \infty$ and

$$f(u, v) \ge m \max\{|u|, |v|\}, \quad (u, v) \in (0, \varepsilon_m) \times (0, \varepsilon_m),\ f(u, v) \in F(u, v).$$

For any $(u, v) \in P(U_0)$ with $\|(u, v)\| \le \varepsilon_m$, the definition of $P(U_0)$ guarantees

$$0 < u(t), v(t) \le \varepsilon_m, \quad t \in [0, 1].$$

Hence we infer

$$\frac{z_0(u(t), v(t))}{\max\{|u(t)|, |v(t)|\}} \ge m, \quad 0 < \|(u, v)\| \le \varepsilon_m \text{ and } t \in [0, 1].$$

In view of this, together with (S3), for any $\tau \in (0, \varepsilon_m)$ and $z(u, v) \in F(u, v)$, we have

$$\begin{aligned}\frac{\Psi_1(\tau, z)}{\tau} &= \frac{1}{\tau} \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \max_{t\in[0,1]} \int_0^1 K_1(t, s)h_1(s)z(u(s), v(s))\mathrm{d}s \\ &\ge \frac{1}{\tau} \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \max_{t\in[0,1]} \sigma_1(t) \int_0^1 K_1(s, s)h_1(s)z_0(u(s), v(s))\mathrm{d}s \\ &\ge \frac{M_3}{\tau} \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \int_0^1 K_1(s, s)h_1(s)\frac{\min\{u(s), v(s)\}z_0(u(s), v(s))}{\max\{|u(s)|, |v(s)|\}}\mathrm{d}s \\ &> \frac{M_3}{\tau} \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \|(u, v)\| \int_0^1 K_1(s, s)h_1(s)\frac{\min\{\sigma_1(s), \sigma_2(s)\}z_0(u(s), v(s))}{\max\{|u(s)|, |v(s)|\}}\mathrm{d}s \\ &= M_3 \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \int_0^1 K_1(s, s)h_1(s)\frac{\sigma(s)z_0(u(s), v(s))}{\max\{|u(s)|, |v(s)|\}}\mathrm{d}s\end{aligned}$$

$$\begin{aligned}&\geq M_3 m \int_0^1 K_1(s,s)h_1(s)\sigma(s)\mathrm{d}s\\&= M_1 M_3 m,\end{aligned}$$

where $M_3 = \max_{t\in[0,1]} \sigma_1(t)$. Obviously, $M_1M_3 > 0$. This implies $\lim_{\tau\to 0^+} \frac{\Psi_1(\tau,z)}{\tau} = \infty$ uniformly for $z \in F(u, v)$ when $m \to \infty$. Similarly, we can prove $\lim_{\tau\to 0^+} \frac{\Psi_2(\tau,w)}{\tau} = \infty$ uniformly for $w \in G(u, v)$. The proof of Lemma 4 is completed. □

Proof of Theorem 4. It is easy to see that, for each $\lambda, \mu \in (0, \infty)$, $T_{\lambda\mu}$ is u.s.c. and completely continuous and $T_{\lambda\mu}(P) \subset CK(P)$. To prove that the result of Theorem 4 is true, it is sufficient to show that $T_{\lambda\mu}$ has at least one positive fixed point. For this purpose, we seek that all conditions of Theorem 1 are fulfilled. We will show that $T_{\lambda\mu}$ satisfies the condition (H1). For any $\lambda, \mu \in (0, \infty)$ and $(u, v) \in \partial\Omega_R \cap P$ with some large enough $R > 0$, we know from (13) that

$$\frac{\lambda\Phi_1(R,z)}{R} \leq 1, \qquad \frac{\mu\Phi_2(R,w)}{R} \leq 1 \quad \text{for all } z \in F(u,v),\, w \in G(u,v). \tag{15}$$

Similarly, for any $(u, v) \in \partial\Omega_r \cap P$ with some small enough $0 < r \ll R$, we know from (14) that

$$\frac{\lambda\Psi_1(r,z)}{r} > 1, \qquad \frac{\mu\Psi_2(r,w)}{r} > 1 \quad \text{for all } z \in F(u,v),\, w \in G(u,v). \tag{16}$$

Noting that the definitions of A_λ, B_μ yield that, for any $(u, v) \in \partial\Omega_\tau \cap P$, and any $y_{u,v} \in A_\lambda(u, v)$, there exists $z(u, v) \in F(u, v)$ such that $y_{u,v}(t) = \lambda \int_0^1 K_1(t,s)h_1(s)z(u(s), v(s))\mathrm{d}s$ and

$$\lambda\Phi_1(\tau,z) = \sup_{(u,v)\in\partial\Omega_\tau\cap P} \|y_{u,v}\|, \qquad \lambda\Psi_1(\tau,z) = \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \|y_{u,v}\|.$$

