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A Note on Barotropic Compressible Quantum Navier-Stokes Equations

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Abstract: The global-in-time existence of weak solutions to the barotropic compressible quantum Navier-Stokes equations has been proved by Jüngel [1] very recently if the viscosity constant is smaller than the scaled Plank constant. This paper extends the results to the case that the viscosity constant equals the scaled Plank constant. By a new estimate on the square root of the solution, apparently not available in paper [1], the semiclassical limit for the viscous quantum Euler equations (which is equivalent to the barotropic compressible quantum Navier-Stokes equations) can be performed, then the results of this paper are obtained easily.

Keywords: Compressible quantum Navier-Stokes equations, Viscous quantum Euler equations, Global existence.

1. Introduction and main results

This paper we investigate the following barotropic compressible quantum Navier-Stokes equations with initial data [1]:

$$n_t + \mathrm{div}(nu) = 0, \quad x \in \mathbf{T}^d, \ \ t > 0, \tag{1.1}$$

$$(nu)_t + \mathrm{div}(nu \otimes u) + \nabla p(n) - 2\varepsilon^2 n \nabla \left(\frac{\triangle \sqrt{n}}{\sqrt{n}} \right) - nf = 2\nu \mathrm{div}(nD(u)), \tag{1.2}$$

$$n(\cdot, 0) = n_0, \quad (nu)(\cdot, 0) = n_0 u_0 \quad \text{in } \mathbf{T}^d, \tag{1.3}$$

where the particle density n and the particle velocity u are unknown variables, the scaled Planck constant $\varepsilon > 0$ and the viscosity constant $\nu > 0$ are physical parameters, the function $p(n) = n^\gamma$ with $\gamma \geq 1$ is the pressure and f describes external forces, $D(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ is the symmetric part of the velocity gradient and $\mathbf{T}^d$ is the d-dimensional torus ($d \leq 3$).

There are a few mathematical results for the Navier-Stokes equations (1.1)-(1.2) with $\varepsilon = 0$ [2-7]. The problem (1.1)-(1.3) with the case $\varepsilon = \nu > 0$ and $d = 1$ has been treated in [8]. It seems that the barotropic compressible quantum Navier-Stokes equations (1.1)-(1.2) with $\varepsilon > 0$ and multidimensional situation has been analytically investigated for the first time in [1]. The main idea of [1] is to transform the quantum Navier-Stokes equations (1.1)-(1.3) by means of the so-called effective velocity $w = u + \nu \nabla \log n$ to the viscous quantum Euler equations:

$$n_t + \mathrm{div}(nw) = \nu \triangle n, \qquad x \in \mathbf{T}^d, \quad t > 0, \tag{1.4}$$

$$(nw)_t + \mathrm{div}(nw \otimes w) + \nabla p(n) - 2\varepsilon_0^2 n \nabla \left(\frac{\triangle \sqrt{n}}{\sqrt{n}} \right) - nf = \nu \triangle (nw), \tag{1.5}$$

$$n(\cdot, 0) = n_0, \qquad (nw)(\cdot, 0) = n_0 w_0 \quad \text{ in } \mathbf{T}^d, \tag{1.6}$$

where $w_0 = u_0 + \nu \nabla \log n_0$ and $\varepsilon_0^2 = \varepsilon^2 - \nu^2$. Then by using the Faedo-Galerkin method and weak compactness techniques the global existence of weak solutions to the viscous quantum Euler equations (1.4)-(1.6) with $\varepsilon_0 > 0$ is shown, which deduces the existence of solutions to the quantum Navier-Stokes equations (1.1)-(1.3) with $\varepsilon > \nu$. This paper we will extend the results to the case $\varepsilon = \nu$.

Our main results are stated as follows:

Theorem 1.1 Let $\varepsilon = \nu > 0$ and other assumptions of Corollary 1.2 in [1] hold, then the results of Corollary 1.2 in [1] are also right.

2. Proof of the results

The main idea of this paper is to perform the semiclassical limit ($\varepsilon_0 \to 0$) for the viscous quantum Euler equations (1.4)-(1.6), which will deduce the results of Theorem 1.1 easily. Let $(n_{\varepsilon_0}, w_{\varepsilon_0})$ be a weak solution of problem (1.4)-(1.6) with $\varepsilon_0 > 0$, which is proved in Theorem 1.1 of [1]. From Lemma 3.1 and (4.2), (4.3) in [1], we can see apparently that

Lemma 2.1

$$\| n_{\varepsilon_0} \|_{L^\infty(0,T;L^\gamma(\mathbf{T}^d))} \leq C, \tag{2.1}$$

$$\| \sqrt{n_{\varepsilon_0}} w_{\varepsilon_0} \|_{L^\infty(0,T;L^2(\mathbf{T}^d))} + \| \sqrt{n_{\varepsilon_0}} \nabla w_{\varepsilon_0} \|_{L^2(0,T;L^2(\mathbf{T}^d))} \leq C, \tag{2.2}$$

Attentively, we can see from Lemma 3.1 and Lemma 4.1 in [1] that the estimates $\| \sqrt{n_{\varepsilon_0}} \|_{L^\infty(0,T;H^1(\mathbf{T}^d))}$, $\| \sqrt{n_{\varepsilon_0}} \|_{L^2(0,T;H^2(\mathbf{T}^d))}$ and $\| \sqrt[4]{n_{\varepsilon_0}} \|_{L^4(0,T;W^{1,4}(\mathbf{T}^d))}$ depend on ε_0. In the following, we will obtain these uniform estimates independent of $\varepsilon_0 \to 0$.