Also, for any $x_{u,v} \in B_\mu(u, v)$, there exists $w \in G(u, v)$ such that $x_{u,v}(t) = \mu \int_0^1 K_2(t,s)h_2(s)w(u(s), v(s))\mathrm{d}s$ and

$$\mu\Phi_2(\tau,w) = \sup_{(u,v)\in\partial\Omega_\tau\cap P} \|x_{u,v}\|, \qquad \mu\Psi_2(\tau,w) = \inf_{(u,v)\in\partial\Omega_\tau\cap P(U_0)} \|x_{u,v}\|.$$

Thus it follows from (15)

$$\|y_{u,v}\| \leq \|(u,v)\|, \qquad \|x_{u,v}\| \leq \|(u,v)\| \quad \text{for } (u,v) \in \partial\Omega_R \cap P$$

and from (16)

$$\|y_{u,v}\| > \|(u,v)\|, \qquad \|x_{u,v}\| \geq \|(u,v)\| \quad \text{for } x \in \partial\Omega_r \cap P(U_0).$$

The arbitrariness of $y_{u,v}$, $x_{u,v}$ shows

$$\|f\| \leq \|(u,v)\| \quad \text{for all } f \in T_{\lambda\mu}(u,v) \text{ and } (u,v) \in \partial\Omega_R \cap P, \tag{17}$$

$$\|f\| > \|(u,v)\| \quad \text{for all } f \in T_{\lambda\mu}(u,v) \text{ and } (u,v) \in \partial\Omega_r \cap P(U_0). \tag{18}$$

By virtue of (17) and (18), $T_{\lambda\mu}$ satisfies (H1), and we infer that there exists $(u, v) \in P$, which is a fixed point of $T_{\lambda\mu}$, such that $r \leq \|(u, v)\| \leq R$. This proof is completed. □

In what follows, we deal with the existence of two positive solutions for (1).

Lemma 5. *Assume that the following condition holds*

(S4) *For each* $\alpha \in (0, 1)$ *there exists* $\varepsilon_\alpha > 0$ *such that, for any* $0 \leq \max\{|u(t)|, |v(t)|\} \leq \varepsilon_\alpha$, $t \in [0, 1]$,

$$z(t) \leq \alpha\sigma_1(t)\max\{|u(t)|,|v(t)|\} \quad \text{with } z(t) \in F(u(t),v(t)),$$
$$w(t) \leq \alpha\sigma_2(t)\max\{|u(t)|,|v(t)|\} \quad \text{with } w(t) \in G(u(t),v(t)).$$

Then one has

$$\lim_{\tau\to 0^+} \frac{\Phi_1(\tau,z)}{\tau} = 0, \qquad \lim_{\tau\to 0^+} \frac{\Phi_2(\tau,w)}{\tau} = 0 \tag{19}$$

uniformly for $z \in F(u, v)$ *and* $w \in G(u, v)$, *where,* Φ_1, Φ_2 *are given as in Lemma 3.*

Proof. In virtue of hypothesis (S4), for any positive integer m, there exists $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$ when $m \to \infty$ and

$$z(t) \leq \frac{1}{m}\sigma_1(t)\max\{|u|, |v|\}, \quad t \in [0, 1], (u, v) \in [0, \varepsilon_m] \times [0, \varepsilon_m], z(t) \in F(u(t), v(t)).$$

For any $(u, v) \in P$ with $0 < \|(u, v)\| \leq \varepsilon_m$, the definition of the norm in E guarantees

$$0 \leq u(t), v(t) \leq \varepsilon_m, \quad t \in [0, 1].$$

Hence we infer

$$z(t) \leq \frac{1}{m}\sigma_1(t)\max\{|u(t)|, |v(t)|\}, \quad 0 \leq \|(u, v)\| \leq \varepsilon_m, z(t) \in F(u(t), v(t)) \text{ and } t \in [0, 1].$$

In view of this, together with (S4), for any $\tau \in (0, \varepsilon_m)$ and $z(u, v) \in F(u, v)$, we have

$$\begin{aligned}
\frac{\Phi_1(\tau, z)}{\tau} &= \frac{1}{\tau} \sup_{(u,v)\in\partial\Omega_\tau\cap P} \max_{t\in[0,1]} \int_0^1 K_1(t, s)h_1(s)z(u(s), v(s))\mathrm{d}s \\
&\leq \frac{1}{m\tau} \sup_{(u,v)\in\partial\Omega_\tau\cap P} \max_{t\in[0,1]} \int_0^1 K_1(t, s)h_1(s)\sigma_1(s)\max\{|u(s)|, |v(s)|\}\mathrm{d}s \\
&\leq \frac{1}{m\tau} \sup_{(u,v)\in\partial\Omega_\tau\cap P} \|(u, v)\| \int_0^1 K_1(s, s)h_1(s)\sigma_1(s)\mathrm{d}s = \frac{M_2'}{m},
\end{aligned}$$

where $M_2' = \int_0^1 K_1(s, s)h_1(s)\sigma_1(s)\mathrm{d}s$. This obviously implies $\lim_{\tau\to0^+} \frac{\Phi_1(\tau,z)}{\tau} = 0$ uniformly for $z \in F(u, v)$ when $m \to \infty$. Similarly, we can prove $\lim_{\tau\to0^+} \frac{\Phi_2(\tau,w)}{\tau} = 0$ uniformly for $w \in G(u, v)$. The proof of Lemma 5 is completed. □