Lemma 2.2

$$\| \sqrt{n_{\varepsilon_0}} \|_{L^\infty(0,T;H^1(\mathbf{T}^d))} \leq C, \tag{2.3}$$

$$\| \sqrt{n_{\varepsilon_0}} \|_{L^2(0,T;H^2(\mathbf{T}^d))} + \| \sqrt[4]{n_{\varepsilon_0}} \|_{L^4(0,T;W^{1,4}(\mathbf{T}^d))} \leq C, \tag{2.4}$$

where $C > 0$ is independent of $\varepsilon_0 \to 0$.

Proof. Use $\frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}}$ as a test function in (1.4) to obtain

$$\int_{\mathbf{T}^d} (n_{\varepsilon_0})_t \frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} dx + \int_{\mathbf{T}^d} \mathrm{div}(n_{\varepsilon_0} w_{\varepsilon_0}) \frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} dx = \nu \int_{\mathbf{T}^d} \triangle n_{\varepsilon_0} \frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} dx. \tag{2.5}$$

Integrating by parts, it follows

$$\int_{\mathbf{T}^d} (n_{\varepsilon_0})_t \frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} dx = -\frac{d}{dt} \int_{\mathbf{T}^d} |\nabla \sqrt{n_{\varepsilon_0}}|^2 dx, \tag{2.6}$$

$$\int_{\mathbf{T}^d} \triangle n_{\varepsilon_0} \frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} dx = \frac{1}{2} \int_{\mathbf{T}^d} n_{\varepsilon_0} |\nabla^2 \log n_{\varepsilon_0}|^2 dx, \tag{2.7}$$

$$\begin{aligned}\int_{\mathbf{T}^d} \mathrm{div}(n_{\varepsilon_0} w_{\varepsilon_0}) \frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} dx &= -\int_{\mathbf{T}^d} n_{\varepsilon_0} w_{\varepsilon_0} \cdot \nabla \left(\frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} \right) dx = -\frac{1}{2} \int_{\mathbf{T}^d} w_{\varepsilon_0} \mathrm{div}(n_{\varepsilon_0} \nabla^2 \log n_{\varepsilon_0}) dx \\ &= \frac{1}{2} \int_{\mathbf{T}^d} \nabla w_{\varepsilon_0} : (n_{\varepsilon_0} \nabla^2 \log n_{\varepsilon_0}) dx = \frac{1}{2} \int_{\mathbf{T}^d} (\sqrt{n_{\varepsilon_0}} \nabla w_{\varepsilon_0}) : (\sqrt{n_{\varepsilon_0}} \nabla^2 \log n_{\varepsilon_0}). \end{aligned} \tag{2.8}$$

The product "A : B" means summation over both indices of the matrices A and B. In (2.7) and (2.8), we have used the identity $2n_{\varepsilon_0} \nabla \left(\frac{\triangle\sqrt{n_{\varepsilon_0}}}{\sqrt{n_{\varepsilon_0}}} \right) = \mathrm{div}(n_{\varepsilon_0} \nabla^2 \log n_{\varepsilon_0})$.

By (2.5)-(2.8) and Cauchy inequality, we get

$$\begin{aligned}\frac{d}{dt} \int_{\mathbf{T}^d} |\nabla \sqrt{n_{\varepsilon_0}}|^2 dx + \frac{\nu}{2} \int_{\mathbf{T}^d} n_{\varepsilon_0} |\nabla^2 \log n_{\varepsilon_0}|^2 dx &= \frac{1}{2} \int_{\mathbf{T}^d} (\sqrt{n_{\varepsilon_0}} \nabla w_{\varepsilon_0}) : (\sqrt{n_{\varepsilon_0}} \nabla^2 \log n_{\varepsilon_0}) \\ &\leq \frac{\nu}{4} \int_{\mathbf{T}^d} n_{\varepsilon_0} |\nabla^2 \log n_{\varepsilon_0}|^2 dx + \frac{1}{4\nu} \int_{\mathbf{T}^d} |\sqrt{n_{\varepsilon_0}} \nabla w_{\varepsilon_0}|^2 dx,\end{aligned}$$

namely,

$$\frac{d}{dt} \int_{\mathbf{T}^d} |\nabla \sqrt{n_{\varepsilon_0}}|^2 dx + \frac{\nu}{4} \int_{\mathbf{T}^d} n_{\varepsilon_0} |\nabla^2 \log n_{\varepsilon_0}|^2 dx \leq \frac{1}{4\nu} \int_{\mathbf{T}^d} |\sqrt{n_{\varepsilon_0}} \nabla w_{\varepsilon_0}|^2 dx. \tag{2.9}$$

Using (2.2) and (2.9), we infer the estimates (2.3) and

$$\int_0^T \int_{\mathbf{T}^d} n_{\varepsilon_0} |\nabla^2 \log n_{\varepsilon_0}|^2 dx dt \leq C \tag{2.10}$$

hold, where $C > 0$ is independent of $\varepsilon_0 \to 0$. By (2.5) in [9] and (2.10), we can conclude (2.4) holds.

Proof of Theorem 1.1: With Lemma 2.1 and Lemma 2.2 at hand, we can perform the semiclassical limit ($\varepsilon_0 \to 0$) for (1.4)-(1.6) by weak compactness techniques as in [1], then the results of Theorem 1.1 hold, we omit the details.

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