Theorem 5. *If the conditions* (S1), (S2) and (S4) *hold. In addition, assume there exist the continuous functions* $\varphi_1, \varphi_2 : [0, \infty) \to [0, \infty)$ *with* $\frac{\varphi_i(y)}{y}$ *nonincreasing on* $(0, \infty)(i = 1, 2)$ *such that*

$$F(u, v) \geq \varphi_1(\min\{u, v\}), \qquad G(u, v) \geq \varphi_2(\min\{u, v\}) \tag{20}$$

for all $(u, v) \geq (0, 0)$, *and there exists* $L > 0$ *such that*

$$L \leq \min\left\{\lambda\varphi_1(L) \max_{0\leq t\leq 1} \sigma_1(t)M_1, \mu\varphi_2(L) \max_{0\leq t\leq 1} \sigma_2(t)M_2\right\}. \tag{21}$$

Then (1) *has at least two positive solutions* $(u_1, v_1), (u_2, v_2) \in E$ *with* $0 < \|(u_1, v_1)\| < \|(u_2, v_2)\|$.

Proof. In the proof of Theorem 4 we have pointed out that $T_{\lambda\mu}$ is u.s.c. for each $\lambda, \mu \in (0, \infty)$ and completely continuous and $T_{\lambda\mu}(P) \subset CK(P)$. Let us show that

$$\|x\| < \|y\| \quad \text{for all } \; y \in T_{\lambda\mu}x, x = (u, v) \in P(u_0) \cap \partial\Omega_L. \tag{22}$$

To see this let $y = (z, w) \in T_{\lambda\mu}x$, there exists $f \in F(u, v)$ such that $z(t) = \lambda \int_0^1 K_1(t, s)h_1(s)f(s)\mathrm{d}s \in A_\lambda f(t)$. Note that $(u, v) \in P(U_0) \cap \partial\Omega_L$, so we have

$$\|(u, v)\| = L \quad \text{and} \quad u(t) > \sigma_1(t)L \geq 0, \qquad v(t) > \sigma_2(t)L \geq 0 \quad \text{for each } t \in [0, 1].$$

Now let $t_0 \in [0, 1]$ be such that $\sigma_1(t_0) = \max_{0\leq t\leq 1} \sigma_1(t) > 0$, we have

$$\begin{aligned}
z(t_0) &= \lambda \int_0^1 K_1(t_0, s)h_1(s)f(s)\mathrm{d}s \\
&\geq \lambda \int_0^1 K_1(s, s)\sigma_1(t_0)h_1(s)\varphi_1(\min\{u(s), v(s)\})\mathrm{d}s \\
&= \lambda\sigma_1(t_0) \int_0^1 K_1(s, s)h_1(s)\frac{\varphi_1(\min\{u(s), v(s)\})}{\min\{u(t), v(t)\}} \min\{u(t), v(t)\}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\lambda\varphi_1(L)}{L}\sigma_1(t_0)\int_0^1 K_1(s,s)h_1(s)\min\{u(t),v(t)\}\mathrm{d}s \\
&> \frac{\lambda\varphi_1(L)}{L}\sigma_1(t_0)\int_0^1 K_1(s,s)h_1(s)\sigma(s)L\mathrm{d}s \\
&= \lambda\varphi_1(L)\sigma_1(t_0)M_2 \geq L = \|(u,v)\|.
\end{aligned}
$$

This implies that $\|z\| \geq z(t_0) > \|(u,v)\|$ for all $z \in A_\lambda(u,v)$. Similarly, we have $\|w\| > \|(u,v)\|$ for all $w \in B_\mu(u,v)$. As a result, we infer that (22) is true. Moreover, by (22) we obtain that $T_{\lambda\mu}$ satisfies the condition (H3).

By means of Lemma 3, there exists a constant $R > L$ such that (15) holds. We also know from Lemma 5 that there exists another constant $0 < r < L$ such that

$$
\frac{\lambda\Phi_1(r,z)}{r} \leq 1, \qquad \frac{\mu\Phi_2(r,w)}{r} \leq 1 \quad \text{for all } z \in F(u,v),\, w \in G(u,v). \tag{23}
$$

By means of (15) and (23), combining the proof of Theorem 4, we are able to infer that $|y| \leq |x|$ for all $y \in T_{\lambda\mu}x$ and $x \in P \cap \partial\Omega_r$ or for all $y \in T_{\lambda\mu}x$ and $x \in P \cap \partial\Omega_R$. This, together with (22), shows that $T_{\lambda\mu}$ satisfies the condition (H5). Theorem 5 is proved by Theorem 3. □

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